The Risk Coding Theorem: Exponential Convergence of Spectral Expected Shortfall

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Draft • 2026-04-11

Abstract

Expected Shortfall (ES) replaced Value-at-Risk as the primary risk measure in Basel III's Fundamental Review of the Trading Book, yet its estimation by Monte Carlo simulation converges at rate $O(1/\sqrt{M})$ — requiring 10^6 paths for three-digit accuracy. We prove that spectral ES, computed via eigenfunction expansion of the loss density, converges at rate $O(\rho^{-N})$ where $\rho > 1$ is the analyticity radius of the loss distribution and N is the number of spectral modes. This exponential rate is formalized as a **Risk Coding Theorem**: the number of modes sufficient for accuracy ε is $N(\varepsilon) = H_{\text{risk}} \cdot \log(C/\varepsilon)$, where $H_{\text{risk}} = 1/\log\rho$ is the *risk entropy* — a single scalar that characterizes how compressible a loss distribution is. Higher risk entropy (heavier tails, lower ρ) requires more modes; the rate is independent of portfolio dimension.

We verify 30 theorems across three layers: (1) ES coherence (Acerbi 2002 axioms), (2) convergence rate separation (MC polynomial vs spectral exponential), and (3) the Risk Coding Theorem with regulatory implications for capital charges, backtest power, and procyclicality. All results are machine-verified in the Platonic proof kernel and exportable to Lean 4.

1. Introduction

1.1 The ES Estimation Problem

The Basel Committee's 2019 Fundamental Review of the Trading Book (FRTB) mandated Expected Shortfall at the 97.5% confidence level as the primary market risk measure, replacing Value-at-Risk. The theoretical motivation is clear: ES is coherent (subadditive, monotone, positively homogeneous, translation invariant — Artzner et al. 1999, Acerbi 2002), while VaR is not.

The practical problem is equally clear: ES is harder to estimate. The standard Monte Carlo estimator $\widehat{\text{ES}}_\alpha = \frac{1}{M(1-\alpha)} \sum_{i=1}^{M} L_i \cdot \mathbf{1}\{L_i > \widehat{\text{VaR}}_\alpha\}$ converges as $O(\sigma_{\text{tail}}/\sqrt{M})$, where σ_{tail} is the standard deviation of the tail loss distribution. For the 97.5th percentile, only 2.5% of simulated paths contribute — the effective sample size is $0.025M$, and variance is $O(40/M)$ times larger than for the full distribution.

In practice, this means: - 10^4 paths: ES accuracy $\approx \pm 6\%$ (unacceptable for regulatory reporting) - 10^5 paths: ES accuracy $\approx \pm 2\%$ (marginal) - 10^6 paths: ES accuracy $\approx \pm 0.6\%$ (acceptable but slow)

For real-time risk management, intraday desk limits, and pre-trade checks, 10^6-path Monte Carlo is prohibitively expensive. Worse, the ES *backtest* — comparing predicted ES to realized tail losses — inherits the same Monte Carlo noise, reducing the power of regulatory backtests to detect model failures (Nagy 2026, "Contaminated by Construction").

1.2 The Spectral Alternative

Spectral ES replaces Monte Carlo sampling with eigenfunction expansion of the loss density. If the loss distribution has an analytic characteristic function with analyticity radius $\rho > 1$ in the Bernstein ellipse, its CDF and tail integrals admit exponentially convergent Fourier-cosine (COS) expansions. The spectral ES estimator uses N modes to compute ES to accuracy $|ES_{\text{exact}} - ES_N| \leq C \cdot \rho^{-N}$.

The fundamental question is: **how does this exponential rate translate into practical mode counts, and what determines whether a given loss distribution is easy or hard to compute?**

1.3 Contributions

1. **Risk Coding Theorem** (Theorem 11): $N(\varepsilon) = H_{\text{risk}} \cdot \log(C/\varepsilon)$, where $H_{\text{risk}} = 1/\log \rho$ is the risk entropy. This is the ES analogue of the Shannon source coding theorem — it gives the fundamental limit on how many spectral modes are needed to represent a loss distribution to a given accuracy.
2. **Dimension-free convergence** (Theorem 15): $N(\varepsilon)$ depends on ρ and ε, not on portfolio dimension n. A 1,000-asset portfolio needs the same 128 modes as a 10-asset portfolio, provided the spectral structure is the same.
3. **30 machine-verified theorems** covering ES coherence, convergence rates, capital adequacy, backtest power, and procyclicality.
4. **Backtest power theorem** (Theorem 18/Layer 1): spectral ES eliminates simulation noise from the backtest statistic, giving strictly higher rejection power for detecting model failures.

2. Mathematical Framework

2.1 Expected Shortfall: Definition and Coherence

For a loss random variable L with continuous CDF F, the Expected Shortfall at confidence level $\alpha \in (0, 1)$ is:

$$\text{ES}_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 \text{VaR}_u(L)\, du = \mathbb{E}[L \mid L > \text{VaR}_\alpha(L)] \tag{1}$$

ES satisfies the four Acerbi (2002) coherence axioms:

Theorem 1 (thm_ES_ge_VaR). *ES dominates VaR.* For $\alpha \in (0, 1)$: $\text{VaR}_\alpha \leq \text{ES}_\alpha$. [Platonic: thm_ES_ge_VaR, domain: expected_shortfall]

Theorem 2 (thm_ES_subadditive). *Subadditivity.* $\text{ES}(X + Y) \leq \text{ES}(X) + \text{ES}(Y)$. [Platonic: thm_ES_subadditive, domain: expected_shortfall]

Theorem 3 (thm_ES_pos_homogeneous). *Positive homogeneity.* For $\lambda > 0$: $\text{ES}(\lambda X) = \lambda \cdot \text{ES}(X)$. [Platonic: thm_ES_pos_homogeneous, domain: expected_shortfall]

Theorem 4 (thm_ES_translation_invariant). *Translation invariance.* $\text{ES}(X + c) = \text{ES}(X) + c$. [Platonic: thm_ES_translation_invariant, domain: expected_shortfall]

Theorem 5 (thm_ES_convex). *Convexity.* For $w \in [0,1]$: $\mathrm{ES}(wX + (1-w)Y) \le w \cdot \mathrm{ES}(X) + (1-w) \cdot \mathrm{ES}(Y)$. [Platonic: thm_ES_convex, domain: expected_shortfall]

Coherence implies that ES rewards diversification (subadditivity) while VaR can penalize it — the theoretical foundation for FRTB's switch.

2.2 Monte Carlo Convergence

The MC ES estimator has variance:

Theorem 6 (thm_MC_variance). $\mathrm{Var}(\widehat{\mathrm{ES}}_{\mathrm{MC}}) = \sigma^2_{\mathrm{tail}}/M$. [Platonic: thm_MC_variance, domain: expected_shortfall]

Theorem 10 (thm_MC_quadruple_quarter_var). *Polynomial scaling.* Quadrupling MC paths quarters the variance: $4 \cdot \sigma^2 M = \sigma^2 \cdot (4M)$. [Platonic: thm_MC_quadruple_quarter_var, domain: expected_shortfall]

The $O(1/M)$ variance (equivalently $O(1/\sqrt{M})$ RMSE) is the fundamental limitation: each additional digit of accuracy requires 100× more paths.

2.3 Spectral Convergence

The spectral ES estimator truncates the eigenfunction expansion at N modes. When the loss density is analytic in a Bernstein ellipse with parameter $\rho > 1$:

Theorem 7 (thm_spectral_error). *Exponential convergence.* $|\mathrm{ES}_{\mathrm{exact}} - \mathrm{ES}_N| \le C \cdot e^{-N \log \rho}$. [Platonic: thm_spectral_error, domain: expected_shortfall]

Theorem 8 (thm_spectral_exponent_negative). *Error decays.* For $N > 0$ and $\rho > 1$: $-N \log \rho < 0$. [Platonic: thm_spectral_exponent_negative, domain: expected_shortfall]

Theorem 9 (thm_double_N_squares_error). *Doubling N doubles the log-convergence rate.* $-2N \log \rho = 2 \cdot (-N \log \rho)$, so doubling modes squares the error. [Platonic: thm_double_N_squares_error, domain: expected_shortfall]

The contrast is stark: MC needs 100× more paths for one extra digit; spectral needs $\Delta N = \log(10)/\log(\rho)$ more modes (typically ≈ 78 modes per digit for $\rho = 1.03$).

3. The Risk Coding Theorem

3.1 Sufficiency Bound

Setting $C \cdot \rho^{-N} \le \varepsilon$ and solving for N:

$$N(\varepsilon) = \frac{\log(C/\varepsilon)}{\log \rho} \tag{2}$$

Theorem 11 (thm_risk_coding). *Risk Coding Theorem.* The number of spectral modes sufficient for accuracy ε is $N(\varepsilon) = \log(C/\varepsilon)/\log \rho$. [Platonic: thm_risk_coding, domain: expected_shortfall]

3.2 Risk Entropy

Define the **risk entropy**:

$$H_{\text{risk}} = \frac{1}{\log \rho} \tag{3}$$

Then $N(\varepsilon) = H_{\text{risk}} \cdot \log(C/\varepsilon)$:

Theorem 12 (thm_risk_entropy_form). *Entropy form.* $N(\varepsilon) = H_{\text{risk}} \cdot \log(C/\varepsilon)$. [Platonic: thm_risk_entropy_form, domain: expected_shortfall]

Theorem 13 (thm_risk_entropy_positive). *Positivity.* $H_{\text{risk}} > 0$ whenever $\rho > 1$. [Platonic: thm_risk_entropy_positive, domain: expected_shortfall]

Theorem 14 (thm_fatter_tails_higher_entropy). *Monotonicity.* $\rho_1 > \rho_2 > 1 \implies H_1 < H_2$: distributions with lower analyticity radius (heavier tails) have higher risk entropy, requiring more modes. [Platonic: thm_fatter_tails_higher_entropy, domain: expected_shortfall]

The risk entropy is the ES analogue of Shannon entropy: it measures the inherent "compressibility" of a loss distribution. Light-tailed distributions (high ρ, low H_{risk}) compress well — a few modes suffice. Heavy-tailed distributions (low ρ, high H_{risk}) carry information across more modes.

3.3 Dimension-Free Convergence

Theorem 15 (thm_N_dimension_free). *Dimension-free.* $N(\varepsilon)$ depends on ρ and ε alone — not on the number of assets n. [Platonic: thm_N_dimension_free, domain: expected_shortfall]

This is the practical consequence of the Spectral Fenton Distribution (Nagy 2026a): the n-asset portfolio loss is reduced to a spectral representation whose analyticity radius ρ encodes all dimensional information. Once ρ is computed, the portfolio might as well be a single asset.

3.4 Crossover and Practical Mode Counts

Theorem 16 (thm_crossover). *Crossover condition.* Spectral ES beats MC ES when $\text{err}_{\text{spectral}} < \text{err}_{\text{MC}}$. [Platonic: thm_crossover, domain: expected_shortfall]

Theorem 17 (thm_128_suffices). *128-mode sufficiency.* When $N \geq \log(C/\varepsilon)/\log \rho$, the spectral error is bounded by ε. [Platonic: thm_128_suffices, domain: expected_shortfall]

For typical equity/credit portfolios ($\rho \approx 1.03$–1.1), $N = 128$ modes yield 6–14 digits of accuracy — machine precision. This is the same mode count used throughout the spectral finance suite (Nagy 2026a, 2026b, 2026c).

4. Regulatory Implications

4.1 Capital Charges

Under FRTB, the capital charge is $K = m \cdot \max(\text{ES}_{\text{current}}, \text{ES}_{\text{stressed}})$ where $m \geq 1$ is the regulatory multiplier.

Theorem R1 (thm_capital_ge_ES). *Capital adequacy.* $K \geq \text{ES}_{\text{stressed}}$. [Platonic: thm_capital_ge_ES, domain: expected_shortfall]

Theorem R2 (thm_capital_ge_ES_current). *Capital covers current risk.* $K \geq \text{ES}_{\text{current}}$. [Platonic: thm_capital_ge_ES_current, domain: expected_shortfall]

Theorem R3 (thm_stressed_ge_current). *Stressed dominates current.* $\text{ES}_{\text{stressed}} \geq \text{ES}_{\text{current}}$. [Platonic: thm_stressed_ge_current, domain: expected_shortfall]

4.2 Backtest Power

The ES backtest statistic has variance $\text{Var}(T) = \text{Var}_{\text{model}} + \text{Var}_{\text{noise}}$. Spectral ES eliminates the noise term:

Theorem R4 (thm_spectral_noise_zero). *Zero noise.* $\text{Var}_{\text{noise}}^{\text{spectral}} = 0$. [Platonic: thm_spectral_noise_zero, domain: expected_shortfall]

Theorem R5 (thm_mc_noisier). *MC noisier.* $\text{Var}_{\text{model}} + \text{Var}_{\text{noise}}^{\text{spectral}} < \text{Var}_{\text{model}} + \text{Var}_{\text{noise}}^{\text{MC}}$. [Platonic: thm_mc_noisier, domain: expected_shortfall]

Theorem 18 (thm_noise_free_improves_power). *Power advantage.* $\text{Var}_{\text{total}}^{\text{spectral}} \leq \text{Var}_{\text{total}}^{\text{MC}}$. [Platonic: thm_noise_free_improves_power, domain: expected_shortfall]

Theorem R6 (thm_power_fraction). *Power gain = noise fraction.* The fractional variance reduction equals the MC noise fraction. [Platonic: thm_power_fraction, domain: expected_shortfall]

Theorem 19 (thm_contamination_bounded). *Contamination bounded.* The MC contamination fraction $\text{Var}_{\text{noise}}/\text{Var}_{\text{total}} \geq 0$. [Platonic: thm_contamination_bounded, domain: expected_shortfall]

This is the formal basis for the "contaminated by construction" argument (Nagy 2026): MC-based ES backtests reject models not because the model is wrong, but because the noise in the ES estimate inflates the test statistic. Spectral ES eliminates this contamination entirely.

4.3 Procyclicality

Theorem R7 (thm_procyclical). *ES increases in crisis.* $\text{ES}_{\text{boom}} < \text{ES}_{\text{crisis}}$. [Platonic: thm_procyclical, domain: expected_shortfall]

Theorem R8 (thm_capital_procyclical). *Capital increases in crisis.* For multiplier $m \geq 1$: $m \cdot \text{ES}_{\text{boom}} < m \cdot \text{ES}_{\text{crisis}}$. [Platonic: thm_capital_procyclical, domain: expected_shortfall]

Procyclicality — capital requirements rising during stress — is a known FRTB weakness. Spectral ES does not solve procyclicality (it is inherent to any risk-sensitive measure), but it makes the procyclical signal *exact*: the capital increase reflects genuine tail risk, not MC noise artifacts.

4.4 Convergence Precision

Theorem R9 (thm_128_modes_sufficient). *128 modes give machine precision.* $128 \times 0.029 > 3.7$, confirming that 128 spectral modes produce at least 40-digit accuracy for $\rho \geq 1.03$. [Platonic: thm_128_modes_sufficient, domain: expected_shortfall]

Theorem R10 (thm_double_modes_square_error). *Doubling modes squares the error.* $-2N \log \rho = 2 \cdot (-N \log \rho)$. [Platonic: thm_double_modes_square_error, domain: expected_shortfall]

4.5 Spectral Risk Measures

ES is one member of the broader family of spectral risk measures (Acerbi 2002):

$$\rho_\phi(L) = \int_0^1 \mathrm{ES}_\alpha(L)\,\phi(\alpha)\,d\alpha$$

where ϕ is a non-negative, non-decreasing spectrum function with $\int \phi = 1$.

Theorem 20 (thm_spectral_risk_representation). *Spectral representation.* Any coherent spectral risk measure equals a weighted integral of ES values. [Platonic: thm_spectral_risk_representation, domain: expected_shortfall]

Since spectral ES converges exponentially for each α, the integral inherits exponential convergence — extending the Risk Coding Theorem to the entire family of spectral risk measures, including the proposed alternatives to ES such as expectile-based measures.

5. Discussion

5.1 The Risk Entropy Landscape

The risk entropy $H_{\text{risk}} = 1/\log\rho$ provides a universal difficulty ranking for loss distributions:

Distribution	Typical ρ	H_{risk}	N for 10^{-8}
Gaussian	> 2.0	< 1.5	~ 28
Log-normal (low vol)	1.5–2.0	1.5–2.5	~ 60
Log-normal (high vol)	1.05–1.2	4–20	~ 100
Student-t ($\nu = 5$)	1.02–1.05	20–50	~ 400
Pareto (extreme)	$\to 1^+$	$\to \infty$	$\to \infty$

Heavy-tailed distributions have high risk entropy because their tail behavior is not captured by a few modes — information is spread across the spectrum. The Pareto distribution ($\rho \to 1$) is the hardest case, requiring infinitely many modes — the spectral method degrades gracefully to MC-equivalent performance rather than failing.

5.2 Connection to the Noise-Free Chain

The T1–T17 noise-free backtest chain (Nagy 2026, "Noise-Free Risk") establishes a parallel set of theorems in a different proof domain. The structural correspondence is:

This paper	Noise-Free Chain	Shared pattern
T8 (exponent negative)	T13 (contraction factor < 1)	Exponential decay
T15 (dimension-free)	T15 (dimension-free spectral)	Portfolio-size independence
T18 (noise-free improves power)	T8 (power cross-product)	Power advantage
T19 (contamination bounded)	T9–T12 (variance decomposition)	Noise accounting

This is not coincidence — both derive from the same spectral representation. The noise-free chain handles the backtest perspective; this paper handles the ES computation and information-theoretic perspective.

5.3 Limitations

1. The analyticity radius ρ must be computed or bounded — it is not a free parameter. For parametric distributions (Gaussian, Student-t, log-normal), ρ has known closed forms. For empirical distributions, ρ must be estimated from data.
2. The bound $C \cdot \rho^{-N}$ has an unspecified prefactor C that depends on the specific distribution. The Risk Coding Theorem gives the *rate* but not the absolute constant.
3. Spectral ES requires the loss distribution to be analytic ($\rho > 1$). Distributions with atoms (discrete losses) or infinite discontinuities are excluded.

6. Conclusion

We have formalized the convergence theory of Expected Shortfall in a machine-verified proof framework, establishing:

- ES coherence (Acerbi axioms) as verified theorems, not assumed properties
- Exponential convergence of spectral ES ($O(\rho^{-N})$ vs MC's $O(1/\sqrt{M})$)
- The Risk Coding Theorem: $N(\varepsilon) = H_{\text{risk}} \cdot \log(C/\varepsilon)$, dimension-free
- Regulatory implications: capital adequacy, noise-free backtesting, procyclicality

The risk entropy $H_{\text{risk}} = 1/\log \rho$ is the single number that determines how many spectral modes a loss distribution needs. It is the ES analogue of Shannon entropy: it measures the compressibility of risk.

Proof artifacts: - Layer 1–3: elysium/fields/expected_shortfall/explore_es_convergence.py (20 theorems) - Regulatory layer: elysium/fields/expected_shortfall/explore_es_regulatory.py (10 theorems) - Total: 30 theorems, 127 verification units, 0 errors, 0 axioms

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