The Fenton Distribution: A Complete Analytical Characterization

PDF, CDF, Moments, Quantiles, Risk Measures, and Entropy for Sums of Correlated Lognormals

Every property a distribution can have — from a finite latent

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Draft

Abstract

We provide the first complete analytical characterization of the distribution of weighted sums of correlated lognormal random variables — the **Fenton distribution** $\mathcal{F}(w, \mu, \sigma, C)$. The distribution is determined by the finite generative latent $(w, \mu, \Sigma) \in \mathbb{R}^{n(n+5)/2}$. Using characteristic function inversion with COS evaluation (Nagy, 2026), we derive closed-form or semi-analytical expressions for every standard distributional property:

1. **Probability density function (PDF):** a finite cosine series in log-space.
2. **Cumulative distribution function (CDF):** a finite sine series in log-space.
3. **Moments:** exact closed-form for all orders.
4. **Cumulants:** exact closed-form for all orders.
5. **Skewness and kurtosis:** exact closed-form.
6. **Characteristic function:** Gauss-Hermite quadrature of directly evaluated $S(z)$.
7. **Quantile function (VaR):** root-finding on the COS CDF.
8. **Expected Shortfall (CVaR):** closed-form from COS coefficients.
9. **Spectral risk measures:** closed-form integral over the quantile function.
10. **Mode:** root-finding on the COS PDF derivative.
11. **Median:** special case of quantile function.
12. **Entropy:** numerical integration of the COS PDF.
13. **Tail behavior:** explicit asymptotic decay rates.

Items 1–2, 7–11 are finite trigonometric expressions — once the COS coefficients $\{A_k\}$ are computed, evaluation is $O(N)$ arithmetic. Items 3–6 are exact closed-form algebraic expressions requiring no numerical approximation. Items 12–13 involve one-dimensional numerical integration on the analytic PDF.

This paper serves as a **distribution reference**: a single document containing every formula needed to work with sums of correlated lognormals, analogous to what Johnson, Kotz, and Balakrishnan (1994) provide for the classical distribution families.

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1. Definition and Setup

1.1 The Fenton Distribution

Definition 1. Let $Y = (Y_1, \dots, Y_n) \sim \mathcal{N}(\mu, \Sigma)$ with $\Sigma = \operatorname{diag}(\sigma)\, C \operatorname{diag}(\sigma)$, where C is a correlation matrix. The random variable

$$S = \sum_{i=1}^{n} w_i \, e^{Y_i}$$

follows the **Fenton distribution** with parameters (w, μ, σ, C):

$$S \sim \mathcal{F}(w, \mu, \sigma, C).$$

The parameter space is $w \in \mathbb{R}^n$, $\mu \in \mathbb{R}^n$, $\sigma \in \mathbb{R}^n_{>0}$, $C \in \mathcal{S}^n_+$ (positive definite correlation matrices).

1.2 Special Cases

Condition	Distribution
$n = 1$, $w_1 = 1$	Lognormal $\mathrm{LN}(\mu_1, \sigma_1^2)$
$n = 1$, $w_1 = -1$	Reflected lognormal
$C = I$, all $w_i > 0$	Sum of independent lognormals
$n \to \infty$ (CLT conditions)	Normal
$\sigma \to 0$	Degenerate at $\sum w_i e^{\mu_i}$

1.3 The COS Representation

The exact CDF is given by characteristic function inversion (Nagy, 2026, Section 4.4). For computation, the COS method evaluates this as a finite trigonometric series parameterized by: - Spectral coefficients $\{A_k\}_{k=0}^{N-1}$, each computable in $O(Q^{\min(n,r)})$ operations (where $r = \operatorname{rank}(\Sigma)$). - Support interval $[a, b]$ in log-space, deterministically computed from the cumulants of $\log S$.

Definition 2 (COS Representation). The Fenton distribution $\mathcal{F}(w, \mu, \sigma, C)$ is represented by the tuple $(A_0, \dots, A_{N-1}, a, b)$ where

$$A_k = \sum_{\ell=1}^{Q^n} w_\ell \cos\!\left(k\pi \frac{\log S(z_\ell) - a}{b - a}\right)$$

and $S(z_\ell) = \sum_{i=1}^{n} w_i \exp(\mu_i + \ell_i^T z_\ell)$ is evaluated **directly** from the finite latent at each Gauss-Hermite node z_ℓ (no Hermite polynomial expansion).

2. Probability Density Function

2.1 The PDF Formula

Theorem 1 (PDF). The PDF of $S \sim \mathcal{F}(w, \mu, \sigma, C)$ for $S > 0$ is:

$$\boxed{f_S(x) = \frac{1}{x(b-a)} \left[A_0 + 2\sum_{k=1}^{N-1} A_k \cos\!\left(k\pi \frac{\log x - a}{b-a} \right) \right]}$$

Proof. The COS density in log-space is $f_X(t) = \frac{1}{b-a}[A_0 + 2\sum_{k=1}^{N-1} A_k \cos(k\pi(t-a)/(b-a))]$ where $X = \log S$. The Jacobian $f_S(x) = f_X(\log x)/x$ gives the result. □

2.2 Properties of the PDF

1. **Non-negativity:** Not automatically guaranteed by the COS truncation for arbitrary N. For the regimes studied ($\sigma \le 2$, $N = 128$), the PDF is empirically non-negative.
2. **Normalization:** $\int_0^\infty f_S(x)\,dx = 1 - O(\varepsilon_N)$ where ε_N is the COS truncation error (exponentially small in N).
3. **Smoothness:** $f_S \in C^\infty(0, \infty)$ (the lognormal sum density is infinitely differentiable on the positive reals).
4. **Tail decay:** $f_S(x) \sim x^{-1} \exp(-(\log x)^2/(2\sigma_{\max}^2))$ as $x \to \infty$, dominated by the highest-volatility component.

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3. Cumulative Distribution Function

3.1 The CDF Formula

Theorem 2 (CDF). The CDF of $S \sim \mathcal{F}(w, \mu, \sigma, C)$ is:

$$\boxed{F_S(x) = u + \sum_{k=1}^{N-1} \frac{2A_k}{k\pi} \sin(k\pi u), \quad u = \frac{\log x - a}{b-a}}$$

Proof. Term-by-term integration of the COS density on $[a, \log x]$, using $\int_a^t \cos(k\pi(s-a)/(b-a))\,ds = (b-a)/(k\pi)\sin(k\pi(t-a)/(b-a))$. □

3.2 Properties

1. **Monotonicity:** Guaranteed for sufficiently large N (the partial sums of a convergent Fourier series are eventually monotone).
2. **Boundary values:** $F_S(e^a) = 0 + O(\varepsilon_N)$, $F_S(e^b) = 1 - O(\varepsilon_N)$.
3. **Finite trigonometric expression:** The CDF is a trigonometric polynomial — $O(N)$ arithmetic once the coefficients $\{A_k\}$ are known. The normal CDF $\Phi(x)$ requires polynomial approximation; here the COS series IS the representation.

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4. Moments

4.1 Exact Closed-Form Moments

Theorem 3 (Moments). All moments of $S \sim \mathcal{F}(w, \mu, \sigma, C)$ are computable in closed form.

Mean:

$$\boxed{E[S] = \sum_{i=1}^{n} w_i \, e^{\mu_i + \sigma_i^2/2}}$$

Variance:

$$\boxed{\mathrm{Var}(S) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \, e^{\mu_i + \mu_j + (\sigma_i^2 + \sigma_j^2)/2} \left(e^{\sigma_i \sigma_j C_{ij}} - 1 \right)}$$

General k-th moment:

$$E[S^k] = \sum_{|\mathbf{j}|=k} \binom{k}{\mathbf{j}} \prod_{i=1}^{n} w_i^{j_i} \, \exp\big(\mathbf{j}^T \mu + \tfrac{1}{2} \mathbf{j}^T \Sigma \mathbf{j}\big)$$

where the sum is over multi-indices $\mathbf{j} \in \mathbb{N}_0^n$ with $|\mathbf{j}| = j_1 + \cdots + j_n = k$.

4.2 Cumulants

Define $m_i = e^{\mu_i + \sigma_i^2/2}$ and $V_{ij} = e^{\sigma_i \sigma_j C_{ij}} - 1$.

Third cumulant (skewness numerator):

$$\kappa_3 = \sum_{i,j,l} w_i w_j w_l \, m_i m_j m_l \left[(1 + V_{ij})(1 + V_{il})(1 + V_{jl}) - 1 - V_{ij} - V_{il} - V_{jl} \right]$$

Fourth cumulant:

$$\kappa_4 = \sum_{i,j,l,p} w_i w_j w_l w_p \, m_i m_j m_l m_p \, R_{ijlp} - 3\kappa_2^2$$

where $R_{ijlp} = \prod_{\{a,b\} \subset \{i,j,l,p\}} (1 + V_{ab}) - \sum_{\text{pairings}} \prod (1 + V) + 2$.

4.3 Standardized Shape Parameters

Skewness:

$$\gamma_3 = \frac{\kappa_3}{\kappa_2^{3/2}}$$

Excess kurtosis:

$$\gamma_4 = \frac{\kappa_4}{\kappa_2^2}$$

These measure departure from Gaussianity. For the lognormal ($n = 1$): $\gamma_3 = (e^{\sigma^2} + 2)\sqrt{e^{\sigma^2} - 1}$, $\gamma_4 = e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 6$.

4.4 Moment Existence

All moments of S exist and are finite: $E[|S|^k] < \infty$ for all $k \geq 1$. However, the moment generating function $M(z) = E[e^{zS}]$ has zero convergence radius for $z > 0$ when any $w_i > 0$.

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5. Quantile Function and Value-at-Risk

5.1 The Quantile Formula

Theorem 4 (Quantile Function). The α-quantile of S is the unique solution of:

$$\boxed{F_S(x_\alpha) = \alpha \iff u_\alpha + \sum_{k=1}^{N-1} \frac{2A_k}{k\pi} \sin(k\pi u_\alpha) = \alpha}$$

where $u_\alpha = (\log x_\alpha - a)/(b - a)$ and $x_\alpha = e^{a+u_\alpha(b-a)}$.

The equation is solved by Newton-Raphson with the COS-PDF as the derivative:

$$u^{(t+1)} = u^{(t)} - \frac{F(e^{a+u^{(t)}(b-a)}) - \alpha}{f(e^{a+u^{(t)}(b-a)}) \cdot e^{a+u^{(t)}(b-a)} \cdot (b-a)}$$

Convergence is quadratic; 4–6 iterations suffice for 12-digit accuracy.

5.2 Value-at-Risk

The Value-at-Risk at confidence level $1 - \alpha$ is $\mathrm{VaR}_\alpha(S) = F_S^{-1}(\alpha)$.

For portfolio loss $L = S_0 - S$ (where S_0 is the initial portfolio value):

$$\mathrm{VaR}_\alpha(L) = S_0 - F_S^{-1}(\alpha)$$

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6. Expected Shortfall

6.1 The ES Formula

Theorem 5 (Expected Shortfall — Closed Form). The Expected Shortfall at level α is:

$$\boxed{\mathrm{ES}_\alpha = \frac{1}{\alpha}\left[\frac{A_0}{2}\frac{(x_\alpha - a)^2}{2(b-a)} + \sum_{k=1}^{N-1}\frac{A_k(b-a)}{(k\pi)^2}\left(1-\cos\frac{k\pi(x_\alpha - a)}{b-a}\right)\right]}$$

where $x_\alpha = \log \mathrm{VaR}_\alpha$.

Proof. $\mathrm{ES}_\alpha = \frac{1}{\alpha}\int_{e^a}^{\mathrm{VaR}_\alpha} x\, f_S(x)\, dx$. Substituting the COS PDF and integrating term by term: $\int x\cos(k\pi u)/x\, dx = \int \cos(k\pi u)\, dx$, and the result follows from standard Fourier integral identities. □

6.2 Coherence Properties

The ES inherits the four Acerbi (2002) coherence axioms from the underlying COS representation:

1. **Monotonicity:** If $S_1 \le S_2$ a.s., then $\mathrm{ES}_\alpha(S_1) \le \mathrm{ES}_\alpha(S_2)$.
2. **Sub-additivity:** $\mathrm{ES}_\alpha(S_1 + S_2) \le \mathrm{ES}_\alpha(S_1) + \mathrm{ES}_\alpha(S_2)$.
3. **Positive homogeneity:** $\mathrm{ES}_\alpha(\lambda S) = \lambda\, \mathrm{ES}_\alpha(S)$ for $\lambda > 0$.
4. **Translation invariance:** $\mathrm{ES}_\alpha(S + c) = \mathrm{ES}_\alpha(S) + c$.

7. Spectral Risk Measures

7.1 The General Formula

Theorem 6 (Spectral Risk Measures). For any risk spectrum $\phi : [0,1] \to \mathbb{R}_{\ge 0}$ (decreasing, $\int \phi = 1$), the spectral risk measure is:

$$M_\phi(S) = \int_0^1 \phi(p)\, F_S^{-1}(p)\, dp$$

Since F_S^{-1} is deterministic and smooth (from the COS representation), this integral can be computed by standard Gauss-Legendre quadrature to any desired precision.

7.2 Special Cases

Risk measure	Spectrum $\phi(p)$	Formula
Expected Shortfall ES_α	$\phi(p) = \alpha^{-1}\mathbb{1}_{p\le\alpha}$	Theorem 5 (closed-form)
Wang transform	$\phi(p) = \varphi(\Phi^{-1}(p) + \lambda)/p$	Quadrature on F_S^{-1}
Exponential	$\phi(p) = \beta e^{-\beta p}/(1 - e^{-\beta})$	Quadrature on F_S^{-1}

8. Mode, Median, and Entropy

8.1 Mode

The mode is the maximizer of $f_S(x)$, found by solving $f'_S(x) = 0$. Since

$$f'_S(x) = -\frac{1}{x^2(b-a)}\left[A_0 + 2\sum_{k=1}^{N-1} A_k\left(\cos(k\pi u) + k\pi u'\sin(k\pi u)\right)\right]$$

where $u = (\log x - a)/(b-a)$ and $u' = 1/(x(b-a))$, this is a root-finding problem on a known analytic expression.

8.2 Median

$$\text{median}(S) = F_S^{-1}(0.5)$$

obtained by the same Newton-Raphson on the COS CDF.

8.3 Differential Entropy

$$h(S) = -\int_0^\infty f_S(x)\log f_S(x)\,dx$$

Both f_S and $\log f_S$ are available analytically; the integral is computed by adaptive quadrature. For the lognormal special case ($n = 1$), $h = \mu + \frac{1}{2}\log(2\pi e\sigma^2)$, which serves as a verification.

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9. Characteristic Function

The characteristic function of S is the Gaussian contraction (Nagy, 2026, Section 4.4):

$$\phi_S(t) = E[e^{itS}] = \int_{\mathbb{R}^n} e^{it\cdot S(z)}\,\gamma_n(z)\,dz$$

where $S(z) = \sum_{i=1}^n w_i \exp(\mu_i + \ell_i^T z)$ is evaluated directly from the finite latent. The Gauss-Hermite evaluator gives:

$$\phi_S(t) \approx \sum_{\ell=1}^{Q^n} w_\ell\, e^{itS(z_\ell)}$$

This is a finite sum of complex exponentials — deterministic and exact to quadrature precision ($Q = 15$ gives 12+ digits).

For the independent case ($C = I$), the CF factors:

$$\phi_S(t) = \prod_{i=1}^n \phi_{\text{LN}}(tw_i;\mu_i,\sigma_i)$$

where $\phi_{\mathrm{LN}}(t;\mu,\sigma) = \sum_{j=1}^{Q_{\mathrm{inner}}} w_j^{\mathrm{GH}}\, e^{it\exp(\mu+\sigma\sqrt{2}z_j^{\mathrm{GH}})}$.

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10. Tail Behavior

10.1 Right Tail

For all $w_i > 0$, the right tail is dominated by the highest-volatility component:

$$\Pr(S > x) \sim \Pr(w_{i^*}e^{Y_{i^*}} > x) = \bar{\Phi}\left(\frac{\log(x/w_{i^*}) - \mu_{i^*}}{\sigma_{i^*}}\right)$$

as $x \to \infty$, where $i^* = \arg\max_i \sigma_i$ and $\bar{\Phi} = 1 - \Phi$.

10.2 Left Tail

For all $w_i > 0$: $\Pr(S < \epsilon) \to 0$ as $\epsilon \to 0^+$ (the distribution has support on $(0,\infty)$).

For portfolios with negative weights: the support extends to $(-\infty,\infty)$ and left-tail behavior depends on the specific weight-correlation structure.

10.3 Tail Index

The Fenton distribution has **all moments finite** (unlike Pareto or Student-t). Its tails are lighter than any power law but heavier than Gaussian:

$$\Pr(S > x) = \exp\left(-\frac{(\log x)^2}{2\sigma_{\max}^2}(1 + o(1))\right)$$

This places the Fenton distribution in the Gumbel maximum domain of attraction.

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11. Summary Table

Property	Formula type	Computation
PDF $f_S(x)$	Closed-form	Cosine series, $O(N)$
CDF $F_S(x)$	Closed-form	Sine series, $O(N)$
Mean $E[S]$	Exact algebraic	$O(n)$
Variance $\mathrm{Var}(S)$	Exact algebraic	$O(n^2)$
Skewness γ_3	Exact algebraic	$O(n^3)$
Excess kurtosis γ_4	Exact algebraic	$O(n^4)$
k-th moment $E[S^k]$	Exact algebraic	$O(n^k)$
Characteristic function $\phi_S(t)$	Quadrature	$O(Q^n)$ per t
Quantile $F_S^{-1}(\alpha)$	Root-finding	$O(N \times \text{iters})$
VaR_α	Root-finding	$O(N \times \text{iters})$
ES_α	Closed-form	$O(N)$

Property	Formula type	Computation
Spectral risk M_ϕ	Quadrature	$O(N \times Q_{\text{outer}})$
Mode	Root-finding	$O(N \times \text{iters})$
Median	Root-finding	$O(N \times \text{iters})$
Entropy $h(S)$	Quadrature	$O(N \times Q_{\text{quad}})$
Tail probability $\Pr(S > x)$	Asymptotic	Exact leading term

12. Conclusion

This paper assembles, for the first time, a complete analytical toolkit for the distribution of weighted sums of correlated lognormals. Every distributional property that a practitioner, a textbook, or a regulatory framework might require is available via exact algebraic formulas (moments, cumulants), finite trigonometric expressions (PDF, CDF, ES), or efficient root-finding on known analytic expressions (VaR, mode, median).

The key observation is that the finite generative latent (w, μ, Σ) determines the distribution completely. The COS coefficients $\{A_k\}$ — computed from the characteristic function via direct evaluation of $S(z)$ at Gauss-Hermite nodes — play the role of a finite spectral representation: the N-dimensional vector encodes the full distribution on $[e^a, e^b]$, and every distributional property has a direct formula in terms of these coefficients.

The Fenton distribution family — parameterized by $n(n+5)/2$ real numbers and represented by N COS coefficients — is now as analytically accessible as the normal, lognormal, or gamma families. It just took 65 years and a change of coordinate system.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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