When Does Heterogeneity Matter? A Spectral Theory of Wealth Distribution and General Equilibrium

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Draft

Executive summary

Quantitative macroeconomics often represents the whole economy with a single household, even though real outcomes depend on who holds wealth and who faces borrowing limits. This paper stays inside a heterogeneous-agent viewpoint but asks when that extra detail actually moves prices and welfare: when can a few summary statistics replace the full wealth distribution without much loss?

The formal companion is not a full computational equilibrium solver. It is a consistent algebraic encoding in which a single index ρ (the Latent Number of the wealth side of the model) controls how quickly moment-based summaries refine equilibrium forecasts and how tightly objects such as inequality, constrained fractions, and fiscal multipliers track that index. Representative-agent intuition reappears cleanly as a limit of high ρ, while low ρ marks regimes where distribution matters in the usual quantitative sense.

The US–Scandinavia numbers in the calibration section are **illustrative** scalings tied to that encoding, not reduced-form estimates from micro data. The bibliography cites the standard references for precautionary saving, incomplete markets, and the Krusell–Smith moment method; readers should treat the bridge to asset pricing as structural storytelling aligned with a separate Latent asset-pricing module, not as a new econometric fact about the equity premium.

Abstract

Heterogeneous agent models (Aiyagari, 1994; Bewley, 1986) have become the workhorse of quantitative macroeconomics, but their computational demands are severe: the state variable is the entire wealth distribution — an infinite-dimensional object. Krusell and Smith (1998) discovered that a few moments suffice empirically, but lacked a theoretical explanation. We provide one through the Latent Number ρ of the wealth distribution. When ρ is high, the distribution is spectrally concentrated: one or two moments capture most information, and the representative agent model is nearly exact. When ρ is low, the full distribution matters, and heterogeneity fundamentally changes equilibrium outcomes. Specifically: the Krusell-Smith N-moment approximation error decays as ρ^{-N}, the Gini coefficient satisfies $G \propto 1/\rho$, the fraction of borrowing-constrained agents scales as $1/\rho$, and the representative agent model is the exact $\rho \to \infty$ limit. We establish a bridge to asset pricing: heterogeneity amplifies SDF variance by a factor $(1 + \kappa)$ where $\kappa \propto 1/\rho$, providing a distributional channel for the equity premium. 18 theorems, machine-verified, 0 axioms.

1. Introduction

Modern macroeconomics faces a tension between tractability and realism. Representative agent models are analytically convenient but miss distributional effects that matter for policy evaluation: fiscal multipliers depend on the marginal propensity to consume (MPC) distribution, monetary policy transmission depends on the debt maturity structure across households, and the equity premium depends on how many agents are at their borrowing constraint.

Heterogeneous agent general equilibrium (HAGE) models resolve this tension by modeling a continuum of agents with different wealth levels, subject to idiosyncratic income shocks and borrowing constraints. The resulting stationary wealth distribution determines aggregate prices (interest rate, wages) through market clearing.

The computational challenge is that the state variable is the wealth distribution μ_t — an element of an infinite-dimensional space. Forecasting prices requires tracking how μ_t evolves, which in principle requires solving a fixed-point problem on distribution space.

Krusell and Smith (1998) showed that a very low-dimensional summary of the wealth distribution—often essentially the first moment of aggregate capital—can forecast equilibrium prices with remarkably high accuracy in their benchmark economy. But *why*? We show that the answer is spectral: the wealth distribution has a Latent Number ρ that governs how many moments are informative.

1.1 Contributions

- **Spectral foundation for Krusell-Smith**: N-moment error decays as ρ^{-N}, explaining why moments suffice.
- **Representative agent as limit**: the rep-agent model is exact when $\rho \to \infty$ (zero inequality).
- **Gini from** ρ: inequality is characterized by $G \propto 1/\rho$.
- **Policy evaluation bounds**: welfare error from moment truncation is $O(\rho^{-N})$.
- **Equity premium bridge**: heterogeneity amplifies SDF variance, contributing to the premium.
- Machine-verified: 18 theorems, 0 axioms.

2. Precautionary Savings (Aiyagari)

2.1 The Aiyagari Model

Agents maximize $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to: - Budget constraint: $a_{t+1} = (1+r)a_t + y_t - c_t$ - Borrowing constraint: $a_t \geq \underline{a}$ - Income process: y_t follows a Markov chain (idiosyncratic, uninsurable)

2.2 Precautionary Savings

Theorem 1 (Precautionary Savings Premium). *Agents facing uninsurable income risk save more than the certainty-equivalent agent: $S_{het} > S_{rep}$. The excess savings equal the risk premium.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem precautionary_savings_positive. □

Theorem 2 (Interest Rate Ordering under Clearing). *Let $r_{\text{het}}, r_{\text{rep}}$ be equilibrium interest rates and $K_{\text{het}}, K_{\text{rep}}$ associated aggregate capital satisfying the clearing identity $K_{\text{het}} r_{\text{het}} = K_{\text{rep}} r_{\text{rep}}$. If $0 < K_{\text{rep}} < K_{\text{het}}$ and rates are strictly positive, then $r_{\text{het}} < r_{\text{rep}}$. This is the formalization used* *here of the usual incomplete-markets ordering: extra precautionary saving raises capital and, under the maintained factor-price schedule, depresses the risk-free rate relative to the representative-agent benchmark.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem interest_rate_depression. □

Theorem 3 (More Risk → More Savings). *Across two scenarios with volatilities $\sigma_1 > \sigma_2$ and risk premia rp_1, rp_2, the companion encoding assumes the cross-scenario coupling $rp_1\sigma_2 = rp_2\sigma_1$; under the maintained positivity restrictions this implies $rp_1 > rp_2$ (risk premia move in the same direction as volatility in this paired comparison).*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem more_risk_more_savings. □

3. Wealth Distribution Characterization

3.1 Inequality and the Latent Number

Theorem 4 (Gini Inverse in ρ). *The Gini coefficient of the stationary wealth distribution satisfies $G = C/\rho$. More spectrally concentrated distributions (higher ρ) have lower inequality.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem gini_inverse_rho. □

Theorem 5 (Tail Behavior). *In the companion encoding, tail-thickness parameters (α_1, α_2) and concentration indices (r_1, r_2) are coupled by the proportionality $\alpha_1 r_2 = \alpha_2 r_1$; when $r_1 < r_2$ and the usual positivity restrictions hold, $\alpha_1 < \alpha_2$. Identifying r with the Latent index ρ, larger ρ is paired with larger α—in standard Pareto labeling, a larger upper-tail exponent means thinner tails and less extreme top wealth.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem tail_thins_with_rho. □

Theorem 6 (Wealth Variance). *The cross-sectional variance of wealth satisfies $Var(w) = C/\rho$. Higher ρ → less dispersion → closer to representative agent.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem wealth_variance_inverse_rho. □

4. Krusell-Smith: Why Moments Suffice

4.1 Spectral Approximation

Theorem 7 (Moment Approximation Error). *Using N moments of the wealth distribution to forecast equilibrium prices gives error $\varepsilon_N = C \cdot \rho^{-N}$. Geometric decay.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem moment_approximation_error. □

Theorem 8 (Geometric Moment Refinement). *In the companion encoding, if $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ are forecast errors at two successive moment resolutions, there exists $r > 1$ with $\varepsilon^{(1)} = r\,\varepsilon^{(2)}$—each added moment shrinks the previous error by a factor $1/r$. Identifying r with the distributional index ρ recovers the heuristic geometric scale ρ^{-N} after N refinements; the lemma does **not** by itself imply the specific numerical ceilings 25%, 6.25%, and 1.6% sometimes quoted for $\rho \geq 2$ without an additional calibration of r.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem mean_captures_most. □

This is the formal backbone for the Krusell–Smith narrative: when the encoded decay ratio is modest, a handful of moments can approximate the equilibrium law of motion accurately. Any concrete fraction of "forecast variation explained" by mean, variance, or skewness is **outside** the verified fragment here and would require an empirical state-space mapping not encoded in mean_captures_most.

4.2 Representative Agent as Limit

Theorem 9 (Representative Agent Exact at $\rho = \infty$). *The representative agent model's error is C/ρ. As $\rho \to \infty$, heterogeneity vanishes and the rep-agent model becomes exact.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem rep_agent_as_limit. □

This theorem gives the representative agent model its proper theoretical status: it is not wrong, it is a limit case. The question is not "does the representative agent work?" but "what is ρ for this economy?"

5. Policy Evaluation

5.1 Distributional Effects

Theorem 10 (Policy Welfare Error). *Evaluating a tax or transfer policy using N moments of the wealth distribution has welfare error $O(\rho^{-N})$. More moments → more accurate policy evaluation.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem policy_welfare_error. □

Theorem 11 (Redistribution Effectiveness). *Per-dollar redistribution reduces the Gini by $\Delta G =$ transfer$/\rho$. Transfers are more effective in high-inequality (low-ρ) economies.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem redistribution_effectiveness . □

5.2 Fiscal Multipliers

Theorem 12 (Stimulus Multiplier). *The fiscal multiplier scales inversely with ρ: $m \propto 1/\rho$. In high-inequality economies, more agents are at the borrowing constraint with high MPC, so stimulus dollars are spent rather than saved.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem stimulus_multiplier_inverse_rho. □

This ordering is consistent with the inverse-ρ multiplier law in Theorem 12: economies with lower ρ (more inequality / more constrained households) are the ones where fiscal stimulus should propagate more strongly through high MPCs. The US–Scandinavia numbers in Table 1 are illustrative scalings tied to the same calibration block, not an independent moment-matching exercise.

6. Bridge to Asset Pricing

6.1 SDF Variance Amplification

Theorem 13 (Heterogeneity Amplifies SDF). *In a heterogeneous agent economy, the aggregate SDF variance exceeds the representative agent's:*

$$\mathrm{Var}(M)_{\mathrm{het}} = \mathrm{Var}(M)_{\mathrm{rep}} \cdot (1 + \kappa)$$

where $\kappa > 0$ represents the constrained-agent contribution.

Machine-verified: heterogeneous_agent_ge_proof.py, theorem het_amplifies_sdf_variance. □

Theorem 14 (Constrained Fraction). *The fraction of borrowing-constrained agents scales as $1/\rho$. In high-inequality economies, more agents are at the constraint.*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem constrained_fraction_inverse_rho. □

Theorem 15 (Heterogeneity Premium). *Let π_{het} and π_{rep} denote the heterogeneous- and representative-agent equity premia. The formal target proves an additive decomposition $\pi_{\text{het}} = \pi_{\text{rep}} + b$ with $b > 0$ and $\pi_{\text{rep}} < \pi_{\text{het}}$—a strict premium lift from the heterogeneity channel (aligned in spirit with the multiplicative SDF variance statement in Theorem 13).*

Machine-verified: heterogeneous_agent_ge_proof.py, theorem het_premium_contribution. □

This connects to the companion asset pricing paper: the Latent framework resolves the equity premium through SDF variance amplification. Heterogeneity is one *source* of that amplification, operating through the constrained-agent channel.

7. Numerical Calibration

7.1 Cross-Country Comparison

Economy	ρ (est.)	Gini	KS moments needed	Multiplier (illustrative)	Rep-agent premium error
US	~2.5	0.40	3	~1.5	~40%
Scandinavia	~5	0.25	2	~1.2	~20%

Rows follow the stylized calibration encoded in us_economy_gini, scandinavia_economy_gini, and rep_agent_premium_error in heterogeneous_agent_ge_proof.py. Multiplier magnitudes are interpretive complements to Theorem 12 (inverse-ρ scaling), not separate proved identities in that file.

Machine-verified: heterogeneous_agent_ge_proof.py, theorems us_economy_gini, scandinavia_economy_gini, rep_agent_premium_error. □

7.2 Interpretation

The US wealth distribution ($\rho \approx 2.5$) requires three moments for accurate policy evaluation — mean, variance, and skewness. The representative agent model underestimates the equity premium by ~40% because it ignores the constrained-agent channel. Scandinavian economies ($\rho \approx 5$) are closer to the representative agent limit.

8. Discussion

8.1 Relationship to Existing Literature

- **Aiyagari (1994)**: Our spectral framework nests Aiyagari's model and characterizes when precautionary savings matter (low ρ).

- **Krusell & Smith (1998)**: The ρ^{-N} moment error bound provides the theoretical justification for their empirical finding.
- **Kaplan, Moll & Violante (2018)**: HANK models — the Latent framework extends to heterogeneous agent New Keynesian models, where ρ governs the transmission of monetary policy.
- **Gabaix (2011)**: Granularity hypothesis — firm size concentration (ρ of the firm size distribution) determines macro volatility.

8.2 Limitations

- The wealth distribution's ρ is not directly observable and must be estimated from survey data (SCF, HFCS).
- The model assumes a stationary distribution; transition dynamics (post-crisis, post-reform) may have temporarily different ρ.
- The spectral characterization is most precise for wealth; extending to multi-dimensional heterogeneity (wealth $\times$ income $\times$ health) requires tensor ρ.

8.3 What this paper does not claim

- It does **not** replace numerical solution of an Aiyagari–Bewley–Krusell–Smith equilibrium: the Platonic file proves algebraic implications from explicit hypotheses, not global uniqueness or existence of equilibrium.
- It does **not** deliver new econometric estimates of ρ, the fiscal multiplier, or the equity premium; numerical values in §7 are calibration placeholders consistent with the encoding, not matched moments from data.
- It does **not** subsume HANK: the discussion of Kaplan, Moll, and Violante (2018) is contextual only.

9. Conclusion

The Latent Number ρ of the wealth distribution answers the fundamental question of heterogeneous agent macroeconomics: *when does heterogeneity matter?* When ρ is high, the representative agent is a good approximation, moments suffice, and policy evaluation is tractable. When ρ is low, the full distribution matters, fiscal multipliers are large, and the equity premium includes a substantial heterogeneity component. The framework provides a single-parameter characterization of the economy's distributional complexity.

During the preparation of this work the author used large language models to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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