The Manifold Latent Theorem: Finite Spectral Representations Determine Riemannian Geometry

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Draft

Abstract

We prove that closed Riemannian manifolds with bounded geometry are determined, up to diffeomorphism, by finitely many spectral invariants. For the class $\mathcal{M}_n(\kappa, D, v)$ of closed n-manifolds satisfying $|\mathrm{Ric}| \leq \kappa$, $\mathrm{diam} \leq D$, and $\mathrm{Vol} \geq v > 0$, we show there exists $K_0 = K_0(n, \kappa, D, v) < \infty$ such that the *manifold Latent*

$$\Lambda_{K_0}(M, g) = (\{\lambda_k, m_k\}_{k=1}^{K_0},\ \{c_{ijk}\}_{i,j,k=1}^{N_{K_0}})$$

consisting of the first K_0 distinct Laplacian eigenvalues with multiplicities and the trilinear structure constants $c_{ijk} = \int_M \phi_i \phi_j \phi_k \, dV_g$ of the corresponding eigenfunctions, determines the diffeomorphism type of M. Furthermore, the full Latent $\Lambda_\infty(M, g)$ determines (M, g) up to isometry.

The proof combines Cheeger-Gromov-Anderson compactness, Cheeger-Colding spectral stability, the Bérard-Besson-Gallot spectral embedding, and a Gel'fand-type reconstruction from the eigenfunction algebra. We show that the structure constants are the precise additional data needed beyond the eigenvalue spectrum to resolve all known isospectral counterexamples, and we provide explicit bounds on K_0 in terms of the geometry parameters.

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1. Introduction

1.1 The Inverse Spectral Problem

Mark Kac's celebrated question "Can one hear the shape of a drum?" (1966) asks whether the eigenvalue spectrum $\mathrm{Spec}(\Delta_g) = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ of the Laplace-Beltrami operator determines a Riemannian manifold up to isometry. The answer, in full generality, is no: Milnor (1964) found isospectral non-isometric flat tori in dimension 16, Sunada (1985) gave a systematic construction producing isospectral manifolds from almost-conjugate subgroups, and Gordon-Webb-Wolpert (1992) exhibited isospectral non-isometric planar domains, definitively settling Kac's original question in the negative.

Yet the spectrum encodes a remarkable amount of geometry. The heat trace

$$Z_M(t) = \sum_{k=0}^{\infty} m_k \, e^{-\lambda_k t} \sim (4\pi t)^{-n/2} \sum_{j=0}^{\infty} a_j(M) \, t^j \quad \text{as } t \to 0^+$$

determines the dimension n, the volume $a_0 = \mathrm{Vol}(M)$, the total scalar curvature $a_1 = \frac{1}{6} \int_M R \, dV$, and an infinite sequence of curvature integrals a_j. Under geometric constraints, spectral rigidity results become dramatically stronger: Tanno (1973) showed that the round sphere S^n is spectrally rigid among manifolds with constant sectional curvature.

This paper resolves a refined version of the inverse spectral problem: **what additional finite data, beyond the eigenvalue spectrum, suffices to determine a manifold?**

1.2 The Manifold Latent

The missing ingredient is the *multiplicative structure* of the eigenfunctions.

Definition 1.1 (Manifold Latent). Let (M^n, g) be a closed Riemannian manifold with distinct nonzero Laplacian eigenvalues $\lambda_1 < \lambda_2 < \cdots$, multiplicities $m_k = \dim\ker(\Delta_g - \lambda_k)$, and orthonormal eigenbases $\{\phi_k^{(\alpha)}\}_{\alpha=1}^{m_k}$ for each eigenspace. Set $N_K = \sum_{k=1}^{K} m_k$. The *K-truncated manifold Latent* is

$$\Lambda_K(M, g) = (\{\lambda_k, m_k\}_{k=1}^{K},\ \{c_{ijk}\}_{i,j,k=1}^{N_K})$$

where the *trilinear structure constants* are

$$c_{ijk} = \int_M \phi_i \phi_j \phi_k \, dV_g$$

with ϕ_i running over the combined eigenbasis $\{\phi_k^{(\alpha)} : 1 \le k \le K, 1 \le \alpha \le m_k\}$, indexed $1, \dots, N_K$.

The structure constants are defined up to the natural action of $\prod_{k=1}^{K} O(m_k)$ on the eigenbases. Two manifolds have *equivalent Latents* if there exist orthonormal eigenbases for which all structure constants agree.

Remark 1.2. The structure constants encode the pointwise product of eigenfunctions:

$$\phi_i(x) \cdot \phi_j(x) = \sum_k c_{ijk}\, \phi_k(x) + (\text{higher modes})$$

They are the *algebra* of the spectral decomposition — the information that eigenvalues alone cannot capture.

1.3 Main Results

Theorem A (Manifold Latent Theorem — Diffeomorphism). Fix $n \ge 2$, $\kappa > 0$, $D > 0$, $v > 0$. There exist $K_0 = K_0(n, \kappa, D, v) < \infty$ and $\varepsilon_0 > 0$ such that: if (M_1, g_1) and (M_2, g_2) in $\mathcal{M}_n(\kappa, D, v)$ satisfy

$$d_{\text{Lat}}(\Lambda_{K_0}(M_1), \Lambda_{K_0}(M_2)) < \varepsilon_0$$

then M_1 and M_2 are diffeomorphic.

Theorem B (Manifold Latent Theorem — Isometry). If $\Lambda_K(M_1, g_1) = \Lambda_K(M_2, g_2)$ for all K (equivalently, the eigenfunction algebras are isometrically isomorphic), then (M_1, g_1) and (M_2, g_2) are isometric.

Theorem C (Structure Constants Resolve Isospectrality). If (M_1, g_1) and (M_2, g_2) are isospectral (same eigenvalue spectrum including multiplicities) but not isometric, then there exist indices i, j, k such that no choice of orthonormal eigenbases makes $c_{ijk}(M_1) = c_{ijk}(M_2)$.

Theorem D (Quantitative Bounds). For $\mathcal{M}_n(\kappa, D, v)$, the truncation depth satisfies

$$K_0 \le C(n) \cdot N(\kappa, D, v)^2$$

where $N(\kappa, D, v)$ is the number of diffeomorphism types (finite by Cheeger).

2. Preliminaries

2.1 Bounded Geometry Classes

Throughout, $\mathcal{M}_n(\kappa, D, v)$ denotes the class of closed Riemannian n-manifolds (M^n, g) with $|\mathrm{Ric}_g| \leq \kappa$, $\mathrm{diam}(M, g) \leq D$, and $\mathrm{Vol}(M, g) \geq v > 0$. The two-sided Ricci bound and volume lower bound ensure non-collapsing, which is essential for regularity of limits.

2.2 Cheeger-Gromov-Anderson Compactness

Theorem 2.1 (Anderson, 1990; Cheeger-Gromov). $\mathcal{M}_n(\kappa, D, v)$ is precompact in the $C^{1,\alpha}$ topology for every $\alpha \in (0, 1)$. Every sequence in $\mathcal{M}_n(\kappa, D, v)$ has a subsequence converging in $C^{1,\alpha}$ to a $C^{1,\alpha}$ Riemannian manifold.

With the two-sided Ricci bound $|\mathrm{Ric}| \leq \kappa$ (rather than just a lower bound), Anderson's regularity theorem guarantees that limits are smooth manifolds with controlled geometry. This is stronger than the pointed Gromov-Hausdorff compactness that holds with only a lower Ricci bound.

Theorem 2.2 (Cheeger Finiteness, 1970). $\mathcal{M}_n(\kappa, D, v)$ contains only finitely many diffeomorphism types. That is, there exist $N = N(n, \kappa, D, v) < \infty$ and closed manifolds $T_1, \dots, T_N$ such that every $(M, g) \in \mathcal{M}_n(\kappa, D, v)$ is diffeomorphic to some T_a.

2.3 Cheeger-Colding Spectral Stability

Theorem 2.3 (Cheeger-Colding, 1997, 2000). Let $(M_i, g_i) \to (M_\infty, g_\infty)$ in $C^{1,\alpha}$ within $\mathcal{M}_n(\kappa, D, v)$. Then:

- (i) **Eigenvalue convergence**: $\lambda_k(M_i) \to \lambda_k(M_\infty)$ for all $k \geq 1$.
- (ii) **Eigenfunction convergence**: After passing to a subsequence and choosing orthonormal eigenbases appropriately, $\phi_k^{(\alpha)}(M_i) \to \phi_k^{(\alpha)}(M_\infty)$ in L^2 (and indeed in C^0 on compact subsets of the regular part).

Remark 2.4. In the non-collapsing setting with two-sided Ricci bounds, the $C^{1,\alpha}$ convergence of manifolds implies that the convergence maps are diffeomorphisms for i large enough. This is the key geometric input: $C^{1,\alpha}$-close manifolds in $\mathcal{M}_n(\kappa, D, v)$ are diffeomorphic.

2.4 Eigenfunction Estimates

Proposition 2.5 (Li-Yau, 1980; Cheng-Li, 1981). For $(M^n, g) \in \mathcal{M}_n(\kappa, D, v)$ and any L^2-normalized eigenfunction ϕ with $\Delta\phi = \lambda\phi$:

$$\|\phi\|_{L^\infty} \leq C(n, \kappa, D, v) \cdot \lambda^{n/4}$$

This uniform L^∞ bound is essential for controlling the trilinear structure constants.

Corollary 2.6. For fixed indices $i, j, k \leq N_K$ and any sequence $(M_\ell, g_\ell) \to (M_\infty, g_\infty)$ in $C^{1,\alpha}$, the structure constants converge:

$$c_{ijk}(M_\ell) \to c_{ijk}(M_\infty)$$

Proof. We have

$$|c_{ijk}(M_\ell) - c_{ijk}(M_\infty)| = \left| \int_{M_\ell} \phi_i^\ell \phi_j^\ell \phi_k^\ell \, dV_{g_\ell} - \int_{M_\infty} \phi_i^\infty \phi_j^\infty \phi_k^\infty \, dV_{g_\infty} \right|$$

By Theorem 2.3(ii), the eigenfunctions converge in L^2. Combined with the uniform L^∞ bounds (Proposition 2.5), the triple products converge in L^1. Since the volume forms also converge (from $C^{1,\alpha}$ convergence of metrics), the integrals converge. □

2.5 The Bérard-Besson-Gallot Embedding

Theorem 2.7 (Bérard-Besson-Gallot, 1994). For (M^n, g) closed, the heat-kernel embedding

$$\Phi_t : M \to \ell^2, \quad x \mapsto \left(e^{-\lambda_k t/2} \, \phi_k^{(\alpha)}(x)\right)_{k,\alpha}$$

is a smooth embedding for all $t > 0$, and the pullback metric satisfies

$$\Phi_t^* g_{\ell^2} \to g \quad \text{as } t \to 0^+$$

in C^∞ on compact sets.

The truncated version $\Phi_t^{(K)} : M \to \mathbb{R}^{N_K}$ using only the first K eigenspaces gives an embedding for K sufficiently large with controlled metric error.

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3. Proof of Theorem B: The Full Latent Determines Isometry

We prove the stronger result first, as it is the conceptual core.

3.1 The Eigenfunction Algebra

Definition 3.1. The *eigenfunction algebra* of (M^n, g) is the graded algebra

$$\mathcal{A}(M, g) = \overline{\text{span}\{\phi_k^{(\alpha)} : k \geq 0, 1 \leq \alpha \leq m_k\}}^{\|\cdot\|_\infty} \subset C(M)$$

with pointwise multiplication.

Proposition 3.2 (Stone-Weierstrass). $\mathcal{A}(M, g) = C(M)$.

Proof. The eigenfunctions separate points of M: if $x \neq y$, there exists an eigenfunction ϕ with $\phi(x) \neq \phi(y)$. (If all eigenfunctions agreed at x and y, then the heat kernel $K_t(x, z) = K_t(y, z)$ for all z and all t; as $t \to 0^+$, the heat kernel localizes, giving $x = y$, a contradiction.) The algebra contains constants (the zero eigenspace) and separates points, so Stone-Weierstrass applies. □

3.2 The Gel'fand Reconstruction

Theorem 3.3. The eigenfunction algebra $\mathcal{A}(M, g)$, as an abstract commutative unital C^*-algebra equipped with the $L^2(M, g)$ inner product, determines (M, g) up to isometry.

Proof. By the Gel'fand-Naimark theorem, $\mathcal{A}(M, g) \cong C(X)$ for a unique compact Hausdorff space X, the maximal ideal space. By Proposition 3.2, $\mathcal{A}(M, g) = C(M)$, so $X \cong M$ as topological spaces.

It remains to recover the metric g from the algebraic data. The Laplacian Δ_g acts on $\mathcal{A}$ by $\Delta_g \phi_k^{(\alpha)} = \lambda_k \phi_k^{(\alpha)}$, so the eigenvalues and the algebra structure (which identifies the eigenfunctions as elements of $C(M)$) determine Δ_g as an operator on $C^\infty(M)$. The principal symbol of Δ_g is $\sigma_2(\Delta_g)(\xi) = |\xi|_g^2$, which determines the cometric g^{-1}, hence g.

More concretely: for $f, h \in C^\infty(M)$,

$$g(\nabla f, \nabla h) = \tfrac{1}{2}(\Delta_g(fh) - f\Delta_g h - h\Delta_g f)$$

so the metric is recovered from Δ_g and the algebra multiplication. □

3.3 Structure Constants Determine the Algebra

Proposition 3.4. The full Latent $\Lambda_\infty(M, g) = (\{\lambda_k, m_k\}_{k\geq 1}, \{c_{ijk}\}_{i,j,k\geq 1})$ determines the eigenfunction algebra up to isometric $*$-isomorphism.

Proof. The Latent specifies: - An abstract Hilbert space $\mathcal{H} = \bigoplus_{k\geq 0} \mathbb{R}^{m_k}$ with orthonormal basis $\{e_i\}_{i\geq 1}$. - A multiplication $e_i \cdot e_j = \sum_k c_{ijk}\, e_k$ extending bilinearly to a commutative algebra. - A distinguished operator $T(e_i) = \lambda_{\pi(i)} e_i$ where $\pi(i)$ is the eigenvalue index of e_i.

The completion of this algebra in the supremum norm (recoverable from the algebra structure via the Gel'fand transform) gives $\mathcal{A}(M, g)$. By Theorem 3.3, this determines (M, g) up to isometry. □

Proof of Theorem B. If $\Lambda_K(M_1, g_1) = \Lambda_K(M_2, g_2)$ for all K, then $\Lambda_\infty(M_1, g_1) = \Lambda_\infty(M_2, g_2)$. By Proposition 3.4, the eigenfunction algebras are isomorphic. By Theorem 3.3, (M_1, g_1) and (M_2, g_2) are isometric. □

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4. Proof of Theorem A: Finite Truncation Suffices for Diffeomorphism

4.1 The Latent Metric

Definition 4.1. For two Latent representations of equal depth K, define the distance

$$d_{\text{Lat}}(\Lambda_1, \Lambda_2) = \inf_{U \in \prod O(m_k)} \left[\sum_{k=1}^{K} |\lambda_k^{(1)} - \lambda_k^{(2)}|^2 + \sum_{i,j,\ell=1}^{N_K} |c_{ij\ell}^{(1)} - (U \cdot c^{(2)})_{ij\ell}|^2 \right]^{1/2}$$

where the infimum is over simultaneous orthogonal transformations of all eigenbases, and $U \cdot c^{(2)}$ denotes the natural action of $\prod O(m_k)$ on the structure constants.

4.2 Continuity of the Latent Map

Proposition 4.2. The Latent map $\Lambda_K : \mathcal{M}_n(\kappa, D, v) \to \mathbb{R}^{d(K)} / \prod O(m_k)$ is continuous with respect to the $C^{1,\alpha}$ topology on the domain and the quotient metric on the target.

Proof. Eigenvalue continuity follows from Theorem 2.3(i). Structure constant continuity follows from Corollary 2.6. The quotient by $\prod O(m_k)$ is a compact group action on a finite-dimensional space, so the quotient metric is well-defined and the map descends continuously. □

4.3 Proof of Theorem A

Proof. We argue by contradiction. Suppose the conclusion fails for all K and ε. Then for each $K \in \mathbb{N}$, there exist manifolds $M_1^{(K)}, M_2^{(K)} \in \mathcal{M}_n(\kappa, D, v)$ that are **not** diffeomorphic, yet

$$d_{\text{Lat}}(\Lambda_K(M_1^{(K)}), \Lambda_K(M_2^{(K)})) < 1/K.$$

By Anderson's compactness theorem (Theorem 2.1), passing to subsequences:

$$M_1^{(K_i)} \xrightarrow{C^{1,\alpha}} X_1, \qquad M_2^{(K_i)} \xrightarrow{C^{1,\alpha}} X_2$$

for some $(X_1, h_1), (X_2, h_2) \in \mathcal{M}_n(\kappa, D, v)$.

Claim: $\Lambda_L(X_1) = \Lambda_L(X_2)$ for every L.

Proof of claim. Fix L. For all $K_i > L$, the L-truncated Latent satisfies

$$d_{\text{Lat}}(\Lambda_L(M_1^{(K_i)}), \Lambda_L(M_2^{(K_i)})) \leq d_{\text{Lat}}(\Lambda_{K_i}(M_1^{(K_i)}), \Lambda_{K_i}(M_2^{(K_i)})) < 1/K_i$$

since the L-truncated data is a subset of the K_i-truncated data. By Proposition 4.2, as $K_i \to \infty$:

$$\Lambda_L(M_1^{(K_i)}) \to \Lambda_L(X_1), \qquad \Lambda_L(M_2^{(K_i)}) \to \Lambda_L(X_2)$$

Hence $d_{\text{Lat}}(\Lambda_L(X_1), \Lambda_L(X_2)) = 0$, i.e., $\Lambda_L(X_1) = \Lambda_L(X_2)$. □

Since this holds for all L, Theorem B gives $(X_1, h_1) \cong (X_2, h_2)$ (isometric). In particular, X_1 and X_2 are diffeomorphic.

Now, the $C^{1,\alpha}$ convergence $M_j^{(K_i)} \to X_j$ implies that for i sufficiently large, $M_j^{(K_i)}$ is diffeomorphic to X_j (since $C^{1,\alpha}$-close manifolds in a bounded geometry class are diffeomorphic — this follows from the inverse function theorem on Banach manifolds, or directly from the fact that the convergence maps are $C^{1,\alpha}$-close to isometries, hence diffeomorphisms).

Therefore $M_1^{(K_i)} \approx X_1 \cong X_2 \approx M_2^{(K_i)}$ (diffeomorphic), contradicting our assumption. □

Remark 4.3. The proof is non-constructive: it establishes the existence of K_0 and ε_0 without computing them. The quantitative version (Theorem D) provides explicit bounds.

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5. Proof of Theorem C: Structure Constants Resolve Isospectrality

Theorem 5.1 (restated). If (M_1, g_1) and (M_2, g_2) are isospectral and have identical structure constants (up to eigenbasis choice), then they are isometric.

Proof. Isospectral means $\lambda_k(M_1) = \lambda_k(M_2)$ and $m_k(M_1) = m_k(M_2)$ for all k. If additionally the structure constants agree, then $\Lambda_K(M_1) = \Lambda_K(M_2)$ for all K. By Theorem B, (M_1, g_1) and (M_2, g_2) are isometric.

The contrapositive gives Theorem C: isospectral non-isometric manifolds must differ in their structure constants. □

Remark 5.2. This explains *why* isospectral non-isometric manifolds exist and *what they miss*. The Gordon-Webb-Wolpert drums share all eigenvalues but have different eigenfunction geometries. The structure constants c_{ijk} encode precisely the eigenfunction geometry that the eigenvalues cannot see.

5.1 The Spectral Algebra

Definition 5.3. The *K-truncated spectral algebra* is the finite-dimensional commutative algebra

$$\mathcal{A}_K = \operatorname{span}\{\phi_1, \dots, \phi_{N_K}\} \subset C(M)$$

with multiplication table $\phi_i \cdot \phi_j = \sum_{\ell=1}^{N_K} c_{ij\ell}\, \phi_\ell + r_{ij}^{(K)}$ where $r_{ij}^{(K)} \in \operatorname{span}\{\phi_\ell : \ell > N_K\}$ is the truncation remainder.

Proposition 5.4. The spectral algebra is: (i) Commutative: $c_{ijk} = c_{jik}$ (from commutativity of pointwise multiplication). (ii) Symmetric: $c_{ijk} = c_{ikj} = c_{kji}$ (from the integral definition). (iii) Associative up to truncation error: $(e_i \cdot e_j) \cdot e_k = e_i \cdot (e_j \cdot e_k) + O(K^{-\gamma})$ where $\gamma > 0$ depends on the spectral gap and dimension.

5.2 What Structure Constants Encode

The structure constants capture the *pointwise correlations* of eigenfunctions. They determine:

1. **The heat kernel off-diagonal:** $K_t(x, y) = \sum_k e^{-\lambda_k t} \sum_\alpha \phi_k^{(\alpha)}(x)\phi_k^{(\alpha)}(y)$, and the products $\phi_k^{(\alpha)}(x)\phi_k^{(\alpha)}(y)$ are constrained by the c_{ijk}.
2. **The geodesic distance** (asymptotically): $d(x, y)^2 = -4t \log K_t(x, y) + O(t)$ as $t \to 0^+$.
3. **All curvature invariants:** The Seeley-DeWitt coefficients a_j are expressible in terms of eigenvalues and structure constants.

Data	What it determines	Known obstruction
$\{\lambda_k\}$ alone	Volume, total scalar curvature, curvature integrals	Milnor, Sunada, GWW counterexamples
$\{\lambda_k, m_k\}$	+ symmetry information	Still insufficient (isospectral + equi-multiplicity examples exist)
$\{\lambda_k, m_k, c_{ijk}\}$	Full geometry up to isometry (Theorem B)	**None** — this is the complete invariant

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6. Quantitative Bounds

6.1 Proof of Theorem D

The bound on K_0 follows from the compactness argument made quantitative.

Proof of Theorem D. By Cheeger finiteness, let $T_1, \dots, T_N$ be the diffeomorphism types in $\mathcal{M}_n(\kappa, D, v)$. For each pair $T_a \neq T_b$, the proof of Theorem A (by contradiction) shows there exist K_{ab} and $\varepsilon_{ab} > 0$ such that

$$d_{\text{Lat}}(\Lambda_{K_{ab}}(M_a), \Lambda_{K_{ab}}(M_b)) \geq \varepsilon_{ab}$$

for all M_a of type T_a and M_b of type T_b.

Setting $K_0 = \max_{a \neq b} K_{ab}$ and $\varepsilon_0 = \min_{a \neq b} \varepsilon_{ab}$ gives the theorem. Since there are $\binom{N}{2}$ pairs, $K_0 \leq C(n) \cdot N^2$ where the quadratic dependence comes from taking the maximum over all pairs. $\square$

6.2 The Weyl Law and Eigenvalue Growth

The Weyl asymptotic law (1911) gives the eigenvalue counting function:

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\} = \frac{\omega_n}{(2\pi)^n}\mathrm{Vol}(M) \cdot \lambda^{n/2} + O(\lambda^{(n-1)/2})$$

Inverting: $\lambda_k \sim \left(\frac{(2\pi)^n}{\omega_n \mathrm{Vol}(M)}\right)^{2/n} k^{2/n}$.

For the BBG embedding to approximate the metric within error ε, we need the spectral tail $\sum_{k>N_K} \lambda_k^{-2}$ controlled, which by Weyl asymptotics is $O(K^{1-4/n})$ for $n \geq 5$ and involves logarithmic terms for $n \leq 4$.

6.3 Explicit Bounds for Low Dimensions

Proposition 6.1. For $\mathcal{M}_3(\kappa, D, v)$:

$$K_0 \leq C_3 \cdot \frac{\kappa^{3/2} D^3}{v} \cdot \left(\log \frac{\kappa D^2}{v^{2/3}}\right)^3$$

The logarithmic factor arises from the borderline convergence of $\sum \lambda_k^{-2}$ in dimension 3.

6.4 Latent Dimension

Definition 6.2. The *Latent dimension* at precision ε is $d_{\mathrm{Lat}}(M, \varepsilon) = \min\{K : \Lambda_K$ determines (M, g) to within ε in

Proposition 6.3. (i) $d_{\mathrm{Lat}}(S^n, \varepsilon) = O(\varepsilon^{-n/2})$ — the round sphere is spectrally efficient. (ii) $d_{\mathrm{Lat}}(M, \varepsilon) \leq C(n, \kappa, D, v) \cdot \varepsilon^{-n}$ for any $M \in \mathcal{M}_n(\kappa, D, v)$. (iii) Among manifolds in a δ-neighborhood of S^n, the round sphere minimizes d_{Lat}.

Part (iii) confirms that **the round sphere has the minimal Latent** — the most compressed spectral representation, which connects to the Ricci flow interpretation: the flow compresses Latent dimension toward this minimum.

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7. The Heat Kernel and Spectral Zeta Function

7.1 The Heat Kernel as Latent Encoder

The heat kernel $K_t(x, y) = \sum_{k\geq 0} e^{-\lambda_k t} \sum_{\alpha=1}^{m_k} \phi_k^{(\alpha)}(x)\, \phi_k^{(\alpha)}(y)$ encodes the full Latent in the following sense:

Proposition 7.1. The function $(t, x, y) \mapsto K_t(x, y)$ for all $t > 0$ and $x, y \in M$ determines the full Latent $\Lambda_\infty(M, g)$.

Proof. The eigenvalues are the poles of $s \mapsto \int_0^\infty t^{s-1} K_t(x, x)\, dt$. The eigenfunctions (up to sign/phase) are recovered from $\lim_{t\to 0^+} e^{\lambda_k t}(K_t(x, \cdot) - \sum_{j<k} e^{-\lambda_j t} P_j(x, \cdot))$ where P_j is the projection onto the j-th eigenspace. The structure constants follow. □

7.2 The Spectral Zeta Function

The spectral zeta function $\zeta_M(s) = \sum_{k\geq 1} m_k \lambda_k^{-s}$ for $\mathrm{Re}(s) > n/2$ extends meromorphically to $\mathbb{C}$ with poles related to the Seeley-DeWitt coefficients:

$$\mathrm{Res}_{s=(n-j)/2}\, \zeta_M(s) = \frac{a_j}{\Gamma((n-j)/2)}$$

The spectral zeta function is the Mellin transform of the heat trace: $\zeta_M(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(Z_M(t) - 1)\, dt$.

Remark 7.2. For the flat torus $\mathbb{T}^n = \mathbb{R}^n/\Lambda$, $\zeta_M(s)$ is an Epstein zeta function — a higher-dimensional analogue of the Riemann zeta function. The deep connections between spectral zeta functions and number-theoretic zeta functions hint at a unified "zeta landscape" that the Latent framework may help clarify.

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8. The Hierarchy of Latent Theorems

The Manifold Latent Theorem is the geometric instance of a broader principle: smooth systems governed by elliptic operators admit finite sufficient representations.

System	Latent representation	Key truncation parameter
Riemannian manifold	**Eigenvalues + multiplicities + structure constants**	**Geometry bounds** (n, κ, D, v)
Portfolio of d assets	Eigenvalues of covariance + Hermite-COS coefficients	$d \cdot \sigma^2_{\max}$
European option (1D)	COS coefficients $\{c_k\}_{k=0}^K$	$\sigma^2 T$
Dynamical system (3-body)	Energy, angular momentum + regularization data	$O(1)$

In every case, the underlying operator (Laplacian, Fokker-Planck, Hamiltonian) has a discrete spectrum, and the truncated spectral data is a sufficient statistic for the system.

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9. Open Problems

Problem 9.1 (Spectrum alone for diffeomorphism). Does the eigenvalue spectrum $\{\lambda_k, m_k\}$ alone (without structure constants) determine the diffeomorphism type within $\mathcal{M}_n(\kappa, D, v)$? This is weaker than isometry and does not contradict known isospectral counterexamples (which are isospectral but of different isometry type, though they may be diffeomorphic).

Problem 9.2 (Sharp bounds). What is the optimal K_0 for specific geometry classes? In particular, what is the Latent dimension of the class of closed hyperbolic 3-manifolds with volume $\leq V$?

Problem 9.3 (Stability). Does the Latent map have a quantitative stability estimate: $d_{\mathrm{GH}}(M_1, M_2) \leq C \cdot d_{\mathrm{Lat}}(\Lambda_K(M_1), \Lambda_K(M_2))^\alpha$ for some exponent $\alpha > 0$?

Problem 9.4 (Non-compact extension). For manifolds with controlled geometry at infinity (asymptotically flat, asymptotically hyperbolic), does a Latent exist using the continuous spectrum (scattering data)?

Problem 9.5 (Topological transitions). If a family M_t undergoes a topology change (connected sum decomposition), does the Latent exhibit a detectable discontinuity?

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10. Conclusion

We have proved that closed Riemannian manifolds with bounded geometry are determined, up to diffeomorphism, by a finite collection of spectral data: eigenvalues, multiplicities, and trilinear structure constants of the eigenfunctions. The full infinite collection determines isometry type. The structure constants are the precise missing ingredient that resolves all known failures of spectral determination.

The proof architecture — compactness + spectral stability + algebraic reconstruction — is robust and suggests extensions to orbifolds, manifolds with boundary, and sub-Riemannian geometries. The quantitative bounds, while not yet optimal, show that the truncation depth K_0 is controlled by the geometry parameters.

The result places spectral geometry within a unified framework: the Manifold Latent Theorem is the geometric topology instance of the universal principle that smooth systems admit finite sufficient representations. The Latent is not a metaphor — it is a point in a finite-dimensional space that encodes an infinite-dimensional geometry. The shape of space is finite-dimensional.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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