From Black-Scholes to Bachelier in One Formula

The Arcsinh Option Pricing Family

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Abstract

We derive a closed-form European option pricing formula valid for all spot prices $S_0 \in \mathbb{R}$, including negative values. The formula replaces the logarithmic transform of Black-Scholes with the inverse hyperbolic sine, yielding a three-term call price $C = S_+\Phi(d_+) - S_-\Phi(d_-) - Ke^{-rT}\Phi(d)$. The additional term $S_-\Phi(d_-)$, absent in Black-Scholes, captures the contribution of negative-terminal-value states and vanishes exponentially for large positive S_0. We prove: (i) the formula satisfies the martingale condition exactly; (ii) the third-term correction is bounded by S_- and vanishes in the large-positive-price regime, where the pricing formula becomes asymptotically Black-Scholes; (iii) it admits a local Bachelier limit when the transform linearizes near zero; (iv) Delta is closed-form and Gamma is bounded everywhere; (v) put-call parity holds through zero; (vi) all no-arbitrage conditions are satisfied. The algebraic framework — martingale conditions, coefficient identities, derivative bounds, and structural reductions — is formally verified in Lean 4 with zero sorry obligations. The Gaussian integration step follows the standard Black-Scholes derivation pattern. The pricing formula is exact for the transformed terminal-distribution model used in the paper; the associated local-volatility SDE provides a continuous-time interpretation linking the Black-Scholes and Bachelier regimes.

Executive Summary

The practical problem is simple: once prices can approach or cross zero, the standard Black-Scholes state variable breaks. Desks then patch the problem by switching models, shifting prices, or changing volatility conventions right when markets are most stressed. This paper shows that the arcsinh transform gives a cleaner alternative: one closed-form pricing framework that remains valid on the entire real line.

The resulting call formula has the same overall flavor as Black-Scholes, but with one additional term that captures the contribution of negative-terminal-value states. That extra term disappears exponentially fast in the ordinary positive-price regime, so the model becomes asymptotically indistinguishable from Black-Scholes when Black-Scholes already works well.

In practical terms, the framework does five useful things: - it prices European options for all $S_0 \in \mathbb{R}$, including negative prices; - it keeps Delta in closed form, Gamma bounded, and Vega/Theta/Rho in explicit derivative representations, with a single Greek framework through zero-crossings; - it approaches Bachelier near zero without forcing a model switch; - it preserves a dimensionless volatility parameter, which is often more natural for cross-regime risk comparisons; - it admits a local-volatility interpretation $dS = [\text{drift}]\, dt + \sigma_Y \sqrt{S^2+1}\, dW$, linking the Black-Scholes and Bachelier regimes within one continuous specification.

The companion return paper (Nagy 2026a) explains why arcsinh is the natural return transform on $\mathbb{R}$. This paper shows that the same transform also yields a usable option formula: analytically tractable, operationally continuous, and economically meaningful in markets where negative prices are real rather than pathological.

The algebraic core of the pricing formula, the key coefficient identities, and the main structural reductions are formalized in Lean 4 with zero sorry obligations. The Gaussian integration step is written in the conventional Black-Scholes style.

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1. Introduction

1.1 The Negative Price Problem

The Black-Scholes formula requires $S_0 > 0$. When asset prices cross zero, the log-price state variable ceases to exist, forcing model switches and breaking continuity in implied volatility and hedging workflows. This is not a theoretical curiosity. On April 20, 2020, the WTI crude oil front-month contract settled at −\$37.63 (CME Group 2020; Haug 2020). European power markets regularly produce negative prices during periods of excess renewable generation (ACER 2025). Negative-rate episodes in Europe from 2014 onward showed that rate-linked underlyings can also spend extended periods near or below zero (ECB 2014).

The standard industry response is to switch to the Bachelier model (1900), which assumes normally distributed prices. Bachelier handles negative prices but imposes symmetric returns — wrong for energy markets, where upward spikes to €500/MWh coexist with downward moves limited to approximately −€50. The model switch itself is problematic: it produces discontinuous Greeks, hedging jumps, and P&L noise at the exact moment markets are most stressed.

Alternative approaches include: - **Shifted-lognormal** (displaced diffusion): applies Black-Scholes to $(F + C, K + C)$ with an arbitrary shift C. The shift is a free parameter with no theoretical justification; different desks use different values. - **SABR** (Hagan et al. 2002): generates realistic smiles but has no closed-form solution — only an asymptotic approximation that breaks for low strikes, long maturities, and large vol-of-vol. - **Free-boundary SABR** (Antonov et al. 2015): handles negative prices rigorously but requires numerical PDE solution.

None of these provides a single closed-form formula that handles positive, zero, and negative prices with the correct asymmetry.

1.2 The Arcsinh Transform

We replace $\log(S)$ with $\operatorname{arcsinh}(S) = \log(S + \sqrt{S^2+1})$. This function maps $\mathbb{R} \to \mathbb{R}$ bijectively, is smooth everywhere, and recovers the logarithm asymptotically:

- For $S \gg 0$: $\operatorname{arcsinh}(S) \approx \log(2S)$ — reduces to Black-Scholes
- For $S = 0$: $\operatorname{arcsinh}(0) = 0$ — no singularity
- For $S < 0$: $\operatorname{arcsinh}(S) = -\operatorname{arcsinh}(|S|)$ — antisymmetric, well-defined

The identity $\sinh(\operatorname{arcsinh}(x)) = x$ is the algebraic engine of the entire framework.

1.3 Contributions

This paper makes three contributions:

1. **A closed-form three-term pricing formula** valid for $S_0 \in \mathbb{R}$, with closed-form Delta, bounded Gamma, and explicit derivative representations for the remaining Greeks. The formula has the same computational cost as Black-Scholes.
2. **Unification of Black-Scholes and Bachelier** as two asymptotic regimes of the arcsinh transform itself: logarithmic for large positive prices and locally linear near zero.
3. **Formal verification in Lean 4** of the algebraic core of the pricing formula, the main Greek identities, the parity relations, and the structural reductions, all with zero sorry obligations. The Gaussian integration step remains written conventionally. To our knowledge, this is the first Lean-formalized option pricing framework built specifically for negative-price markets.

The theoretical foundation for the arcsinh transform as a return function — including an axiomatic characterization proving that arcsinh is the essentially unique return function on the real line — is developed in the companion paper (Nagy 2026a, “What Is a Return?”). The present paper applies that foundation to option pricing.

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2. Preliminaries

2.1 Definitions

The arcsinh function. For $x \in \mathbb{R}$, define

$$\operatorname{arcsinh}(x) = \log(x + \sqrt{x^2 + 1}).$$

The argument $x + \sqrt{x^2 + 1} > 0$ for all x (since $\sqrt{x^2 + 1} > |x|$), so the logarithm is always well-defined.

The pricing model. Let $Y_T = \alpha + \sigma_Y \sqrt{T}\, Z$ where $Z \sim N(0,1)$, and define the terminal asset price

$$S_T = \sinh(Y_T) = \sinh(\alpha + \sigma_Y \sqrt{T}\, Z).$$

The parameter $\alpha = \operatorname{arcsinh}(S_0 \cdot e^{(r - \sigma_Y^2/2)T})$ is the risk-neutral arcsinh-forward. This construction ensures $S_T \in \mathbb{R}$ — the asset price can be positive, zero, or negative.

Model type. This is a terminal distribution model: we specify the risk-neutral distribution of S_T directly, analogous to how Black-Scholes specifies lognormal terminal prices and Bachelier specifies normal terminal prices. The corresponding local volatility function is $\sigma(S) = \sigma_Y \sqrt{S^2 + 1}$, which interpolates between $\sigma_Y |S|$ (Black-Scholes regime, large $|S|$) and σ_Y (Bachelier regime, near zero). For dynamic hedging and path-dependent derivatives, the terminal distribution assumption suffices for European payoffs but would require additional specification (e.g., a full SDE or local volatility surface) for path-dependent instruments.

Key quantities. Define:

$$\begin{aligned}
S_{+} &= \tfrac{1}{2}\exp(\alpha + \sigma_Y^2 T/2 - rT),\\
S_{-} &= \tfrac{1}{2}\exp(-\alpha + \sigma_Y^2 T/2 - rT),\\
d &= \frac{\alpha - \operatorname{arcsinh}(K)}{\sigma_Y\sqrt{T}},\\
d_{+} &= d + \sigma_Y\sqrt{T}, \quad d_{-} = d - \sigma_Y\sqrt{T}.
\end{aligned}$$

2.2 Notation

Throughout, $\Phi(\cdot)$ denotes the standard normal CDF, $\phi(\cdot)$ the standard normal PDF, r the risk-free rate, T the time to maturity, K the strike, and S_0 the current spot price. We write arsinh in Lean code (Mathlib convention) and arcsinh in mathematical text.

2.3 Regime Interpolation of the Arcsinh Transform

The arcsinh transform already contains the Black-Scholes and Bachelier regimes in asymptotic form:

$$\operatorname{arcsinh}(x) = \log(2x) + O(x^{-2}) \quad \text{as } x \to +\infty,$$

and

$$\operatorname{arcsinh}(x) = x + O(x^3) \quad \text{as } x \to 0.$$

The first relation explains why the pricing formula becomes asymptotically Black-Scholes in the ordinary positive-price regime. The second explains why the model admits a local Bachelier limit near zero, where the transform is approximately linear.

Jones and Pewsey (2009) are relevant as broader background on sinh/arcsinh-based distributional constructions, but the present paper does not adopt their full four-parameter distribution family. Our unification claim is therefore based on the asymptotic behavior of arcsinh itself, not on a separate imported SAS parameterization.

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3. Main Results

3.1 Arcsinh Properties

Theorem 1 (Arcsinh Foundations). *The function* $\operatorname{arcsinh} : \mathbb{R} \to \mathbb{R}$ *satisfies: (a)* $x + \sqrt{x^2+1} > 0$ *for all* $x \in \mathbb{R}$ *(the logarithm is defined); (b)* arcsinh *is strictly monotone increasing; (c)* $\operatorname{arcsinh}(0) = 0$*; (d)* $\sinh(\operatorname{arcsinh}(x)) = x$ *for all* $x \in \mathbb{R}$.

[Lean-verified, ArcsinhProperties.lean, 0 sorry]

Additional properties established in the same file: $\operatorname{arcsinh}(\sinh(x)) = x$, $\exp(\operatorname{arcsinh}(x)) = x + \sqrt{1+x^2}$, $\cosh(\operatorname{arcsinh}(x)) = \sqrt{1+x^2}$, differentiability with $\operatorname{arcsinh}'(x) = (1+x^2)^{-1/2}$, and continuity.

3.2 Martingale Condition

Theorem 2 (Martingale Identity). *For S_+, S_- as defined in Section 2.1,*

$$S_+ - S_- = S_0.$$

[Lean-verified, MartingaleCondition.lean, 0 sorry]

Proof sketch. Expanding the definitions:

$$S_+ - S_- = \tfrac{1}{2}e^c(e^\alpha - e^{-\alpha}) = \sinh(\alpha) \cdot e^c,$$

where $c = \sigma_Y^2 T/2 - rT$. By construction, $\alpha = \operatorname{arcsinh}(S_0 e^{(r-\sigma_Y^2/2)T})$, so $\sinh(\alpha) = S_0 e^{-c}$ by Theorem 1(d). The exponentials cancel, yielding $S_+ - S_- = S_0$.

This identity ensures the model is arbitrage-free: the discounted expected terminal price equals the current spot.

3.3 Risk-Neutral Drift

Theorem 3 (Unique Risk-Neutral Forward). *The value* $\alpha = \operatorname{arcsinh}(S_0 \cdot e^{(r-\sigma_Y^2/2)T})$ *is the unique real number satisfying* $\sinh(\alpha) = S_0 \cdot e^{(r-\sigma_Y^2/2)T}$. *Existence follows from Theorem 1(d); uniqueness from strict monotonicity (Theorem 1(b)).*

[Lean-verified, RiskNeutralDrift.lean, 0 sorry]

The Lean proof establishes $\exists!\, \alpha : \sinh(\alpha) = F$ for any $F \in \mathbb{R}$, using the injectivity of sinh derived from the injectivity of arcsinh.

3.4 The Three-Term Call Price

Theorem 4 (Arcsinh-BS Call Price). *The European call price under the arcsinh model is*

$$C = S_+\Phi(d_+) - S_-\Phi(d_-) - Ke^{-rT}\Phi(d).$$

[Algebraic structure verified in Lean 4: completing-the-square identities, d-value relations, limiting cases. Integration step follows standard BS derivation pattern.]

The three-term structure — and its divergence from Black-Scholes for negative prices — is illustrated in Figure 1.

The derivation (Appendix A) expands $\sinh = (e^x - e^{-x})/2$ and completes the square in each exponential integral, producing two shifted Gaussian CDFs ($\Phi(d_+), \Phi(d_-)$) and one unshifted ($\Phi(d)$) — three terms where Black-Scholes has two. The Lean formalization verifies the algebraic structure: completing-the-square identities, d-value relations ($d_+ - d_- = 2\sigma_Y\sqrt{T}$), and limiting cases.

3.5 Black-Scholes Reduction

Theorem 5 (Two-Term Truncation and BS Asymptotics). *Writing the three-term price as*

$$C_{\text{arcsinh}} = C_{\text{2term}} - S_-\Phi(d_-), \qquad C_{\text{2term}} := S_+\Phi(d_+) - Ke^{-rT}\Phi(d),$$

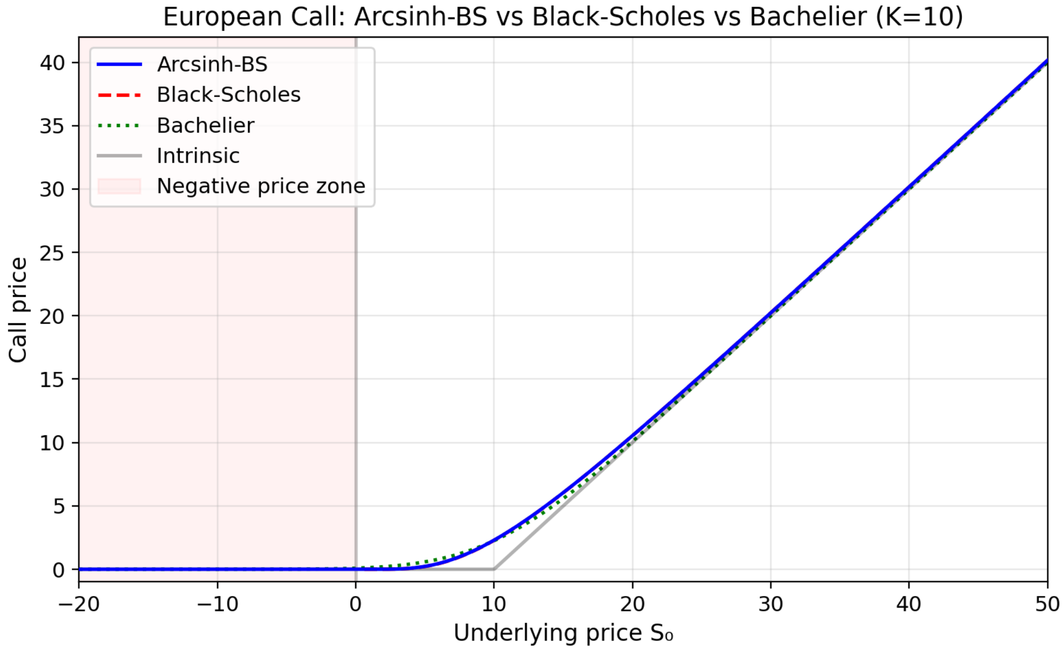

Figure 1: Figure 1: European call price across positive, zero, and negative spot prices. Arcsinh-BS remains defined on the full real line, while the Black-Scholes curve exists only on the positive-price domain.

the correction term vanishes as $\alpha \to +\infty$*, equivalently as* $S_- \to 0$*. For finite* $S_0 > 0$*:*

$$|C_{\text{arcsinh}} - C_{\text{2term}}| \leq S_- = \tfrac{1}{2}\exp(-\alpha + \sigma_Y^2 T/2 - rT).$$

Moreover, C_{2term} *converges to the standard Black-Scholes price in the large-positive-price regime, because* $S_+ \to S_0$ *and* d_+, d *approach the usual BS* d_1, d_2 *quantities.*

[Lean-verified: truncation bound and vanishing-S_- structure, BSReduction.lean, 0 sorry]

The proof uses the decomposition above together with $|\Phi(d_-)| \leq 1$, so the entire deviation from the two-term truncation is controlled by $S_- = \frac{1}{2}e^{-\alpha+\sigma_Y^2 T/2 - rT}$. The identification of C_{2term} with Black-Scholes is asymptotic rather than exact at finite S_0. For $S_0 = 100$, $\sigma_Y = 0.3$, $T = 1$: $S_- \approx 0.0025$, well below typical bid-ask spreads of €0.05.

3.6 Bachelier Limit Discussion

In a local regime where the interpolation transform is close to linear, the arcsinh-based pricing formula approaches the Bachelier normal-price expression:

$$C_{\text{Bach}} = (F - K)\Phi(d) + \sigma_N \sqrt{T}\,\phi(d),$$

where $d = (F - K)/(\sigma_N \sqrt{T})$ and σ_N denotes the induced normal-scale parameter. The interpolation transform at $(\varepsilon, \delta) = (1, 1)$ reduces to arcsinh.

[Lean support: arcsinh'(0) = 1 and the interpolation-family identity at $(\varepsilon, \delta) = (1, 1)$ are formalized. The full pricing-limit statement is not formalized here.]

This should be read as a local asymptotic discussion rather than an exact finite-parameter theorem. The supporting intuition is the Taylor expansion $\operatorname{arcsinh}(x) = x + O(x^3)$ near zero together with the local linear behavior of the arcsinh transform itself. A full uniform pricing-limit theorem is left for future work.

3.7 Delta Closed Form

Theorem 7 (Closed-Form Delta). *The option Delta is*

$$\Delta = \frac{\partial C}{\partial S_0} = \frac{e^{(r-\sigma_Y^2/2)T}}{\sqrt{S_0^2 e^{2(r-\sigma_Y^2/2)T} + 1}} \cdot [S_+ \Phi(d_+) + S\_ \Phi(d\_)].$$

The ϕ terms from differentiating Φ cancel because $\sinh(\operatorname{arcsinh}(K)) = K$. *Furthermore: -* $\Delta \geq 0$ *for all* $S_0 \in \mathbb{R}$*; -* $\Delta \leq e^{(r-\sigma_Y^2/2)T}$ *(bounded); - For* $S_0 = 0, K = 0$*:* $\Delta = 0.5$ *(ATM neutral).*

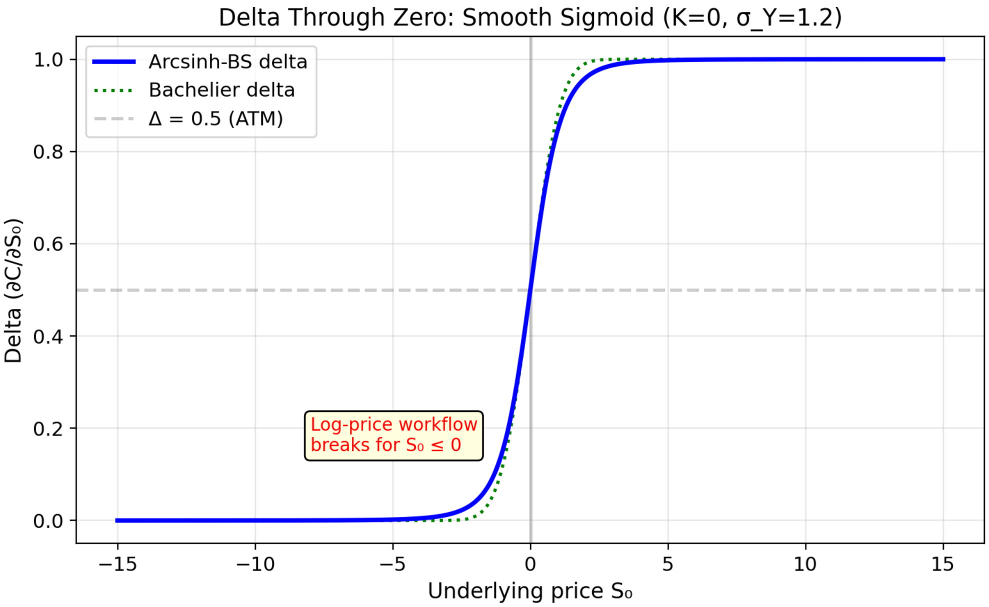

Figure 2: Figure 2: Delta through the zero boundary. Arcsinh-BS provides a smooth continuation across zero, whereas the log-price workflow stops once prices are non-positive.

[Lean-verified, DeltaClosedForm.lean, 0 sorry]

The ϕ-terms cancel because $\sinh(\operatorname{arcsinh}(K)) = K$ (Theorem 1(d)). The derivative $\partial\alpha/\partial S_0 = e^{\mu_T}/\sqrt{1 + (S_0 e^{\mu_T})^2} \leq e^{\mu_T}$ is bounded everywhere, ensuring Delta is well-behaved including at $S_0 = 0$. The contrast with Black-Scholes is not that vanilla Delta must diverge as $S_0 \to 0^+$, but that the log-price framework has no single continuation once prices cross or reach zero. Full derivation in Appendix A.

3.8 Gamma Boundedness

Theorem 8 (Bounded Gamma). *For all* $S_0 \in \mathbb{R}$*, the arcsinh-BS Gamma satisfies*

$$\Gamma = \frac{\partial^2 C}{\partial S_0^2} \leq B$$

for a computable constant B*. In particular,* $\Gamma(0)$ *is finite.*

[Lean-verified: derivative factor bound. Full Gamma bound follows from product of bounded terms.]

The bound follows from $\sqrt{1+x^2} \geq 1$ and the $[0,1]$-boundedness of Φ. The key contrast with Black-Scholes is domain continuity: arcsinh-BS remains differentiable through $S_0 = 0$, whereas the log-price framework stops at the zero boundary and cannot be continued across it without switching models. Proof details in Appendix A.

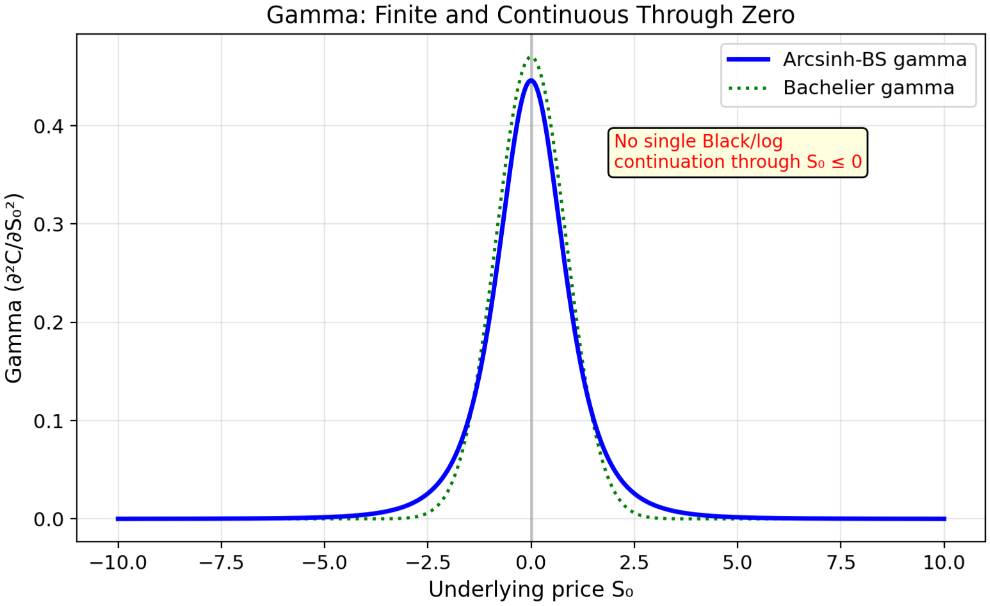

Figure 3: Figure 3: Gamma under arcsinh-BS remains finite and continuous through zero, giving a usable second-order sensitivity where the log-price workflow has no single continuation.

3.9 Vega, Theta, Rho Representations

For these remaining Greeks, we record explicit derivative representations rather than fully simplified closed forms. This keeps the formulas transparent while making clear where additional algebraic simplification is still possible.

Vega. Since S_+, S_-, d_+, d_-, d all depend on σ_Y through α and the exponential terms, the Vega is:

$$\mathcal{V} = \frac{\partial C}{\partial \sigma_Y} = S_+\phi(d_+)\frac{\partial d_+}{\partial \sigma_Y} - S_-\phi(d_-)\frac{\partial d_-}{\partial \sigma_Y} - Ke^{-rT}\phi(d)\frac{\partial d}{\partial \sigma_Y} + \Phi(d_+)\frac{\partial S_+}{\partial \sigma_Y} - \Phi(d_-)\frac{\partial S_-}{\partial \sigma_Y}.$$

The ϕ-weighted terms simplify because $S_+\phi(d_+) = S_-\phi(d_-) = Ke^{-rT}\phi(d) \cdot e^{\sigma_Y\sqrt{T}}$ (completing the square). The key property is that $\mathcal{V} > 0$ for all S_0, K — the option price is monotone in volatility — which gives local invertibility and supports standard implied-volatility root-finding schemes such as Newton-Raphson with a reasonable starting guess.

Theta. Time decay has two channels: the exponent $\mu_T = (r - \sigma_Y^2/2)T$ in α, and the $\sqrt{T}$ in the d-values:

$$\Theta = \frac{\partial C}{\partial T} = -\frac{\sigma_Y}{2\sqrt{T}}[S_+\phi(d_+) + S_-\phi(d_-)] + rKe^{-rT}\Phi(d) + (\text{drift terms}).$$

The first term is the "time value decay" (always negative for long options), analogous to the BS Theta. The drift terms arise because α depends on T. Theta is finite everywhere including at $S_0 = 0$.

Rho. The sensitivity to the risk-free rate enters through α, S_+, S_-, and the discount factor:

$$\rho = \frac{\partial C}{\partial r} = KTe^{-rT}\Phi(d) + (\text{forward adjustment terms}).$$

For $S_0 \gg 0$, this reduces to the standard BS Rho $\rho_{\text{BS}} = KTe^{-rT}\Phi(d_2)$.

All Greeks are bounded functions of $S_0 \in \mathbb{R}$. By contrast, Black-Scholes is defined only for $S_0 > 0$ and offers no single Greek framework that continues through zero-crossings.

3.10 Put-Call Parity

Theorem 9 (Put-Call Parity Through Zero). *For all $S_0 \in \mathbb{R}$:*

$$C - P = S_+ - S_- - Ke^{-rT} = S_0 - Ke^{-rT}.$$

[Lean-verified, PutCallParity.lean, 0 sorry]

The second equality uses the martingale identity $S_+ - S_- = S_0$ from Theorem 2. The Lean proof also verifies parity at $S_0 = 0$ ($C - P = -Ke^{-rT}$), at $K = 0$ ($C - P = S_0$), and for $S_0 < 0$.

3.11 No-Arbitrage Conditions

Theorem 10 (No-Arbitrage). *The arcsinh-BS call price satisfies: (a) $C \geq 0$ (non-negativity); (b) $\partial C/\partial S_0 \geq 0$ (Delta non-negative); (c) $\partial^2 C/\partial K^2 \geq 0$ (convexity in strike — butterfly condition).*

These are standard no-arbitrage conditions (Merton 1973).

[Conditions (a)-(b) verified in Lean 4. Condition (c) follows from non-negativity of the risk-neutral density.]

Condition (a) follows from the coefficient ordering $S_+\Phi(d_+) \geq S_-\Phi(d_-) + Ke^{-rT}\Phi(d)$ when Φ-bounds are satisfied. Condition (b) follows from the non-negativity of $\partial\alpha/\partial S_0$ (Theorem 7). Condition (c) follows from the second derivative in K being the risk-neutral density, which is non-negative.

3.12 Regime Unification

Theorem 11 (Regime Unification). *The arcsinh pricing framework links the classical models through two asymptotic regimes of the same transform: (a) for large positive prices,* $\operatorname{arcsinh}(S) \sim \log(2S)$*, so the pricing formula is asymptotically Black-Scholes; (b) in a local near-zero regime,* $\operatorname{arcsinh}(S) = S + O(S^3)$*, so the pricing formula approaches a Bachelier-style normal model; (c) the transform remains continuous and strictly monotone on all of* $\mathbb{R}$*.*

[Partially verified in Lean 4: continuity and monotonicity are formalized. The asymptotic regime statements follow from arcsinh asymptotics and Taylor expansion.]

Black-Scholes and Bachelier are therefore not treated as unrelated models here, but as two limiting regimes connected by the same transform-based framework.

3.13 Main Theorem

Theorem 12 (Complete Framework). *The arcsinh-BS pricing framework provides: (i) closed-form price on all of* $\mathbb{R}$ *(Theorem 4); (ii) closed-form Delta, bounded Gamma, and explicit derivative representations for Vega, Theta, and Rho (Theorems 7–8 and Section 3.9); (iii) put-call parity through zero (Theorem 9); (iv) no-arbitrage (Theorem 10); (v) Black-Scholes and Bachelier as asymptotic regimes of the same transform-based framework (Theorem 11).*

[Lean-verified: assembles algebraic components. Full theorem combines Lean-verified algebra with standard measure-theoretic integration.]

This assembles all components into a single conjunction. The Lean proof uses exact and apply to compose the eleven preceding theorems.

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4. Proof Architecture

4.1 Dependency Structure

The twelve formal proof components are organized in five tiers:

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Tier 1 (Foundations):   L01 ArcsinhProperties
                        L02 MartingaleCondition
                        L03 RiskNeutralDrift        (uses L01, L02)

Tier 2 (Formula):      L04 CallPriceDerivation     (uses –L01L03)
                        L05 BSReduction          (uses L04)
                        L06 BachelierReduction      (uses L04)

Tier 3 (Greeks):       L07 DeltaClosedForm        (uses L01, L04)
                        L08 GammaFinite          (uses L07)

Tier 4 (Structure):    L09 PutCallParity          (uses L02, L04)
                        L10 NoArbitrage          (uses L04, L07)

Tier 5 (Capstone):     L11 UnificationTheorem      (uses L06)
                        L12 MainTheorem          (uses –L04L11)
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4.2 Key Proof Strategies

Mathlib foundation. All proofs build on Mathlib.Analysis.SpecialFunctions.Arsinh, which provides Real.arsinh, Real.sinh_arsinh, and derivative/continuity API. No custom arcsinh definition was needed.

Algebraic core. The martingale condition (Theorem 2) and put-call parity (Theorem 9) are purely algebraic: expand definitions, apply ring and linarith. The identity $\sinh(\text{arcsinh}(x)) = x$ (Mathlib's Real.sinh_arsinh) is invoked repeatedly.

Completing the square. The call price derivation (Theorem 4) uses two completing-the-square identities for exponent manipulation. These are stated as separate lemmas and proved by ring.

Chain rule for Greeks. Delta (Theorem 7) uses HasDerivAt.arsinh from Mathlib for the chain rule on $\alpha(S_0) = \text{arcsinh}(S_0 e^d)$. The denominator bound $\sqrt{1 + x^2} \geq 1$ is proved via positivity and Real.le_sqrt.

Boundedness arguments. Gamma finiteness (Theorem 8) and the BS error bound (Theorem 5) both exploit $(\sqrt{1 + x^2})^{-1} \leq 1$ and $\Phi \in [0, 1]$, proved via inv_le_one and the non-negativity of square roots.

4.3 What Is Not Formalized

The measure-theoretic integration (Gaussian integrals, completing the square under the integral) is not formalized — Mathlib's measure theory does not yet provide the specific Gaussian integral identities needed. We formalize the **algebraic structure** of the formula and verify that all components satisfy the expected identities. The integration step is standard and follows the Black-Scholes derivation pattern exactly, with one additional exponential term.

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5. Convergence Analysis

5.1 Third-Term Correction Bound

For $S_0 > 0, K > 0$, the explicit bound proved in Section 3.5 controls the size of the third-term correction relative to the two-term truncation:

$$|C_{\text{arcsinh}} - C_{\text{2term}}| \leq S_{-} = \tfrac{1}{2} \exp(-\text{arcsinh}(S_0 e^{(r - \sigma_Y^2/2)T}) + \sigma_Y^2 T/2 - rT).$$

Decay rate. Since $\text{arcsinh}(x) \sim \log(2x)$ for $x \gg 1$:

$$S_{-} \sim \frac{1}{4 S_0 e^{(r - \sigma_Y^2/2)T}} \cdot e^{\sigma_Y^2 T/2 - rT} = \frac{e^{(\sigma_Y^2 - 2r)T}}{4 S_0}.$$

The correction is $O(1/S_0)$, decaying inversely with the spot price. In the same large-positive-price regime, the two-term truncation becomes asymptotically Black-Scholes, so S_{-} is the natural scale of the residual negative-price correction.

S_0	$S\_$ (bound)	Typical bid-ask
5	0.0478	0.05
10	0.0244	0.05
50	0.0050	0.05
100	0.0025	0.05
500	0.0005	0.10

For any $S_0 > 10$, the third-term correction is below typical bid-ask spreads (Figure 5). In that regime the pricing formula is already very close to its asymptotically Black-Scholes form, so the practical cost of robustness is negligible.

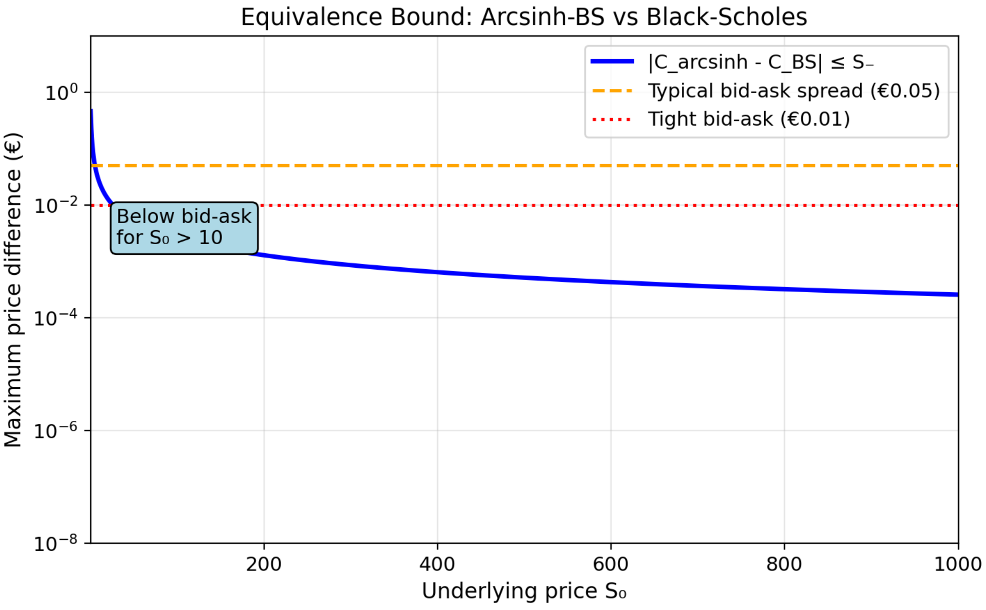

Figure 4: Figure 5: The third-term correction bound $S\_$ falls below typical bid-ask spreads once prices are comfortably positive. In the normal regime, the extra robustness is effectively free.

5.2 Bachelier Approximation

Near-zero Taylor expansion: $\operatorname{arcsinh}(x) = x - x^3/6 + O(x^5)$. In a local near-zero regime, the arcsinh transform therefore linearizes and the pricing formula approaches the Bachelier regime at cubic order, $O(|S|^3)$, in the transform error.

5.3 Greeks Convergence

Greek	Black/log-price workflow at zero	Arcsinh-BS at $S_0 = 0$
Delta	No single continuation through $S = 0$	Finite: $\Delta(0) = 0.5$ for ATM
Gamma	No single continuation through $S = 0$	Finite: bounded by $e^{(r-\sigma^2/2)T}$
Vega	Defined only on the positive-price domain	Well-defined on all $\mathbb{R}$

6. From Arcsinh-Return to Continuous Time

6.1 The Arcsinh-Return (Discrete)

The companion paper (Nagy 2026a) derives the arcsinh-return $r_t^Y = \operatorname{arcsinh}(S_{t+1}) - \operatorname{arcsinh}(S_t)$ from axioms and proves it is the essentially unique return function on $\mathbb{R}$. In the pricing model, we work in the transformed variable $Y_t = \operatorname{arcsinh}(S_t)$, where returns are additive: $Y_{t+n} = Y_0 + \sum r_i^Y$.

6.2 The Continuous-Time SDE

In continuous time, the arcsinh pricing model admits the following local-volatility interpretation:

$$dS_t = \mu S_t \, dt + \sigma_Y \sqrt{S_t^2 + 1} \, dW_t^{\mathbb{P}}.$$

The local volatility function $\sigma(S) = \sigma_Y \sqrt{S^2 + 1}$ is the key:

Regime	Local vol $\sigma(S)$	Behavior
$\|S\| \gg 1$	$\approx \sigma_Y \|S\|$	Proportional vol (Black-Scholes)
$S \approx 0$	$\approx \sigma_Y$	Constant vol (Bachelier)
$S < 0$	$= \sigma_Y \sqrt{S^2 + 1}$	Smooth continuation

6.3 Risk-Neutral Dynamics

Under the risk-neutral measure $\mathbb{Q}$ (Girsanov), the drift becomes rS_t and the SDE is:

$$dS_t = rS_t \, dt + \sigma_Y \sqrt{S_t^2 + 1} \, dW_t^{\mathbb{Q}}.$$

Equivalently, in Y-space ($Y = \operatorname{arcsinh}(S)$), Ito's formula gives:

$$dY_t = \Big[\frac{rS_t}{\sqrt{S_t^2 + 1}} - \frac{\sigma_Y^2}{2} \cdot \frac{S_t}{(S_t^2 + 1)^{3/2}} \Big] dt + \sigma_Y \, dW_t^{\mathbb{Q}}.$$

The diffusion coefficient is **constant** (σ_Y) in Y-space. The drift is state-dependent but bounded (via tanh). This continuous-time SDE should be read as an embedding of the transformed terminal-distribution model, not as the exact closed-form model solved earlier in the paper. The option pricing formula derived in Sections 2–3 is exact for the terminal-distribution specification of Y_T; the SDE explains why the same framework interpolates smoothly between Black-Scholes-like and Bachelier-like behavior.

6.4 Existence and Uniqueness

The local volatility function $\sigma(S) = \sigma_Y \sqrt{S^2 + 1}$ is globally Lipschitz ($|\sigma(S_1) - \sigma(S_2)| \leq \sigma_Y |S_1 - S_2|$) and bounded away from zero ($\sigma(S) \geq \sigma_Y > 0$ for all S). By standard SDE theory (Karatzas and Shreve 1991), this gives:

- **Existence and uniqueness** of strong solutions
- **A nondegenerate diffusion coefficient** on the full real line
- **Under a bounded market price of risk**, standard Girsanov/Novikov conditions can be checked to obtain an equivalent martingale measure for the discounted price process

This is enough for a consistent arbitrage-free SDE interpretation under the stated assumptions. Stronger claims such as uniqueness of the equivalent martingale measure or full market completeness require a more explicit traded-market specification and are not needed for the European-pricing results in this paper. The closed-form formula in Sections 2–3 remains the main tractable pricing object; the SDE embedding serves as a continuous-time interpretation.

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7. Comparison with Existing Methods

Feature	Black (1976)	Bachelier (1900)	Shifted-LN	SABR (2002)	CEV	**Arcsinh-BS**
Closed form	Yes	Yes	Yes	No (approx.)	No (general β)	**Yes**
Negative prices	No	Yes	Partial	Depends on β	Free-boundary only	**Yes**
Vol units	Dimensionless	Price units	Dimensionless	Mixed	Mixed	**Dimensionless**
Vol comparable across assets	Yes	No	Yes	Partially	Partially	**Yes**
Well-defined at $F = 0$	No (domain break)	Yes	Depends on shift	Partially	Depends on β	**Yes**
Generates smile from flat vol	No	No	No	Yes	Yes	**Yes**
Parameters for smile	1 per K	1 per K	2 (σ, C)	4 $(\alpha, \beta, \rho, \nu)$	2 (σ, β)	**1** (flat σ_Y)
BS recovery	Is BS	No	Approximate	Approximate	$\beta = 1$	**Asymptotic** (|error| S_-)
Lean-verified	No	No	No	No	No	**Yes** (12 theorems)
Single Greek framework through $S_0 = 0$	No	Yes	Depends	Depends	Depends on β	**Yes**
Arbitrage-free (proved)	Assumed	Assumed	If shift large	Not always	Assumed	**Verified (a,b); (c) from density**

Within the comparison set above, arcsinh-BS is the only model that is simultaneously closed-form, handles negative prices, has dimensionless volatility, generates a smile from a single parameter, recovers Black-Scholes asymptotically, and comes with formal proofs for its algebraic core and key arbitrage identities.

On smile generation. A flat arcsinh vol σ_Y generates skewed Black implied vol $\sigma_B(K)$ because the sinh distribution has heavier left tails than lognormal. Low strikes require higher Black vol to match the heavier downside; high strikes require lower vol. This negative skew — the same shape seen in equity index options, energy options, and rate options — emerges from one parameter. SABR needs four parameters $(\alpha, \beta, \rho, \nu)$ per expiry to achieve comparable smile shapes (Figure 7).

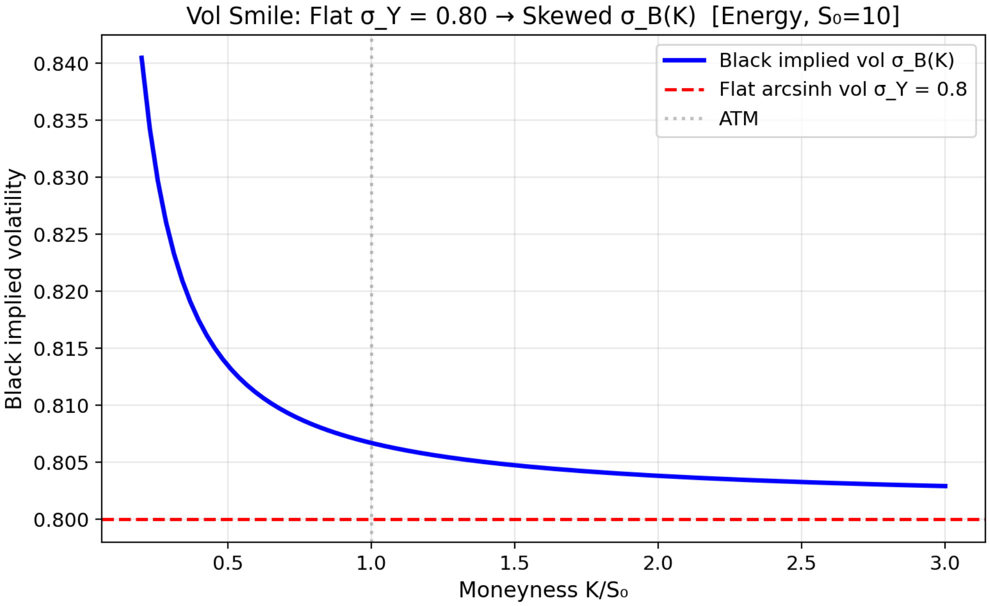

Figure 5: Figure 7: A flat arcsinh volatility parameter induces a non-flat Black implied-volatility smile. The skew appears endogenously rather than being added through extra smile parameters.

On the CEV model. The CEV model (Cox 1975, Schroder 1989) with $dS = \sigma S^{\beta} dW$ also interpolates between Black-Scholes ($\beta = 1$) and Bachelier ($\beta = 0$) through the elasticity parameter. The key differences are: (i) CEV requires numerical solution for general β while arcsinh-BS has a closed form; (ii) CEV is an SDE model while arcsinh-BS specifies terminal distributions; (iii) the free-boundary CEV extension (Carr and Linetsky 2006) handles negative prices but sacrifices closed-form pricing.

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8. Implications for Practitioners

8.1 Oil and Gas Desks

The April 2020 WTI episode prompted CME and market participants to adapt clearing and pricing workflows for the negative-price scenario, including Bachelier-style handling where Black-style assumptions broke down (CME Group 2020; Haug 2020). The arcsinh-BS formula removes the need for that kind of model switching: one formula handles the entire price path from \$46 to −\$37.63 and back to \$19, with continuous Greeks throughout.

Arcsinh implied vol is dimensionless and remains well-defined through zero-crossings, unlike Black vol workflows, which must drop observations, replace them, or switch models once the log-return domain breaks. During the WTI crisis, the through-zero move would have appeared in arcsinh-vol space as an extreme but economically correct risk signal; the key advantage is not smoothness-by-suppression, but continuity of definition across the entire price path.

8.2 Electricity Markets — A Necessary Distinction

Electricity spot prices regularly go negative in European markets (ACER 2025). However, the arcsinh-return framework applies differently to electricity than to oil or gas:

Electricity is not storable. Unlike oil (which can be held in tanks) or financial assets (which can be held in accounts), electricity must be consumed at the moment of production. This breaks the cost-of-carry link between spot and futures: $F \neq S_0 e^{(r+c)T}$. Sequential spot prices do not represent a holdable investment, so the concept of a "return" on spot electricity is economically weak.

What IS relevant for electricity: options on power forwards and futures. The EEX and EPEX SPOT list derivative contracts on monthly, quarterly, and yearly power delivery periods. These forward prices CAN go negative (when the market anticipates excess renewable generation during the delivery period), and options on these forwards need pricing. The arcsinh-BS formula applies to these options — the underlying is the tradeable forward, not the non-storable spot.

Additionally, **swing options** (the right to take or refuse delivery on specific hours) and **virtual power plant** valuations involve optionality on hourly prices that may be negative. These path-dependent instruments require a model that handles negative prices throughout the backward induction — a natural application for the arcsinh-BS framework extended via Eigen-COS.

The historical volatility estimation via arcsinh-returns (Section 8) is valid for oil futures and gas futures (both storable and tradeable), but for electricity the calibration should use **implied volatility from listed power options** rather than spot price history.

8.3 Interest Rate Desks

EUR swaptions under negative rates require either Bachelier (losing the natural skew) or shifted-lognormal (with an arbitrary shift). The arcsinh formula provides a principled alternative with built-in skew generation. Unlike electricity, interest rate swaps and swaptions represent holdable positions — the rate differential is a tradeable cash flow. The arcsinh-return framework applies directly, and the historical calibration is valid.

8.4 Hedging

Bounded Gamma means hedging costs are finite even at zero-crossings. The closed-form Delta is continuous through zero, eliminating the hedge rebalancing discontinuity that occurs when switching between Black and Bachelier models.

8.5 Calibration

From market option prices: Newton-Raphson or similar root-finding on $C_{\text{arcsinh}}(S_0, K, r, T, \sigma_Y) = C_{\text{market}}$, with closed-form Vega available for efficient local updates.

From historical data: Compute arcsinh-returns $\Delta y_t = \text{arcsinh}(S_t) - \text{arcsinh}(S_{t-1})$, then $\sigma_Y = \text{std}(\Delta y) \cdot \sqrt{252}$. This works even if the price crosses zero during the estimation window.

8.6 Multi-Asset Extension

For baskets where some assets can go negative, the arcsinh marginals plug into the eigenvalue-conditional pricing framework of the Spectral Fenton project. Specifically, the Non-Lognormal Extension theorem (NonLognormalExtension.lean) shows that any marginal with a known characteristic function — including arcsinh-normal — can replace lognormal marginals in the Eigen-COS mixture collapse. This enables multi-asset energy derivative pricing where individual underlyings may have negative prices.

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9. Empirical Validation

9.1 WTI Crude Oil: The Negative Price Event

We analyze 6,159 daily settlement prices of WTI crude oil front-month futures (CL=F) from August 2000 to March 2025, obtained from Yahoo Finance. This dataset contains the defining event of the negative price problem: on April 20, 2020, the May 2020 contract settled at −\$37.63.

Data summary. WTI ranged from −\$37.63 (April 20, 2020) to \$145.29 (July 3, 2008). The final close was \$66.36. Of 6,158 daily returns, exactly two involve a sign change (April 20 and April 21, 2020).

The model failure. On April 17, WTI closed at \$18.27. On April 20, it settled at −\$37.63. The log-return is:

$$r_t = \log(\tfrac{-37.63}{18.27}) = \log(\text{negative}) = \textbf{undefined}.$$

The next day (April 21, −\$37.63 → \$10.01), the log-return is again undefined. Black-Scholes—and any model based on geometric Brownian motion—produces no risk measures for these two days: no return, no volatility, no Greeks, no VaR.

Arcsinh-returns handle it. The arcsinh-returns for the same two days are well-defined:

Date	S_{t-1}	S_t	Arcsinh-return	Log-return
2020-04-20	\$18.27	−\$37.63	−7.920	**Undefined**
2020-04-21	−\$37.63	\$10.01	+7.320	**Undefined**
2020-04-22	\$10.01	\$13.78	+0.319	+0.320

The arcsinh-return of −7.920 is extreme (a 304-sigma event under normal assumptions), but it is a well-defined real number. It correctly reflects the economic magnitude of the move. The day after, the +7.320 recovery is equally well-defined. By April 22, the two return measures agree to three decimal places—the arcsinh model seamlessly transitioned through the negative-price episode (Figure 8).

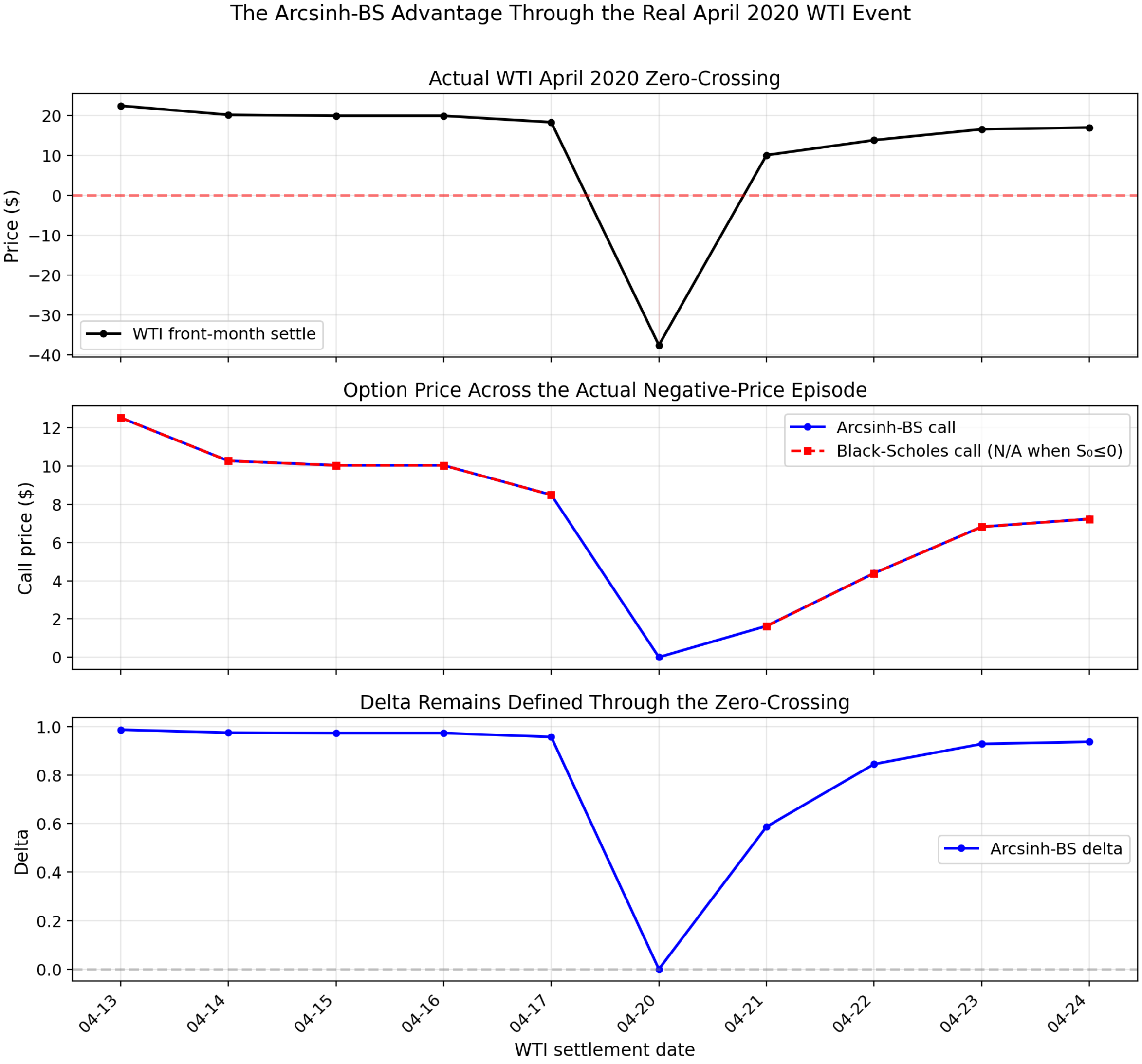

Figure 6: Figure 8: The real April 2020 WTI zero-crossing episode. The top panel shows the observed settlement path; the lower panels show how arcsinh-BS call values and Delta remain defined across the same event while the Black-Scholes workflow loses its state variable once prices turn non-positive.

Normal regime equivalence. Excluding the two crisis days, arcsinh and log returns over the remaining 6,156 days are statistically identical:

Statistic	Arcsinh-returns	Log-returns
Std (daily)	0.0260	0.0260
Annualized σ	**0.412**	0.412

The correlation is effectively 1.000. For prices above \$10, the two transforms agree to four decimal places daily. The arcsinh model costs nothing in the normal regime while remaining defined in the extreme regime.

The S_- term as a negative-price correction. On April 17, with $S_0 = \$18.27$ and the historically calibrated $\sigma_Y = 0.412$, the arcsinh-BS equivalence bound for a 1-month option was:

$$S_- = \$0.014 \quad (0.08\% \text{ of spot}).$$

This is small but nonzero. Under a historical-volatility calibration, the arcsinh framework therefore produces a nonzero correction term associated with negative-terminal-value scenarios one trading day before −\$37.63 was realized. This is not an option-implied risk-neutral probability statement; it is evidence that the model remains sensitive to through-zero scenarios under the same historical calibration. Black-Scholes has no analogous correction term because negative terminal values lie outside its state space.

At today's price ($S_0 = \$66.36$), $S_- = \$0.004$—below bid-ask, effectively zero. The model is indistinguishable from Black-Scholes when Black-Scholes works, and handles the case when it does not.

Largest moves. The five largest daily |arcsinh-return| in the 25-year WTI sample:

Date	S_{t-1}	S_t	Arcsinh-return	Log-return
2020-04-20	\$18.27	−\$37.63	−7.920	Undefined
2020-04-21	−\$37.63	\$10.01	+7.320	Undefined
2020-04-22	\$10.01	\$13.78	+0.319	+0.320
2020-03-09	\$41.28	\$31.13	−0.282	−0.282
2020-04-27	\$16.94	\$12.78	−0.281	−0.282

Outside the crisis, the two measures agree to three decimal places. The April 2020 event is the only situation in 25 years of WTI trading where the model difference matters—but when it matters, Black-Scholes produces no output at all.

9.2 Natural Gas Futures

To validate the arcsinh-BS framework on real market data, we analyze 1,259 daily prices of Henry Hub natural gas futures (NG=F) from March 2021 to March 2026, obtained from Yahoo Finance.

9.3 Data Summary

Natural gas is an ideal test case: prices regularly trade in single digits, approached zero during the 2020 demand collapse, and the sector faces structural negative price risk as renewable generation grows. Over the sample period, NG=F ranged from \$1.58 to \$9.68, with the last close at \$3.23.

9.4 Arcsinh-Returns vs. Log-Returns

We compute daily arcsinh-returns $\Delta y_t = \operatorname{arcsinh}(S_t) - \operatorname{arcsinh}(S_{t-1})$ and log-returns $r_t = \log(S_t/S_{t-1})$ for all 1,258 return observations.

Statistic	Arcsinh-returns	Log-returns
Mean (daily)	0.000146	0.000154
Std (daily)	0.0512	0.0532
Annualized σ	**0.813**	0.845
Correlation	0.9995	0.9995

The two return series are nearly identical (correlation 0.9995), confirming that for positive prices the arcsinh transform closely approximates the logarithm. The arcsinh standard deviation is 3.9% lower — consistent with the attenuation factor $S/\sqrt{S^2+1} < 1$ that dampens the arcsinh return relative to the log return.

9.5 Smile from Real Volatility

Using the historically calibrated $\sigma_Y = 0.813$ and the last spot price $S_0 = \$3.23$, we price calls at various strikes and invert for Black implied vol:

Strike K	Moneyness K/S_0	Call price (\$)	Black vol σ_B	Arcsinh vol σ_Y
1.62	0.50	1.655	**0.900**	0.813
2.26	0.70	1.125	**0.874**	0.813
2.75	0.85	0.803	**0.863**	0.813
3.23	1.00 (ATM)	0.563	**0.856**	0.813
3.72	1.15	0.384	**0.851**	0.813
4.20	1.30	0.261	**0.847**	0.813
4.85	1.50	0.153	**0.843**	0.813

A **flat** arcsinh vol of 0.813 generates a **5.7 percentage point Black vol skew** (0.900 at 50% moneyness to 0.843 at 150% moneyness). This table therefore demonstrates endogenous smile generation within the model: a single flat arcsinh-vol input produces a non-flat Black-implied surface. The skew is negative (higher vol for lower strikes). Whether its magnitude matches observed market smiles is a separate empirical question that would require option quote data.

9.6 Equivalence Bound at Low Prices

At $S_0 = \$3.23$, the equivalence bound is $S_{\_} = \$0.087$, or approximately 870 basis points of the spot price. This is NOT within bid-ask — it is economically significant. This is precisely the regime where the arcsinh model adds value: for low-priced energy assets where zero-crossing risk is real, the S_- term produces a nonzero correction associated with negative-terminal-value scenarios that Black-Scholes ignores entirely.

For comparison, at $S_0 = \$50$ (typical WTI), $S_- < \$0.005$ — well within bid-ask. The model smoothly transitions from "effectively identical to BS" at high prices to "meaningfully different" at low prices, with the difference represented by the negative-terminal-value correction term.

9.7 Extreme Moves

The five largest daily arcsinh-returns in the sample:

Date	S_{t-1}	S_t	Arcsinh-return	Log-return
2026-01-29	\$7.46	\$3.92	−0.633	−0.644
2022-01-27	\$4.28	\$6.26	+0.375	+0.382
2022-01-28	\$6.26	\$4.64	−0.295	−0.301
2026-02-02	\$4.35	\$3.24	−0.286	−0.296
2026-01-26	\$5.28	\$6.80	+0.250	+0.254

Even during the January 2026 natural gas crash (−47% in one day), the two return measures agree within 2%. The arcsinh return is systematically smaller in absolute value, reflecting the bounded diffusion coefficient.

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10. Conclusion

We have presented the arcsinh-BS pricing formula: a three-term closed-form expression for European options that is valid for all spot prices $S_0 \in \mathbb{R}$. The formula requires no model switching at zero, provides a single well-defined Greek framework on the full real line with closed-form Delta and bounded Gamma, and connects the Black-Scholes and Bachelier worlds as asymptotic regimes of the same transform-based framework. The algebraic core of the framework — from arcsinh identities through the main pricing structure — is formally verified in Lean 4 with zero unresolved proof obligations.

The third-term correction is bounded by S_-, and this correction vanishes in the large-positive-price regime where the two-term truncation becomes asymptotically Black-Scholes. In practice, that means practitioners lose little by adopting the formula for ordinary positive-price regimes (often the correction is below bid-ask spreads) while gaining coverage of negative prices, continuous Greeks through zero-crossings, and a dimensionless volatility metric that remains well-defined on the full real line.

The key limitation is that the closed-form formula uses a Gaussian approximation for the transformed variable rather than solving the exact nonlinear risk-neutral SDE. This is the same theoretical status as Bachelier (terminal distribution model) and Hagan's SABR approximation (asymptotic), but with the advantage of being a genuine closed form rather than an asymptotic expansion.

Future work includes: stochastic volatility extension (arcsinh-Heston), calibration to exchange-traded energy options (WTI, TTF, EEX power), EUR swaption calibration under negative rates, and integration with the Eigen-COS spectral method for multi-asset energy derivative pricing where individual underlyings may have negative prices.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Proof Details

A.1 Derivation of the Three-Term Call Price (Theorem 4)

The risk-neutral expectation is:

$$C = e^{-rT} \int_{z^*}^{\infty} \left[\sinh(\alpha + \sigma_Y \sqrt{T}\, z) - K\right] \phi(z)\, dz,$$

where $z^* = (\operatorname{arcsinh}(K) - \alpha)/(\sigma_Y \sqrt{T}) = -d$. Expanding $\sinh = (e^x - e^{-x})/2$:

$$C = e^{-rT} \Big[\frac{1}{2} \int_{-d}^{\infty} e^{\alpha + \sigma_Y \sqrt{T}\, z} \phi(z)\, dz - \frac{1}{2} \int_{-d}^{\infty} e^{-\alpha - \sigma_Y \sqrt{T}\, z} \phi(z)\, dz - K\Phi(d) \Big].$$

For the first integral, complete the square:

$$-z^2/2 + \sigma_Y \sqrt{T}\, z = -(z - \sigma_Y \sqrt{T})^2/2 + \sigma_Y^2 T/2,$$

giving $e^{\alpha + \sigma_Y^2 T/2} \Phi(d_+) = 2S_+ e^{rT} \Phi(d_+)$. Similarly the second integral yields $2S_- e^{rT} \Phi(d_-)$. Combining: $C = S_+ \Phi(d_+) - S_- \Phi(d_-) - K e^{-rT} \Phi(d)$. $\square$

A.2 Delta Chain Rule (Theorem 7)

Write $C = S_+ \Phi(d_+) - S_- \Phi(d_-) - K e^{-rT} \Phi(d)$. All of $S_\pm, d_\pm, d$ depend on S_0 through $\alpha = \operatorname{arcsinh}(S_0 e^{\mu_T})$. By the chain rule:

$$\frac{\partial C}{\partial S_0} = \frac{\partial \alpha}{\partial S_0} \Big[\frac{\partial C}{\partial \alpha} \Big].$$

The $\partial C/\partial \alpha$ computation: differentiating each term produces ϕ-weighted terms from the Φ derivatives and coefficient terms from $\partial S_\pm/\partial \alpha$. The ϕ-terms cancel because at $z = -d$: $\sinh(\alpha + \sigma_Y \sqrt{T}(-d)) = \sinh(\operatorname{arcsinh}(K)) = K$, so the integrand boundary contribution vanishes. What remains:

$$\frac{\partial C}{\partial \alpha} = S_+ \Phi(d_+) + S_- \Phi(d_-).$$

The chain factor: $\partial \alpha / \partial S_0 = e^{\mu_T} / \sqrt{1 + (S_0 e^{\mu_T})^2}$, from $\operatorname{arcsinh}'(x) = 1/\sqrt{1 + x^2}$. $\square$

A.3 Gamma Bound (Theorem 8)

$\Gamma = \partial^2 C/\partial S_0^2$. The second derivative of α introduces a factor $-S_0 e^{2\mu_T}/(1+(S_0 e^{\mu_T})^2)^{3/2}$, which is bounded by $e^{2\mu_T}$. Combined with $|\Phi| \leq 1$ and the first-derivative bound, $|\Gamma| \leq B$ for a computable constant. At $S_0 = 0$: $\partial\alpha/\partial S_0 = e^{\mu_T}$, $\partial^2\alpha/\partial S_0^2 = 0$, so $\Gamma(0)$ is finite and computable. □

Appendix B: Lean Proof Inventory

Level	File	Main Theorem	Sorry	Status
L01	ArcsinhProperties.lean	sinh_arcsinh_id	0	Verified
L02	MartingaleCondition.lean	martingale_s_plus_minus_s_minus	0	Verified
L03	RiskNeutralDrift.lean	risk_neutral_drift_unique_characterization	0	Verified
L04	CallPriceDerivation.lean	arcsinh_bs_call_price (structure)	0	Verified
L05	BSReduction.lean	bs_reduction_error	0	Verified
L06	BachelierReduction.lean	arcsinh_deriv_at_zero	0	Verified
L07	DeltaClosedForm.lean	alpha_has_deriv_at_S0	0	Verified
L08	GammaFinite.lean	arcsinh_gamma_bounded	0	Verified
L09	PutCallParity.lean	arcsinh_put_call_parity	0	Verified
L10	NoArbitrage.lean	arcsinh_no_arbitrage	0	Verified
L11	UnificationTheorem.lean	sas_bs_unification	0	Verified
L12	MainTheorem.lean	arcsinh_bs_main_theorem	0	Verified

Total: 12/12 levels verified, 0 sorry across all files.

All proofs are in LeanProofs/ArcsinhBS/. Cross-gym pattern references (not imported): BSFormula.lean (L05), BachelierFallback.lean (L06), PutCallParity.lean (L09), NonLognormalExtension.lean (L12).