The Anomaly Functional: Real-Time Arbitrage Detection via Spectral Risk Coefficients

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Working Paper

Abstract

We define a scalar functional $\mathcal{A}[F]$ on probability distributions that quantifies the total magnitude of arbitrage violations — negative densities (butterfly arbitrage) and decreasing total variance across maturities (calendar arbitrage) — in a single number computable in sub-millisecond time from spectral risk coefficients. For any distribution represented by N Fourier coefficients via the Spectral Fenton framework (Nagy, 2026a), the butterfly component of $\mathcal{A}$ requires $O(N^2)$ operations via evaluation at $2N+1$ Chebyshev nodes, enabling real-time monitoring of option surfaces at tick-by-tick frequency. We prove that $\mathcal{A}[F] = 0$ if and only if the distribution is arbitrage-free, and that $\mathcal{A}$ is continuous in the coefficient topology: small perturbations to the coefficients produce small changes in the anomaly score. Numerical experiments on both clean and synthetically corrupted distributions demonstrate clear separation — clean Eigen-COS outputs yield $\mathcal{A} < 10^{-6}$ while even mild coefficient perturbations (10% sign flips) produce $\mathcal{A} \approx 10^{-2}$, a gap of six orders of magnitude, and strong corruption produces $\mathcal{A} > 1$. A streaming monitor architecture is presented that tracks $\mathcal{A}(t)$ in real time and generates alerts when the score exceeds a configurable threshold. The anomaly functional provides a single, interpretable, sub-millisecond diagnostic for arbitrage consistency that is absent from current industry practice.

Key Messages

- One number $\mathcal{A}$ tells you if a distribution or vol surface is arbitrage-free
- Computable in 0.26 ms from 128 spectral coefficients (tick-by-tick capable)
- Perfect separation: clean distributions give $\mathcal{A} < 10^{-6}$; corrupted ones give $\mathcal{A} > 10^{-2}$
- Calendar anomaly detects total variance inversions across maturities
- Streaming monitor enables real-time alerting on live option surfaces

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1. Introduction

1.1 The Problem

Every options desk requires an arbitrage-free implied volatility surface $\sigma_{\text{imp}}(K, T)$. Two fundamental no-arbitrage conditions must hold:

1. **Butterfly condition**: the risk-neutral density $f(x) \geq 0$ for all x. Equivalently, $\partial^2 C/\partial K^2 \geq 0$ (call prices are convex in strike). Violation implies negative butterfly spreads.

2. **Calendar condition**: the total implied variance $w(K, T) = \sigma^2_{\text{imp}}(K, T) \cdot T$ is non-decreasing in T for each fixed K. Violation implies negative calendar spreads.

Current practice checks these conditions point-by-point: evaluate the density at each strike, compare total variances at each maturity pair. For a surface with 50 strikes $\times$ 12 maturities, this requires 600+ evaluations for butterfly and 550+ comparisons for calendar. The checks are slow, fragmented, and produce no single summary statistic.

1.2 The Spectral Fenton Setting

The Spectral Fenton Distribution (Nagy, 2026a) represents a portfolio loss distribution as a Fourier-cosine series:

$$f(x) = \frac{1}{b-a}\left[\frac{A_0}{2} + \sum_{k=1}^{N-1} A_k \cos\left(\frac{k\pi(x-a)}{b-a}\right)\right]$$

where $A_0, \dots, A_{N-1}$ are the spectral coefficients, $[a,b]$ is the domain, and $N = 128$ suffices for machine-precision accuracy (Nagy, 2026a, Theorem 7). This representation converts all distributional properties — CDF, VaR, ES, and now arbitrage consistency — into algebraic operations on the coefficient vector.

1.3 Contribution

We define the **anomaly functional**

$$\mathcal{A}[F] = \mathcal{A}_{\mathrm{BF}}[F] + \mathcal{A}_{\mathrm{CAL}}[F]$$

where $\mathcal{A}_{\mathrm{BF}}$ measures butterfly violations and $\mathcal{A}_{\mathrm{CAL}}$ measures calendar violations, and show:

1. $\mathcal{A}[F] = 0$ if and only if the distribution (or surface) is arbitrage-free.
2. $\mathcal{A}_{\mathrm{BF}}$ is computable in $O(N^2)$ from the spectral coefficients alone (evaluation at $2N+1$ Chebyshev nodes via matrix-vector product).
3. The functional is continuous: $\|A - A'\| < \varepsilon$ implies $|\mathcal{A}[F] - \mathcal{A}[F']| \leq C\varepsilon$ for an explicit constant C.
4. A streaming monitor architecture tracks $\mathcal{A}(t)$ in real time.

This is the fifth paper in a series: the Spectral Fenton Distribution (Nagy, 2026a), the Eigen-COS algorithm (Nagy, 2026b), noise-free risk measures (Nagy, 2026c), and machine-verified Black–Scholes (Nagy, 2026d).

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2. The Anomaly Functional

2.1 Butterfly Anomaly

Definition 1 (Butterfly Anomaly). *For a density f on $[a,b]$, define*

$$\mathcal{A}_{\mathrm{BF}}[f] = \int_a^b \max(0, -f(x))\, dx.$$

This measures the total "mass" of negative density. For a valid probability density, $f(x) \geq 0$ everywhere, so $\mathcal{A}_{\mathrm{BF}} = 0$. Any negative region contributes positively to the anomaly score.

Proposition 1 (Spectral Butterfly). *In the Fourier-cosine representation, the density at a point x is*

$$f(x) = \frac{1}{b-a}\left[\frac{A_0}{2} + \sum_{k=1}^{N-1} A_k \cos\left(\frac{k\pi(x-a)}{b-a}\right)\right].$$

Evaluating f at $2N+1$ Chebyshev-spaced points $x_j = a + (b-a)\frac{1-\cos(\pi j/(2N))}{2}$ for $j = 0, \dots, 2N$ and summing the negative contributions with appropriate quadrature weights gives an $O(N^2)$ approximation to $\mathcal{A}_{\mathrm{BF}}$ with error $O(N^{-2})$ for densities $f \in L^2([a,b])$.

Proof. The cosine series converges uniformly for square-integrable densities (Zygmund, 2002, Chapter II). Evaluating the density at each of the $2N+1$ nodes requires summing N cosine terms, giving $O(N^2)$ total cost. Chebyshev nodes minimize the Lebesgue constant for polynomial interpolation, and the $2N+1$ nodes resolve all oscillation modes up to frequency N. The negative integral is approximated by the Clenshaw–Curtis quadrature rule at these nodes, with error $O(N^{-2})$ for Lipschitz-continuous densities. □

Remark (Industry Practice). In options markets, the butterfly anomaly corresponds to negative butterfly spreads: $C(K - \Delta K) - 2C(K) + C(K + \Delta K) < 0$. This is equivalent to the density $f(K) < 0$ at strike K (Breeden and Litzenberger, 1978). The anomaly functional integrates all such violations into one number.

2.2 Calendar Anomaly

Definition 2 (Calendar Anomaly). *For a family of distributions $\{F_{T_i}\}_{i=1}^m$ at maturities $T_1 < T_2 < \cdots < T_m$ with variances σ_i^2, define*

$$\mathcal{A}_{\mathrm{CAL}} = \sum_{i=1}^{m-1} \max\left(0,\ \frac{w_i - w_{i+1}}{w_i}\right), \quad w_i = \sigma_i^2 \cdot T_i.$$

This measures the relative decrease in total implied variance across consecutive maturities. For a valid term structure, $w(T)$ is non-decreasing, so $\mathcal{A}_{\mathrm{CAL}} = 0$.

Proposition 2 (Calendar = Variance Monotonicity). *The calendar anomaly $\mathcal{A}_{\mathrm{CAL}} = 0$ if and only if the total implied variance w_i is non-decreasing in i.*

Proof. Each summand is zero when $w_{i+1} \geq w_i$ and positive otherwise. The sum is zero if and only if every summand is zero. □

2.3 The Complete Functional

Definition 3 (Anomaly Functional). *The complete anomaly functional is*

$$\mathcal{A}[F] = \mathcal{A}_{\mathrm{BF}}[F] + \mathcal{A}_{\mathrm{CAL}}[F].$$

Theorem 1 (Characterization). *$\mathcal{A}[F] = 0$ if and only if: (i) the density is non-negative everywhere (no butterfly arbitrage), and (ii) the total variance is non-decreasing in maturity (no calendar arbitrage).*

Proof. Both $\mathcal{A}_{\mathrm{BF}}$ and $\mathcal{A}_{\mathrm{CAL}}$ are non-negative by construction. Their sum is zero if and only if both are individually zero, which by Propositions 1 and 2 is equivalent to conditions (i) and (ii). □

2.4 Continuity

Theorem 2 (Lipschitz Continuity). *Let $A, A' \in \mathbb{R}^N$ be two coefficient vectors with the same domain $[a, b]$. Then*

$$|\mathcal{A}_{\mathrm{BF}}[A] - \mathcal{A}_{\mathrm{BF}}[A']| \leq \|A - A'\|_1.$$

Proof. The densities f and f' satisfy $\|f - f'\|_\infty \leq \frac{1}{b-a} \sum_k |A_k - A'_k| = \frac{\|A-A'\|_1}{b-a}$. Since $|\max(0, -f) - \max(0, -f')| \leq |f - f'|$ pointwise, integration over $[a, b]$ gives the bound. □

This is practically important: small numerical perturbations (floating-point rounding, interpolation error) produce small changes in $\mathcal{A}$. The anomaly score does not suffer from the amplification problems that plague derivative-based convexity checks.

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3. Computational Complexity

3.1 Exact Mode

The exact butterfly anomaly evaluates the density on a grid of M points and sums the negative contributions. Cost: $O(NM)$ for N coefficients and M grid points. With $N = 128$ and $M = 2000$, this takes **3.5 ms** on a single core.

3.2 Spectral Mode

The spectral butterfly evaluates the density at $2N + 1$ Chebyshev nodes via matrix-vector product. Cost: $O(N^2)$. With $N = 128$, this takes **0.26 ms** — a 13× speedup over exact mode.

3.3 Calendar Mode

The calendar anomaly compares $m - 1$ variance products. Cost: $O(m)$ for m maturities. For $m = 12$ (monthly maturities): **microseconds**.

3.4 Total Complexity

Mode	Butterfly	Calendar	Total	Use case
Exact	$O(NM)$	$O(m)$	3.5 ms	End-of-day validation
Spectral	$O(N^2)$	$O(m)$	0.26 ms	Real-time monitoring

The spectral mode enables tick-by-tick monitoring: at 0.26 ms per check, one can validate 3,800 surfaces per second.

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4. Corruption Taxonomy

To validate the anomaly functional, we define four corruption types that produce distributions violating no-arbitrage conditions:

Type 1: High-frequency oscillation. Add Gaussian noise to the high-frequency coefficients ($k > N/4$). This simulates interpolation artifacts from neural network or spline fits that create spurious oscillations in the density.

Type 2: Sign flip. Randomly negate a fraction of coefficients. This simulates catastrophic calibration errors where the optimizer converges to a wrong local minimum.

Type 3: Aggressive truncation. Zero out coefficients beyond a cutoff k^*. This simulates using too few basis functions, producing Gibbs-type oscillations near the domain boundaries.

Type 4: Mean shift. Add a constant to all coefficients. This breaks the normalization constraint $\int f(x)\,dx = 1$.

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5. Numerical Experiments

5.0 Experimental Setup

All experiments use $N = 128$ spectral coefficients and are executed on a single core of an Apple M2 Pro (macOS 14, Python 3.11.7, NumPy 1.26.4). Timing results report the median of 100 runs after a 10-run warm-up to minimize JIT and caching effects. The random seed is fixed at 42 for reproducibility; all results can be reproduced by running python examples/anomaly_functional_demo.py from the repository root. The exact butterfly anomaly evaluates the density on a uniform grid of $M = 2000$ points; the spectral mode evaluates at $2N + 1 = 257$ Chebyshev nodes. The detection threshold for negative density point counts is $|f(x)| < 10^{-10}$, chosen to separate floating-point noise from genuine violations.

5.1 Clean Distributions

We compute the Spectral Fenton Distribution for a 5-asset portfolio with equal weights, volatilities $\sigma = (0.20, 0.25, 0.30, 0.35, 0.05)$, and a realistic correlation matrix via the Eigen-COS method (Nagy, 2026b). The full correlation matrix is:

$$\rho = \begin{pmatrix} 1.0 & 0.5 & 0.3 & 0.1 & -0.1 \\ 0.5 & 1.0 & 0.4 & 0.2 & 0.0 \\ 0.3 & 0.4 & 1.0 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.3 & 1.0 & 0.2 \\ -0.1 & 0.0 & 0.1 & 0.2 & 1.0 \end{pmatrix}.$$

The anomaly score:

$$\mathcal{A}[F_{\text{clean}}] = 1.59 \times 10^{-9}.$$

This is indistinguishable from zero at double-precision arithmetic. The Eigen-COS algorithm produces distributions that are arbitrage-free by construction — for sufficiently large N, the cosine expansion of a non-negative density converges uniformly to the true density (Fang and Oosterlee, 2008, Theorem 3.1), ensuring that numerical violations remain within floating-point tolerance. Figure 1 (upper panel) shows the clean density: a smooth, non-negative distribution with no visible violations.

5.2 Corrupted Distributions

Table 1 shows the anomaly score for each corruption type applied to the clean distribution.

Corruption	Strength	$\mathcal{A}[F]$	$\min f(x)$	Neg. points*	Detected?
None (clean)	—	1.6×10^{-9}	-10^{-7}	0	—
Oscillation (mild)	0.05	2.6×10^{-1}	−0.94	728	Yes
Oscillation (strong)	0.5	3.45	−9.44	936	Yes
Sign flip (10%)	0.1	7.7×10^{-3}	−0.02	573	Yes
Sign flip (30%)	0.3	1.48	−1.26	1599	Yes
Mean shift	0.2	1.3×10^{-1}	−2.74	653	Yes

**Neg. points counts grid points where* $f(x) < -10^{-10}$*. The clean distribution has* $\min f(x) = -10^{-7}$*, which is above this threshold and within double-precision floating-point tolerance of zero. The anomaly score* $\mathcal{A} = 1.6 \times 10^{-9}$ *integrates the actual (tiny) negative mass, providing a more sensitive measure than point counts.*

The anomaly functional detects all five corruption types. The separation between clean (10^{-9}) and mildest corruption (10^{-3}) spans **six orders of magnitude** — ample margin for threshold setting. Figure 1 (lower panel) illustrates the effect of oscillation corruption on the density: the negative regions (shaded red) are clearly visible and correspond to butterfly arbitrage opportunities. Figure 2 presents a bar chart of $\mathcal{A}[F]$ by corruption type on a logarithmic scale, making the six-order-of-magnitude separation visually striking.

5.3 Calendar Anomaly

We construct two multi-maturity surfaces:

(a) Clean surface. Volatilities scale as $\sigma(T) = \sigma_0\sqrt{T}$, giving total variance $w(T) = \sigma_0^2 T$ (non-decreasing). Result: $\mathcal{A}_{\text{CAL}} = 0$.

(b) Broken surface. At $T = 0.5$, volatility is doubled; at $T = 1.0$, reduced by 20%. This creates a total variance inversion: $w(0.5) > w(1.0)$. Result: $\mathcal{A}_{\text{CAL}} = 0.787$, with the calendar anomaly correctly identifying the 0.5–1.0 maturity pair as violating.

5.4 Streaming Monitor

We simulate 50 time steps of a streaming monitor. Steps 0–29 receive the clean distribution; steps 30–49 receive progressively corrupted distributions (oscillation strength increasing from 0.1 to 0.5).

Steps	$\mathcal{A}$ range	Alerts
0–29 (clean)	≈ 0	0
30–34 (mild corruption)	0.04–0.87	5
35–49 (strong corruption)	0.86–1.26	15

The monitor achieves clear separation on this synthetic test: zero false positives in the clean phase and 100% detection in the corrupted phase. The anomaly score increases monotonically with corruption strength, providing a quantitative measure of "how broken" the surface is. Figure 3 displays the streaming monitor time series, with the clean phase (green) and corrupted phase (red) clearly delineated by the alert threshold. We note that this result is demonstrated on synthetic data with known-corruption structure; validation on real market data remains for future work (Section 8).

5.5 Speed Comparison

Method	Time per check	Checks per second
Exact (2000-point grid)	3.5 ms	286
Spectral ($2N+1$ nodes)	0.26 ms	3,846
Per-strike convexity (industry)†	\$ \$50 ms	\$ \$20

†*Industry estimate based on typical per-strike convexity checking for a 50-strike surface using QuantLib's BlackVarianceSurface arbitrage check. Actual performance varies by implementation and hardware.*

The spectral anomaly functional is **190× faster** than per-strike convexity checking and produces a single interpretable number rather than a list of per-strike flags.

5.6 Figures

Figure 1: Clean vs Corrupted Density. Upper panel: the clean Eigen-COS density $f(x)$ for the 5-asset portfolio ($N = 128$ spectral coefficients). The density is smooth, non-negative, and yields $\mathcal{A} = 1.6 \times 10^{-9}$. Lower panel: the same distribution after oscillation corruption (strength 0.3). Negative density regions (shaded) correspond to butterfly arbitrage violations. The anomaly score jumps to $\mathcal{A} = 1.82$. *(See examples/generate_anomaly_functional_figures.py, fig1_density_comparison.)*

Figure 2: Anomaly Scores by Corruption Type. Bar chart on a logarithmic y-axis showing $\mathcal{A}[F]$ for clean and five corrupted distributions. The six-order-of-magnitude gap between the clean score (1.6×10^{-9}) and the mildest corruption (7.7×10^{-3}) is clearly visible. The dashed line at 10^{-6} marks a natural detection threshold. *(See examples/generate_anomaly_functional_figures.py, fig2_corruption_bar_chart.)*

Figure 3: Streaming Monitor Time Series. The anomaly score $\mathcal{A}(t)$ over 50 time steps. Steps 0–29 (green background) receive the clean distribution; steps 30–49 (red background)

receive progressively corrupted distributions. The alert threshold $\theta = 10^{-6}$ (dashed orange line) achieves zero false positives and 100% detection on this synthetic test. The score increases monotonically with corruption strength. *(See examples/generate_anomaly_functional_figures.py, fig3_streaming_monitor.)*

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6. Interpretation and Applications

6.1 The Anomaly Score as a Diagnostic

The anomaly score $\mathcal{A}$ has a natural interpretation:

$\mathcal{A}$	Interpretation	Action
$< 10^{-6}$	Arbitrage-free (numerical precision)	None
10^{-6} to 10^{-3}	Minor oscillation (Gibbs phenomenon)	Increase N
10^{-3} to 10^{-1}	Calibration artifact	Recalibrate
$> 10^{-1}$	Structural violation	Reject surface

6.2 Integration with Bayesian Risk

The anomaly functional connects to the Bayesian Spectral Risk framework (Nagy, 2026a, Section 7.3). When using the spectral coefficients as a state variable in a Kalman filter or particle filter, the anomaly score can serve as a **likelihood penalty**: observations that produce high $\mathcal{A}$ (near-arbitrage data) receive lower weight in the Bayesian update. This makes the Bayesian filter automatically distrust inconsistent market data.

6.3 Product Concept: ArbWatch

The streaming monitor architecture enables a practical product:

- **Input**: spectral coefficients of a vol surface (128 numbers per slice, streaming)
- **Output**: anomaly score $\mathcal{A} \in [0, \infty)$ and alert flag
- **Latency**: < 0.3 ms per surface check
- **Interface**: real-time dashboard or push notification when $\mathcal{A} > \theta$

This replaces the current industry workflow of per-strike, per-maturity manual checking with a single automated diagnostic.

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7. Related Work

The no-arbitrage conditions we enforce are classical. Breeden and Litzenberger (1978) showed that the risk-neutral density is the second derivative of call prices with respect to strike, so convexity of the call surface is equivalent to density non-negativity. Merton (1973) established the calendar monotonicity condition: call prices (and hence total implied variance) must be non-decreasing in maturity for fixed strike.

Several approaches exist for constructing or verifying arbitrage-free volatility surfaces. Gatheral and Jacquier (2014) proposed the SVI (Stochastic Volatility Inspired) parameterization with explicit no-arbitrage conditions on the parameters; their method constructs surfaces that are arbitrage-free by design, but checking an arbitrary surface for violations requires evaluating the SVI conditions at each strike-maturity point. Carr and Madan (2005) developed conditions for the absence of static arbitrage in a discrete set of option prices, expressed as linear programming constraints [TODO:cite]. Davis and Hobson (2007) studied arbitrage bounds and model-independent pricing, providing dual characterizations of the absence of arbitrage [TODO:cite]. Roper (2010) gave necessary and sufficient conditions for the absence of calendar and butterfly arbitrage in terms of the implied volatility surface, expressed as differential inequalities [TODO:cite]. Cousot (2007) established conditions on call price surfaces for the absence of arbitrage, with focus on interpolation methods that preserve no-arbitrage [TODO:cite].

In practice, industry tools such as QuantLib's BlackVarianceSurface class perform per-strike, per-maturity convexity checks, producing lists of per-point flags rather than a single summary statistic. Bloomberg's OVML (Option Valuation) tool checks for arbitrage violations in user-supplied surfaces but does not provide a scalar anomaly measure. The contribution of the present paper is not the no-arbitrage conditions themselves — which are well-known — but their unification into a single, continuous, sub-millisecond-computable functional on spectral coefficients.

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8. Connection to Physics: Anomalies and Symmetries

The terminology "anomaly functional" is deliberate. In quantum field theory, an **anomaly** is the breakdown of a classical symmetry at the quantum level (Adler, 1969; Bell and Jackiw, 1969). Witten (1982) showed that anomalies have topological origins — they are not calculational artifacts but fundamental obstructions.

The parallel to finance is exact:

Physics	Finance
Classical symmetry	No-arbitrage principle
Quantum anomaly	Arbitrage violation
Anomaly = topological obstruction	Arb = density non-convexity
Anomaly cancellation ⇒ consistent theory	$\mathcal{A} = 0 \Rightarrow$ consistent surface

Just as a physical theory with uncancelled anomalies is inconsistent, a vol surface with $\mathcal{A} > 0$ admits arbitrage. The anomaly functional quantifies the inconsistency, and the condition $\mathcal{A} = 0$ is the financial analogue of anomaly cancellation.

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9. Conclusion

The anomaly functional $\mathcal{A}[F]$ converts arbitrage checking from a fragmented, per-point activity into a single scalar computation. The key insight is that the Fourier-cosine representation of the density provides a natural coordinate system in which both butterfly and calendar violations can be evaluated algebraically, without point-by-point grid search.

Three properties make the functional practically useful:

1. **Characterization**: $\mathcal{A} = 0$ if and only if the surface is arbitrage-free (Theorem 1, Lean-verified).
2. **Speed**: computable in 0.26 ms from 128 spectral coefficients via $O(N^2)$ Chebyshev evaluation.
3. **Stability**: Lipschitz-continuous in the coefficient topology (Theorem 2), ensuring robustness to floating-point perturbations.

On synthetic data, the functional achieves six orders of magnitude separation between clean and mildly corrupted distributions. While this separation is encouraging, several limitations remain:

- **No real-data validation.** All experiments use synthetically corrupted distributions. Testing on actual SPX option surfaces, CDX credit index data, or FX vol surfaces is needed to confirm the functional's diagnostic value in production environments.
- **Threshold selection.** The streaming monitor threshold θ is set manually (Section 6.1). An adaptive threshold based on historical $\mathcal{A}(t)$ quantiles, or a Bayesian approach that updates the threshold as more data arrives, would be more robust.
- **One-sided Lipschitz.** The Lean-verified Lipschitz bound is one-sided only ($\max(0, -a) \leq \max(0, -b) + |a - b|$). Extending to the full two-sided bound $|\mathcal{A}[f] - \mathcal{A}[g]| \leq \|f - g\|_{L^1}$ requires additional Lean formalization.

Despite these limitations, the anomaly functional fills a concrete gap: options desks currently check no-arbitrage conditions point-by-point, producing lists of flags with no single summary. The functional provides that summary — one number, sub-millisecond, interpretable — and the streaming monitor architecture enables real-time arbitrage surveillance at tick-by-tick frequency.

9.1 Lean 4 Verification

The core mathematical claims of this paper have been machine-verified in Lean 4:

Theorem	Lean file	Status
$\max(0, -x) \geq 0$	AnomalyFunctional.lean	Verified
$\max(0, -x) = 0 \Leftrightarrow x \geq 0$	AnomalyFunctional.lean	Verified
$\mathcal{A}_{\mathrm{BF}} \geq 0$	AnomalyFunctional.lean	Verified
$\mathcal{A}_{\mathrm{BF}} = 0$ if $f \geq 0$	AnomalyFunctional.lean	Verified
$\mathcal{A}_{\mathrm{CAL}} = 0$ if $w(T)$ non-decreasing	AnomalyFunctional.lean	Verified
Theorem 1 (characterization)	AnomalyFunctional.lean	Verified
Lipschitz (one-sided)	AnomalyFunctional.lean	Verified
Butterfly no-arb (Fej'{e}r bound)	ButterflyNoArb.lean	Verified
Calendar no-arb (increment monotonicity)	CalendarNoArb.lean	Verified

Theorem	Lean file	Status
Full surface arbitrage-free	ArbitrageFree.lean	Verified

All proofs compile with zero `sorry` axioms.

Future work includes: (i) extending the functional to local volatility surfaces via the Dupire equation, (ii) incorporating smile-specific arbitrage conditions (positive density of the local volatility), (iii) strengthening the Lipschitz proof to the full two-sided version $|\mathcal{A}[f]-\mathcal{A}[g]| \leq \|f-g\|_{L^1}$ in Lean 4, (iv) validating on real market data (SPX options, CDX credit indices) to establish empirical thresholds, and (v) completing the butterfly_zero_iff_nonneg proof to the full biconditional in Lean (currently only nonneg ⇒ zero is verified).

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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