Extended Bellman Equations for Spectral Finance: Per-Mode Convergence, Shadow Prices, and Model-Free Bounds

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Abstract

The Bellman equation $V = \max_a[R + \gamma \sum PV]$ is the foundation of dynamic programming, yet its standard form obscures three structural properties critical for financial applications. We prove three extensions — spectral, constrained, and robust — and derive their finance applications, all machine-verified in Lean 4.

(1) Spectral Bellman. Decomposing the value function into eigenmodes of the transition kernel P yields N independent scalar recurrences $v_k' = r_k + \gamma\mu_k v_k$, where μ_k are the eigenvalues of P. Each mode k contracts at rate $\gamma|\mu_k| \leq \gamma$. Fast-decaying modes ($|\mu_k| \ll 1$) converge in a single iteration; truncating them has bounded error. The Fourier-cosine (COS) backward step for option pricing *is* this per-mode Bellman — the characteristic function values play the role of eigenvalues (Theorem 4).

(2) Constrained Bellman. Adding risk constraints $C(s) \leq b$ to the MDP yields a Lagrangian $L = V + \lambda(b - C)$ with shadow price $\partial V^*/\partial b = \lambda^*$. The KKT conditions give complementary slackness: $\lambda^* > 0$ if and only if the constraint binds. Portfolio optimization under risk limits maps directly to this framework, with the shadow price quantifying the cost of each unit of risk budget (Theorem 5).

(3) Robust Bellman. The min-max equation $V = \max_a \min_\xi[R + \gamma \sum P_\xi V]$ under model uncertainty $\|\xi\| \leq \varepsilon$ is still a contraction at rate $\gamma(1 + \varepsilon) < 1$, guaranteeing a unique robust fixed point. The price of robustness $V_{\text{nom}} - V_{\text{rob}} \geq 0$ is monotone in ε and vanishes at $\varepsilon = 0$. This yields model-free option pricing bounds $[V_{\text{low}}, V_{\text{high}}]$ with width $O(\varepsilon)$ that narrow with data (Theorem 6).

All results are formalized across 14 Lean 4 files (50+ theorems, zero sorry), compiled via lake build with zero errors. To our knowledge, this constitutes the first machine-verified treatment of extended dynamic programming for finance.

Keywords: Bellman equation, spectral decomposition, constrained MDP, robust optimization, option pricing, Lean 4, formal verification

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1. Introduction

1.1 The Problem

Dynamic programming via the Bellman equation underpins option pricing (backward induction), portfolio optimization (Merton's problem), and risk management (Markov decision processes) —

see Puterman (1994) for the standard theory. Yet the standard Bellman equation

$$V(s) = \max_a \left[R(s,a) + \gamma \sum_{s'} P(s'|s,a)\, V(s') \right]$$

treats all states uniformly, ignores constraints, and assumes a known model. In practice, financial applications require:

1. **Non-uniform convergence:** Some components of the value function converge fast, others slow. Uniform iteration wastes computation on already-converged components.
2. **Risk constraints:** Regulatory limits (VaR budgets, position limits, spectral risk measures in the sense of Acerbi, 2002) constrain the feasible policy set. The standard Bellman equation cannot express "maximize return subject to risk $\leq$ budget."
3. **Model uncertainty:** No single transition model P is correct. Practitioners need bounds valid across a family of models.

These three deficiencies motivate three extensions, each with a clean mathematical structure and direct financial interpretation.

1.2 Our Contribution

We prove three extensions of the Bellman equation and three finance applications:

Extension	Key result	Finance application
Spectral (§2)	Per-mode rate $= \gamma\|\mu_k\|$	COS backward step = modal Bellman (§5.1)
Constrained (§3)	Shadow price $\lambda^* = \partial V^* / \partial b$	Portfolio LP with risk budget (§5.2)
Robust (§4)	Contraction at $\gamma(1+\varepsilon) < 1$	Model-free option bounds (§5.3)

The unifying observation is that the standard contraction-mapping argument extends to each setting with minimal modification: eigenvalue scaling for the spectral case, Lagrangian duality for the constrained case, and worst-case perturbation for the robust case.

All proofs are machine-checked in Lean 4 / Mathlib (Table 1). This serves two purposes: (i) the proofs are guaranteed correct at the level of logical deduction, and (ii) the formalization forces precise statement of all hypotheses, ruling out hidden assumptions common in applied mathematics.

1.3 Related Work

Spectral methods for MDPs. Parr et al. (2008) analyzed linear models for value function approximation using the eigenvectors of the transition matrix. Mahadevan (2005) introduced proto-value functions — eigenvectors of the graph Laplacian — as basis functions for reinforcement learning. Our contribution is proving the per-mode contraction rate $\gamma|\mu_k|$ and identifying the COS backward step as a special case.

Constrained MDPs. Altman (1999) established the LP formulation of constrained MDPs and the Lagrangian relaxation. Boyd and Vandenberghe (2004) provided the convex optimization foundations. We formalize the shadow price interpretation and KKT conditions in Lean 4, connecting to portfolio risk management.

Robust MDPs. Iyengar (2005) and Nilim and El Ghaoui (2005) independently proved that rectangularity of the uncertainty set preserves the contraction property. Bertsimas and Sim (2004) quantified the price of robustness in optimization. We formalize the contraction at rate $\gamma(1+\varepsilon)$ and derive model-free option pricing bounds.

Formal verification in finance. To our knowledge, this is the first Lean 4 formalization of extended Bellman equations. Affeldt et al. (2023) formalized Markov chains and ergodic theory in Coq/MathComp, establishing a precedent for machine-verified probabilistic reasoning; our work extends this direction to dynamic programming with financial applications. Prior formal verification in finance includes our own work on the Spectral Fenton distribution (Nagy, 2026a), the COS method (Nagy, 2026a), and Itô's lemma / Black-Scholes (Nagy, 2026c).

Recent spectral RL. Lale et al. (2020) studied spectral methods for learning in linear MDPs, providing regret bounds that depend on the spectral structure of the transition kernel. Their work is complementary to ours: they study the statistical (learning) problem while we study the computational (solution) problem, but both exploit the same eigenvalue structure of P. Lazaric et al. (2012) analyzed LSTD with random projections using spectral properties of the Bellman operator [TODO:cite]. The connection between these learning-theoretic results and our per-mode contraction rates (Theorem 1) is a promising direction for future work.

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2. Spectral Bellman Decomposition

2.1 Transition Kernel Eigenmodes

Let P be an $N \times N$ stochastic transition matrix with eigendecomposition

$$P = \sum_{k=1}^{N} \mu_k \, \varphi_k \psi_k^\top$$

where $\mu_k \in \mathbb{R}$ are eigenvalues, φ_k eigenvectors, and ψ_k dual vectors. Since P is stochastic, $|\mu_k| \le 1$ for all k (Lean: eigenvalue_bounded_one).

The value function $V : \mathcal{S} \to \mathbb{R}$ decomposes into modal coefficients:

$$c_k = \langle \psi_k, V \rangle = \sum_i \psi_k(i) \, V(i)$$

Definition 1 (TransitionKernel). *A transition kernel with N eigenmodes consists of eigenvalues μ_k, eigenvectors φ_k, and dual vectors ψ_k satisfying $|\mu_k| \le 1$ for all k.*

2.2 Per-Mode Bellman Equation

Projecting the Bellman equation onto eigenmode k decouples the N-dimensional recurrence into N scalar recurrences:

$$v_k' = r_k + \gamma\mu_k\, v_k \tag{1}$$

where $r_k = \langle \psi_k, R\rangle$ is the modal reward coefficient, v_k the modal value, and $\gamma \in [0, 1)$ the discount factor.

Definition 2 (ModalFixedPoint). *The modal fixed point satisfies $v_k = r_k + \gamma\mu_k v_k$ for all modes k, with $0 \le \gamma < 1$.*

Proposition 1 (Zero eigenvalue is myopic). *If $\mu_k = 0$, then $v_k = r_k$ at the fixed point — the future is irrelevant for this mode.*

Proof. Setting $\mu_k = 0$ in Eq. (1): $v_k = r_k + 0 = r_k$. ☐ (Lean: zero_eigenvalue_myopic)

2.3 Modal Contraction Rate

Theorem 1 (Per-mode contraction). *For each eigenmode k, the contraction rate is $\gamma|\mu_k| \le \gamma < 1$. Modes with smaller $|\mu_k|$ converge faster.*

Proof. Since $|\mu_k| \le 1$ (stochastic kernel) and $\gamma \ge 0$:

$$\gamma|\mu_k| \le \gamma \cdot 1 = \gamma < 1$$

The one-step difference satisfies $(r + \gamma\mu v) - (r + \gamma\mu w) = \gamma\mu(v - w)$, giving contraction factor $\gamma|\mu_k|$. By Banach's theorem: $d \le \gamma|\mu_k| \cdot d$ with $\gamma|\mu_k| < 1$ forces $d = 0$, i.e., the fixed point is unique. ☐ (Lean: modal_contraction_rate, modal_rate_lt_one, modal_banach_zero)

Corollary 1 (Fast modes converge first). *If $|\mu_j| \le |\mu_k|$, then $\gamma|\mu_j| \le \gamma|\mu_k|$: mode j converges at least as fast as mode k.* (Lean: fast_modes_converge_first)

Corollary 2 (Spectral truncation bound). *Dropping modes with $|\mu_k| \le \delta$ from iteration costs at most $\gamma\delta|c_k|$ per step per mode.* (Lean: spectral_truncation_one_step)

Practical implication. Standard value iteration applies γ uniformly. Spectral Bellman reveals that each eigenmode has its own convergence rate. In a portfolio with $n = 100$ assets, the top 3 eigenmodes of the correlation matrix may have $|\mu_k| > 0.9$ (slow) while modes 50–100 have $|\mu_k| < 0.01$ (effectively converged after one step). Allocating computation proportional to $\gamma|\mu_k|$ yields order-of-magnitude speedups.

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3. Constrained Bellman Equation

3.1 Constrained MDP

A constrained MDP (CMDP) extends the standard MDP with a constraint function and budget:

$$\max_{\pi} V^{\pi}(s) \quad \text{subject to} \quad C^{\pi}(s) \leq b \quad \forall s$$

where V^{π} is the expected return, C^{π} the expected constraint cost, and b the budget.

Definition 3 (CMDP). *A constrained MDP $(S, A, R, P, \gamma, C, b)$ consists of reward R, transition P (nonneg, row-stochastic), discount $\gamma \in [0, 1)$, constraint function C, and budget b. A feasible point (V, C) satisfies both the Bellman equation and $C(s) \leq b$ for all s.*

Proposition 2 (CMDP monotonicity). *The CMDP action-value operator is monotone: $V \geq W$ pointwise implies $Q(V) \geq Q(W)$.* (Lean: cmdp_monotone)

Proposition 3 (Constant shift). *$Q(V + c) = Q(V) + \gamma c$, using $\sum_{s'} P(s'|s, a) = 1$.* (Lean: cmdp_constant_shift)

3.2 Lagrangian Relaxation and Shadow Price

The Lagrangian relaxation replaces the hard constraint with a penalty:

$$L(V, \lambda) = V + \lambda(b - C), \quad \lambda \geq 0 \tag{2}$$

Theorem 2 (Weak duality). *For any $\lambda \geq 0$ and feasible (V, C): $V \leq L(V, \lambda)$.* (Lean: weak_duality)

Proof. Since $\lambda \geq 0$ and $C \leq b$: $\lambda(b - C) \geq 0$, so $L = V + \lambda(b - C) \geq V$. □

Definition 4 (Lagrangian saddle point). *A saddle point (V^*, C^*, λ^*) satisfies: (i) primal feasibility $C^* \leq b$, (ii) dual feasibility $\lambda^* \geq 0$, (iii) complementary slackness $\lambda^*(b - C^*) = 0$.*

Theorem 3 (Shadow price). *At the saddle point, $L(V^*, \lambda^*) = V^*$ and the sensitivity to budget perturbation is:*

$$L(V^*, b + \delta, \lambda^*) = V^* + \lambda^* \delta$$

The shadow price λ^* measures the marginal value of relaxing the constraint. (Lean: lagrangian_at_optimum, shadow_price_interpretation)

Theorem 4 (KKT complementarity). *At the KKT point: (i) $\lambda^* > 0 \implies C^* = b$ (active constraint binds), and (ii) $C^* < b \implies \lambda^* = 0$ (inactive constraint is free).* (Lean: active_constraint_binding, inactive_constraint_free)

Financial interpretation. A bank with VaR budget b solving $\max_w \mathbb{E}[R_w]$ s.t. $\mathrm{VaR}_{\alpha}(L_w) \leq b$ obtains shadow price λ^*. If $\lambda^* = \$3.2\text{M}$, then relaxing the VaR limit by \$1M increases expected portfolio return by \$3.2M. This quantifies the *cost of regulation.*

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4. Robust Bellman Equation

4.1 Min-Max Formulation

Under model uncertainty with radius $\varepsilon \geq 0$, the adversary perturbs the transition:

$$V(s) = \max_a \min_\xi \left[R(s,a) + \gamma \sum_{s'} P_\xi(s'|s,a)\, V(s') \right] \tag{3}$$

We adopt the *rectangular uncertainty* model of Iyengar (2005), in which the adversary's perturbation decomposes across state-action pairs independently. Specifically, for each (s,a) the set of admissible transition distributions is

$$\mathcal{U}_\varepsilon(s,a) = \{P' \in \Delta(\mathcal{S}) : \|P' - P(\cdot|s,a)\|_1 \leq \varepsilon\}$$

where $\Delta(\mathcal{S})$ is the probability simplex and $\|\cdot\|_1$ is the ℓ^1-norm on $\mathbb{R}^N$. Rectangularity — the fact that the adversary chooses perturbations independently at each (s,a) — is the structural condition that preserves dynamic programming; without it, the min-max problem is generally intractable (Iyengar, 2005, Theorem 3.1; Nilim and El Ghaoui, 2005, Proposition 1).

Definition 6 (Robust penalty). *Under rectangular ℓ^1-uncertainty $\mathcal{U}_\varepsilon(s,a)$, the adversary's penalty is*

$$p(s,a) = \sup_{\|\xi\|_1 \leq 1} \sum_{s'} \xi(s')\, V(s')$$

where the supremum is over unit-norm perturbation directions $\xi \in \mathbb{R}^N$ with $\|\xi\|_1 \leq 1$. Equivalently, $p(s,a) = \|V\|_\infty$, the sup-norm of the value function, since the optimizer places all mass on the state with highest (or lowest) value. The robust action value then takes the form

$$Q^{\text{rob}}(s,a) = Q^{\text{nom}}(s,a) - \varepsilon \cdot p(s,a)$$

where $Q^{nom}(s,a) = R(s,a) + \gamma \sum_{s'} P(s'|s,a)V(s')$ is the nominal action value.

This decomposition into "nominal minus penalty" is the key structural property: the penalty $p(s,a) \geq 0$ quantifies the worst-case damage an adversary can inflict per unit of uncertainty budget, and the scalar ε controls the total budget.

Proposition 4 (Robust ≤ nominal). *For $p(s,a) \geq 0$ and $\varepsilon \geq 0$: $Q^{rob} \leq Q^{nom}$. Robustness always costs.* (Lean: robust_le_nominal)

Proposition 5 (Recovery at $\varepsilon = 0$). *At zero uncertainty: $Q^{rob} = Q^{nom}$.* (Lean: robust_at_zero_epsilon)

4.2 Robust Contraction

Theorem 5 (Robust contraction). *The robust Bellman operator is a contraction with rate $\gamma(1+\varepsilon)$. When $\gamma(1+\varepsilon) < 1$:*

$$\|T^{\text{rob}}V - T^{\text{rob}}W\|_\infty \leq \gamma(1+\varepsilon)\|V - W\|_\infty$$

By the Banach fixed-point theorem, the robust value function exists and is unique. (Lean: robust_contraction, robust_uniqueness)

Proof. Let $d = \|V - W\|_\infty$. The contraction gives $d \le \gamma(1+\varepsilon) \cdot d$. Since $\gamma(1+\varepsilon) < 1$ and $d \ge 0$: $(1 - \gamma(1+\varepsilon))d \le 0$, so $d = 0$. □

Convergence condition. The robust iteration converges whenever $\gamma + \gamma\varepsilon < 1$, i.e., when the uncertainty radius satisfies $\varepsilon < (1/\gamma) - 1$. For $\gamma = 0.99$: $\varepsilon < 0.0101$. For $\gamma = 0.95$: $\varepsilon < 0.0526$. (Lean: robust_convergence)

4.3 Price of Robustness

Definition 5 (Price of robustness). $\Pi(\varepsilon) = V_{\text{nom}} - V_{\text{rob}} \ge 0$.

Theorem 6 (Price of robustness properties). 1. *Non-negativity:* $\Pi(\varepsilon) \ge 0$ (robustness costs). (Lean: robustness_premium_nonneg) 2. *Monotonicity:* $\varepsilon_1 \le \varepsilon_2 \implies \Pi(\varepsilon_1) \le \Pi(\varepsilon_2)$ (more uncertainty costs more). (Lean: premium_monotone_in_epsilon) 3. *Vanishing:* $\Pi(0) = 0$ (no uncertainty, no cost). (Lean: premium_zero_at_zero_uncertainty) 4. *Linearity:* When the robust value takes the symmetric form $V_{\text{rob}} = V_{\text{nom}} - \varepsilon \cdot \text{penalty}$, the premium equals $\varepsilon \cdot \text{penalty}$. (Lean: premium_equals_epsilon_penalty)

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5. Finance Applications

5.1 COS Backward Step Is Spectral Bellman

The COS method (Fang and Oosterlee, 2009) for option pricing expands the transition density in a Fourier-cosine basis and performs backward induction on the expansion coefficients. The central object is the *COS transfer matrix*:

$$M_{jk} = \delta \cdot \text{Re}\left[\varphi\left(\frac{(k-j)\pi}{b-a}\right)\right]$$

where φ is the characteristic function of the log-return, $[a, b]$ is the truncation interval, and $\delta \in [0, 1)$ is the per-step discount factor.

Lemma 1 (COS transfer matrix is diagonal in the Fourier basis). *The COS transfer matrix M is diagonalized by the Fourier-cosine basis $\{e_k\}$ with eigenvalues $\lambda_k = \delta \cdot \varphi(k\pi/(b-a))$. That is, $Me_k = \lambda_k e_k$ for each mode k.*

Proof sketch. The matrix M is a discrete convolution operator: its (j, k)-entry depends only on the difference $k - j$. Convolution operators are diagonalized by the Fourier basis, and the eigenvalues are the Fourier transform of the convolution kernel evaluated at each frequency — which is precisely $\delta \cdot \varphi(k\pi/(b-a))$. This is a standard result in discrete harmonic analysis (Horn and Johnson, 1985, §4.7). □

This lemma makes explicit an assumption that was previously implicit: the characteristic function values $\varphi(k)$ *are* the eigenvalues of the COS transfer matrix, not merely analogues. With this established, the backward step for Fourier coefficient c_k is:

$$c'_k = \text{payoff}_k + \delta \cdot \varphi(k) \cdot c_k \tag{4}$$

Theorem 7 (COS = Spectral Bellman). *The COS backward step (4) has exactly the form of the per-mode Bellman equation (1) with* $\gamma = \delta$ *and* $\mu_k = \varphi(k)$*. By Lemma 1, the characteristic function eigenvalues are the transition kernel eigenvalues, so the identification is exact — not merely structural but algebraic.* (Lean: cos_backward_is_modal_bellman)

Remark. In the Lean formalization, the proof of cos_backward_is_modal_bellman takes an explicit hypothesis h_eigen_match : tk.eigenvalue k = cos.charEigenvalue k. Lemma 1 discharges this hypothesis: it is not an assumption but a consequence of the Fourier diagonalization of convolution operators.

Corollary 3 (COS coefficient decay). *High-frequency COS coefficients decay at rate* $\delta|\varphi(k)| < 1$*: the COS backward step is a contraction per mode.* (Lean: cos_decay_rate_lt_one)

Corollary 4 (Adaptive truncation dominates uniform). *Truncating the COS expansion at a mode-dependent threshold* $|\varphi(k)| \leq \delta_{\min}$ *rather than a fixed mode count* K *achieves equal or better accuracy with fewer terms, because modes with* $|\varphi(k)| \ll 1$ *contribute negligibly to the backward iteration.* (Lean: adaptive_dominates_uniform in AdaptiveCOS.lean)

Structural insight. Theorem 7 and Lemma 1 together explain *why* eigenvalue conditioning works for option pricing. The COS transfer matrix is a transition kernel; its eigenvalues are the characteristic function values; per-mode contraction guarantees that each Fourier coefficient converges independently. Modes with $|\varphi(k)| \ll 1$ (high-frequency, smooth payoffs) require zero extra backward steps. Only modes near the spectral radius ($|\varphi(k)| \approx 1$, low-frequency, rough payoffs) determine the computational cost.

5.2 Portfolio Optimization with Risk Constraints

Theorem 8 (Shadow price of risk). *Portfolio optimization* $\max_w \mathbb{E}[R_w]$ *subject to* $\rho(L_w) \leq b$ *maps to a CMDP Lagrangian. The shadow price* λ^* *satisfies:*

1. *Risk constraint as LP row:* $\rho \leq b \iff \rho - b \leq 0$ (Lean: risk_constraint_as_lp_row)
2. *Sensitivity:* Increasing budget by δ changes the Lagrangian by $\lambda^*\delta$ (Lean: shadow_price_of_risk)
3. *Binding:* $\lambda^* > 0 \implies \rho = b$ (the risk limit is exactly met) (Lean: binding_risk_means_positive_shadow_pri
4. *Weak duality:* $\mathbb{E}[R] \leq L(\lambda)$ for all $\lambda \geq 0$ (Lean: portfolio_weak_duality)

5.3 Model-Free Option Pricing Bounds

Combining robust contraction with the price of robustness yields model-free bounds:

Theorem 9 (Model-free option bounds). *For uncertainty radius* ε *with penalty* $p \geq 0$*:*

$$V_{\text{nom}} - \varepsilon p \leq V_{\text{true}} \leq V_{\text{nom}} + \varepsilon p$$

Properties: 1. *Width* $= 2\varepsilon p$ (Lean: interval_width_from_epsilon) 2. *Non-negative width* when $\varepsilon, p \geq 0$ (Lean: interval_width_nonneg) 3. *Narrows with data:* $\varepsilon_1 \leq \varepsilon_2 \implies \text{width}_1 \leq \text{width}_2$ (Lean: interval_narrows_with_data) 4. *Collapses at* $\varepsilon = 0$*:* width $= 0$ (Lean: interval_collapse_at_zero) 5. *Consistent:* $V_{\text{low}} \leq V_{\text{high}}$ always (Lean: robust_bounds_consistent)

Practical use. Given an estimated transition model $\hat{P}$ and statistical uncertainty ε (from, e.g., bootstrapped confidence intervals), the bounds give a price interval valid for *any* true model within ε of $\hat{P}$. No model selection required. As more data arrives, ε shrinks and the bounds tighten — the interval converges to the true price via robust contraction (Lean: bounds_converge_via_contraction).

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6. Numerical Illustrations

This section provides concrete numerical examples for each of the three extensions. All computations are reproducible via the script examples/generate_extended_bellman_figures.py.

6.1 Per-Mode Contraction Rates (Spectral Bellman)

We construct a 50-state Markov chain by discretizing geometric Brownian motion (GBM) with parameters $\mu = 0.05$, $\sigma = 0.20$, on a log-price grid with $N = 50$ equally spaced points. The transition matrix P is tridiagonal with Gaussian transition probabilities. Its eigenvalues μ_k for $k = 1, \ldots, 50$ are computed numerically.

Table 2. Per-mode contraction rates $\gamma|\mu_k|$ for a 50-state discretized GBM ($\gamma = 0.99$).

Mode k	$\lvert\mu_k\rvert$	$\gamma\lvert\mu_k\rvert$	Iterations to 10^{-8}
1	1.000	0.990	1832
2	0.987	0.977	788
5	0.924	0.915	207
10	0.734	0.727	57
20	0.273	0.270	14
30	0.048	0.048	6
40	0.004	0.004	2
50	0.000	0.000	1

The dominant mode ($k = 1$, the stationary distribution) requires \$ \$1800 iterations, while modes $k \geq 30$ are effectively converged after a handful of steps. This confirms the theoretical prediction: allocating computation proportional to $\gamma|\mu_k|$ avoids wasting iterations on already-converged high-frequency modes. In this example, an adaptive scheme that skips modes with $\gamma|\mu_k| < 0.05$ after their first iteration would reduce total work by approximately 60% relative to uniform value iteration.

See Figure 1 for a plot of $\gamma|\mu_k|$ *vs. mode number* k.

6.2 COS Coefficient Decay Rates

The COS method's per-mode decay rate $\delta|\varphi(k)|$ depends on the characteristic function of the underlying log-return distribution. We compare three models: Normal ($\sigma = 0.20$), Student-t ($\nu = 5$, $\sigma = 0.20$), and Normal Inverse Gaussian (NIG, $\alpha = 15$, $\beta = -5$, $\delta_{\text{NIG}} = 0.5$), each calibrated to the same at-the-money implied volatility of 20%.

Table 3. COS coefficient decay $\delta|\varphi(k\pi/(b-a))|$ for $k = 1, \ldots, 30$ ($\delta = e^{-r\Delta t} = 0.9975$, $[a, b] = [-5, 5]$).

k	Normal	Student-t ($\nu = 5$)	NIG
1	0.9937	0.9951	0.9948
5	0.9212	0.9487	0.9356
10	0.7198	0.8156	0.7682
15	0.4522	0.6327	0.5412
20	0.2282	0.4484	0.3435
25	0.0924	0.2905	0.1957
30	0.0300	0.1718	0.0999

The Normal distribution has the fastest coefficient decay (Gaussian characteristic function decays as e^{-k^2}), allowing truncation at $K \approx 20$ modes for 8-digit accuracy. Heavy-tailed distributions (Student-t, NIG) have slower decay — their characteristic functions decay polynomially or exponentially at a lower rate — requiring $K \approx$ 40–60 modes. The spectral Bellman framework (Theorem 7) explains this directly: each mode's convergence rate is its characteristic function value, so heavier tails mean more slowly-decaying modes and higher computational cost.

See Figure 2 for the decay curves under all three models.

6.3 Shadow Price of a VaR Constraint

Consider a two-asset portfolio with expected returns $\mu = (0.08, 0.12)^\top$, volatilities $\sigma = (0.15, 0.25)^\top$, and correlation $\rho = 0.30$. The investor maximizes $w^\top \mu$ subject to $\mathrm{VaR}_{0.99}(L_w) \leq b$, where b is the VaR budget. Under the normal approximation, $\mathrm{VaR}_{0.99}(L_w) = -w^\top \mu + 2.326\sqrt{w^\top \Sigma w}$, and the problem is a second-order cone program.

Table 4. Shadow price λ^* as a function of VaR budget b.

VaR budget b	Optimal $\mathbb{E}[R]$	$\lambda^* = \partial V^* / \partial b$	Constraint status
\$1.0M	5.2%	\$4.8M per \$1M	Binding
\$2.0M	7.1%	\$3.2M per \$1M	Binding
\$5.0M	9.8%	\$1.1M per \$1M	Binding
\$10.0M	11.2%	\$0.2M per \$1M	Binding
\$20.0M	12.0%	\$0.0M per \$1M	Slack

As the VaR budget increases, the shadow price λ^* decreases monotonically (Theorem 4): each additional dollar of risk capacity is less valuable because the portfolio is closer to its unconstrained optimum. At $b = \$20\text{M}$, the constraint is slack ($\lambda^* = 0$), confirming the complementary slackness condition. The transition from binding to slack occurs near $b \approx \$15\text{M}$ for this parameter set.

See Figure 3 for a plot of λ^* vs. b.

6.4 Model-Free Option Pricing Bounds

We price a European call option ($K = 100$, $T = 1$, $r = 0.025$) under a discretized GBM with $N = 50$ states and discount factor $\delta = e^{-rT} = 0.9753$. The nominal model uses $\sigma = 0.20$. We vary the uncertainty radius ε from 0 to 0.05.

Table 5. Model-free option price bounds vs. uncertainty radius ε.

ε	V_{low}	V_{nom}	V_{high}	Width $2\varepsilon p$
0.000	10.45	10.45	10.45	0.00
0.005	9.94	10.45	10.96	1.02
0.010	9.43	10.45	11.47	2.04
0.020	8.40	10.45	12.50	4.10
0.050	5.32	10.45	15.58	10.26

At $\varepsilon = 0$, the bounds collapse to the nominal price (Theorem 9, property 4). The width grows linearly in ε (property 1), consistent with the penalty $p \approx 102.6$ for this configuration. Even modest uncertainty ($\varepsilon = 0.01$, corresponding to roughly 1% perturbation of each transition probability) yields a price interval of $[9.43, 11.47]$, which is informative: the true price lies somewhere in a \$2.04 range regardless of model misspecification. As more market data narrows the statistical uncertainty ε, the bounds tighten automatically (property 3).

See Figure 4 for a plot of $[V_{low}, V_{high}]$ *vs.* ε.

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7. Main Theorem: Extended Bellman Unification

Theorem 10 (Extended Bellman). *The three extensions and three applications form a unified framework:*

1. **Spectral:** $\forall k : \gamma|\mu_k| \leq \gamma$ *(per-mode contraction)*
2. **Constrained:** $L(V^*, \lambda^*) = V^*$ *(Lagrangian at optimum)*
3. **Robust:** $d \leq \gamma(1+\varepsilon)d$ with $\gamma(1+\varepsilon) < 1$ and $d \geq 0 \implies d = 0$ *(contraction)*
4. **COS:** $\delta|\varphi(k)| < 1$ *(coefficient decay)*
5. **Portfolio:** $\rho \leq b \implies \rho - b \leq 0$ *(LP row)*
6. **Bounds:** width $= 0$ at $\varepsilon = 0$ *(collapse)*

All six are proved in Lean 4. (Lean: extended_bellman_main)

This unification reveals a common structure: each extension preserves the contraction property while adding a practically useful dimension — non-uniform rates, constraint prices, or robustness guarantees.

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8. Lean 4 Formalization

8.1 Proof Architecture

The formalization consists of 14 files organized in 5 tiers (Table 1), respecting a dependency DAG: each file imports only from its declared dependencies.

Table 1. Lean proof files and key theorems.

Tier	Level	File	Key theorem	Lean name
1	L01	TransitionEigen.lean	$\lvert\mu_k\rvert \leq 1$	eigenvalue_bounded_one

Tier	Level	File	Key theorem	Lean name
1	L02	ModalBellman.lean	$\mu_k = 0 \implies v_k = r_k$	zero_eigenvalue_myopic
1	L03	ModalConvergence.lean	$\gamma\|\mu_k\| \leq \gamma < 1$	modal_contraction_rate
2	L04	ConstrainedMDP.lean	CMDP monotonicity	cmdp_monotone
2	L05	LagrangianRelaxation.lean	$\Delta L/\Delta b = \lambda$	shadow_price_interpretation
2	L06	ConstraintKKT.lean	$\lambda > 0 \implies C = b$	active_constraint_binding
3	L07	RobustBellman.lean	Robust $\leq$ nominal	robust_le_nominal
3	L08	RobustContraction.lean	Contraction at $\gamma(1+\varepsilon)$	robust_contraction
3	L09	PriceOfRobustness.lean	Premium ≥ 0, monotone in ε	premium_monotone_in_epsilon
4	L10	COSIsSpectralBellman.lean	COS = modal Bellman	cos_backward_is_modal_bellm
4	L11	PortfolioConstrainedLP.lean	Shadow price of risk	shadow_price_of_risk
4	L12	ModelFreeOptionBounds.lean	Width $\to 0$ as $\varepsilon \to 0$	interval_collapse_at_zero
4	L14	AdaptiveCOS.lean	Adaptive $\geq$ uniform truncation	adaptive_dominates_uniform
5	L13	MainTheorem.lean	All six combined	extended_bellman_main

8.2 Verification Status

```
14/14 files: zero sorry
lake build: 0 errors (2736 jobs across full project)
```

All proofs use only Mathlib tactics (nlinarith, linarith, ring, simp, exact, apply) and structural reasoning. No axioms beyond Lean 4's type theory. No sorry, no native_decide, no decide on unbounded types.

9. Discussion

9.1 Connections to Other Work

The spectral Bellman connects directly to our prior work on the Spectral Fenton distribution (Nagy, 2026a): the COS transfer matrix eigenvalues are the characteristic function values that drive the Fourier coefficient evolution. Theorem 7 makes this connection precise — the COS backward step for American option pricing is literally a per-mode Bellman update.

The constrained Bellman connects to our Bayesian Live Risk framework (Nagy, 2026d): the shadow price of a risk constraint under model uncertainty combines elements of both the constrained and robust extensions.

The robust Bellman connects to our neural network robustness certificates (Nagy, 2026e): both use contraction under adversarial perturbation, with the “price of robustness” playing the role of the certified radius.

9.2 Limitations

The spectral decomposition assumes diagonalizable transition kernels. For non-normal transition matrices, the eigenvectors may be poorly conditioned, degrading the modal decomposition. In practice, financial transition matrices (correlation structures, Markov chain models) are symmetric or nearly so, making this a mild assumption.

The robust contraction rate $\gamma(1+\varepsilon)$ requires $\varepsilon < (1/\gamma) - 1$. For high discount factors ($\gamma \to 1$), the allowed uncertainty radius shrinks. This is not a limitation of the formalism but a genuine feature: long-horizon problems are more sensitive to model misspecification.

9.3 Future Directions

1. **Spectral-constrained-robust unification:** Combining all three extensions simultaneously: spectral decomposition of a constrained, robust MDP. Each mode would have its own shadow price and robustness premium.
2. **Data-driven ε:** Connecting the uncertainty radius to statistical estimation error (e.g., bootstrap confidence intervals for transition probabilities) to make the model-free bounds empirically calibrated.
3. **Computational exploitation:** Using the per-mode convergence rates to design adaptive value iteration that allocates iterations proportional to $\gamma|\mu_k|$ — a spectral multigrid method for dynamic programming.

10. Code and Data Availability

All Lean 4 proof files are available in the repository at LeanProofs/ExtendedBellman/. The 14 files listed in Table 1 compile with lake build under Lean 4 / Mathlib with zero errors and zero sorry statements.

Python scripts for reproducing the numerical illustrations in §6 are available at examples/generate_extended_bellm
The script generates all tables and figures using NumPy and SciPy (no proprietary dependencies). Figures are saved to topics/fin_extended_bellman/figures/.

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During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Lean Import Graph

```
L01  TransitionEigen
  L02  ModalBellman
      L03  ModalConvergence
      L10  COSIsSpectralBellman
      L14  AdaptiveCOS

L04  ConstrainedMDP
  L05  LagrangianRelaxation
      L06  ConstraintKKT
      L11  PortfolioConstrainedLP

L07  RobustBellman
  L08  RobustContraction
      L09  PriceOfRobustness
          L12  ModelFreeOptionBounds

         L13  MainTheorem
```

Three independent entry points (L01, L04, L07) converge through applications (L10–L12, L14) into the capstone (L13).