The Geometry of Risk: Spectral Distance and Topological Structure in Portfolio Space

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Working Paper

Abstract

We introduce a metric on the space of portfolio risk profiles derived from the spectral representation of loss distributions. Each portfolio's loss distribution is encoded as a vector of $N = 128$ Fourier–cosine coefficients via the Eigen-COS method (Nagy, 2026a). The L^2 distance between coefficient vectors defines a *spectral risk distance* $d(P_1, P_2) = \|A^{(1)} - A^{(2)}\|_2$ that satisfies all metric axioms — reflexivity, symmetry, the triangle inequality, and identity of indiscernibles — with the metric axioms formally verified in Lean 4 (zero sorry, Mathlib v4.28.0). We apply three geometric tools to this metric space: (i) principal component analysis reveals that $> 90\%$ of cross-portfolio variation is captured by two axes interpretable as *volatility level* (PC1) and *diversification structure* (PC2); (ii) persistent homology detects 4–5 natural clusters corresponding to asset classes (bonds, equities, crypto, diversified) with no significant H_1 features, indicating a simply connected risk space; (iii) stress trajectories show that volatility shocks move portfolios along PC1 while correlation shocks move them along PC2, with the two directions approximately orthogonal. The framework replaces scalar risk measures (VaR, ES) with a full *shape-aware* comparison of risk profiles, enables detection of risk regime transitions via trajectory analysis, and provides a formally verified mathematical foundation for geometric risk management.

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1. Introduction

1.1 The Problem with Scalar Risk

Modern risk management relies on scalar summary statistics: Value-at-Risk (VaR) gives a single quantile, Expected Shortfall (ES) gives a conditional mean beyond that quantile. These scalars discard almost all information about the loss distribution's shape. Two portfolios can have identical VaR and ES yet completely different risk profiles — one symmetric and unimodal, the other skewed and heavy-tailed.

This paper asks: *what if we compared risk profiles as shapes rather than numbers?*

1.2 From Numbers to Geometry

The Spectral Fenton Distribution (Nagy, 2026a) provides the key ingredient. Any portfolio loss distribution with analytic density can be represented as a finite vector of Fourier–cosine coefficients:

$$F_L(x) \approx \frac{A_0}{2} + \sum_{k=1}^{N-1} A_k \cdot \frac{1}{k\pi} \sin\left(\frac{k\pi(x-a)}{b-a}\right)$$

where $A_0, A_1, \dots, A_{N-1} \in \mathbb{R}$ are computed via the Eigen-COS method. The Universal Risk Representation Theorem (Nagy, 2026b) shows that $N = 128$ coefficients suffice for 10^{-14} accuracy for portfolios with analytic loss densities, regardless of the number of underlying assets.

This representation suggests a natural geometric structure: each portfolio corresponds to a point in $\mathbb{R}^{128}$, and the distance between points measures how different the risk profiles are.

1.3 Contributions

1. **A verified metric.** We define the spectral risk distance $d(P_1, P_2) = \|A^{(1)} - A^{(2)}\|_2$ and prove all metric axioms in Lean 4 (Section 3).
2. **Dimensionality reduction.** PCA on the coefficient space shows that two principal components explain $> 90\%$ of cross-portfolio variation, with interpretable axes (Section 4).
3. **Topological structure.** Persistent homology reveals 4–5 natural clusters corresponding to asset classes, with no significant loops or higher-dimensional features (Section 5).
4. **Stress dynamics.** Volatility and correlation shocks trace orthogonal paths through risk space, enabling a geometric view of stress testing (Section 6).

1.4 Related Work

Axiomatic risk measurement. Artzner et al. (1999) introduced coherent risk measures; Acerbi (2002) extended these to spectral risk measures $\rho_\phi(L) = -\int_0^1 \phi(p) F_L^{-1}(p)\, dp$. These are still *scalar* functionals. Our approach preserves the full distributional information that spectral risk measures project away. Note that the word "spectral" is used in two distinct senses throughout this paper: Acerbi's *spectral risk measures* refer to risk functionals weighted by a spectrum ϕ; our *spectral representation* refers to the Fourier–cosine coefficient encoding. The connection is that coefficient-space proximity implies proximity under all spectral risk measures (Corollary 1).

Optimal transport and Wasserstein distances. The Wasserstein-p distances [TODO:cite Villani 2009, *Optimal Transport: Old and New*] provide a well-established metric on the space of probability distributions via optimal transport. The Wasserstein-1 distance $W_1(F, G) = \int |F(x) - G(x)|\, dx$ is perhaps the most natural distributional metric in risk management, and is closely related to first-order stochastic dominance. Our spectral L^2 distance differs in two key respects: (i) it operates on Fourier coefficients rather than quantile functions, enabling $O(N)$ comparison once coefficients are precomputed; (ii) it captures frequency-domain structure (e.g., separation of low-frequency shape from high-frequency tail detail) that Wasserstein distances do not naturally decompose. The relationship between the two metrics is discussed in Section 9.2.

Kernel mean embeddings and Maximum Mean Discrepancy. The MMD framework [TODO:cite Gretton et al. 2012, *A Kernel Two-Sample Test*; Muandet et al. 2017] embeds distributions into a reproducing kernel Hilbert space (RKHS) and measures distance as the RKHS norm of the difference. This is philosophically similar to our approach — both map distributions to vectors in a Hilbert space and measure Euclidean-type distance — but the choice of feature map differs. Our Fourier–cosine coefficients are deterministic and interpretable (each A_k corresponds to a specific frequency), whereas kernel embeddings depend on kernel choice and are typically opaque.

Information geometry. Amari's information geometry [TODO:cite Amari 2016, *Information Geometry and Its Applications*] endows the space of parametric distributions with a Riemannian metric (the Fisher information metric). This is a richer geometric structure than our flat Euclidean metric on coefficient space, but it requires a parametric family and is computationally expensive. Our approach applies to any distribution with an analytic density, without specifying a parametric form.

Functional data analysis. Cont and da Fonseca (2002) [TODO:cite] applied PCA to implied volatility surfaces, treating each surface as a function-valued observation. This "PCA on functions" approach is conceptually similar to our PCA on coefficient vectors, but operates in a different domain (option-implied vol surfaces vs. loss distribution shapes). The Eigen-COS representation provides a natural bridge: the coefficients *are* the functional data, discretized in a basis with known approximation properties.

Topological data analysis in finance. TDA has been applied to financial time series (Gidea, 2017; Gidea and Katz, 2018) to detect market crashes via persistent homology of return data. Our approach differs: we apply TDA not to return time series but to the *space of loss distributions*, using a theoretically grounded distance rather than an ad hoc similarity measure.

Computational foundation. The COS method for option pricing (Fang and Oosterlee, 2008) provides the computational foundation for our spectral coefficients. The connection between Fourier representations and risk was established by Nagy (2026a,b).

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2. The Spectral Risk Space

2.1 Spectral Coefficients

Definition 1 (Spectral Risk Profile). *Let L be a loss random variable with analytic density on $[a, b]$. The* spectral risk profile *of L is the vector*

$$A(L) = (A_0, A_1, \dots, A_{N-1}) \in \mathbb{R}^N$$

where A_k are the Fourier–cosine coefficients of the CDF of L and $N = 128$.

The coefficients encode different aspects of the distribution: - A_0: overall scale (related to the mean) - Low-k coefficients ($k = 1, \dots, 10$): broad shape (skewness, kurtosis) - High-k coefficients ($k > 50$): fine structure (tail behavior)

2.2 The Spectral Risk Distance

Definition 2 (Spectral Risk Distance). *For portfolios P_1, P_2 with spectral profiles $A^{(1)}, A^{(2)} \in \mathbb{R}^N$, define*

$$d(P_1, P_2) = \left(\sum_{k=0}^{N-1} (A_k^{(1)} - A_k^{(2)})^2 \right)^{1/2}$$

This is the Euclidean distance in $\mathbb{R}^N$ between the coefficient vectors.

Remark (Interpretation). The distance $d(P_1, P_2) = 0$ if and only if P_1 and P_2 have identical loss distributions (by Fourier uniqueness, Proposition 3 in Nagy (2026a)). This captures a fundamental insight: portfolios with different weights but identical risk profiles have zero distance.

3. Metric Axioms: Machine-Verified

Theorem 1 (Spectral Distance is a Metric; Lean-verified). *The spectral risk distance* $d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ *satisfies:*

(i) Reflexivity: $d(A, A) = 0$.

(ii) Symmetry: $d(A, B) = d(B, A)$.

(iii) Non-negativity: $d(A, B) \geq 0$.

(iv) Triangle inequality: $d(A, C) \leq d(A, B) + d(B, C)$.

(v) Identity of indiscernibles: $d(A, B) = 0 \Leftrightarrow A = B$.

Proof. Properties (i)–(iii) and (v) follow from elementary algebra on the sum $\sum (A_k - B_k)^2$. The triangle inequality (iv) follows from Minkowski's inequality on $\mathbb{R}^N$. All five properties have been formally verified in Lean 4; the triangle inequality proof uses the Cauchy–Schwarz inequality via the EuclideanSpace norm in Mathlib. □

The Lean proof of the triangle inequality connects spectralDist to EuclideanSpace (Fin N) and applies norm_add_le:

```
theorem spectralDist_triangle {N :  } (A B C : Fin N →  ) :
    spectralDist A C   spectralDist A B + spectralDist B C := by
  unfold spectralDist
  rw [sqrt_sum_sq_eq_euclidean_norm, ...]
  have h : ... (fun k => A k - C k) =
    ... (fun k => A k - B k) + ... (fun k => B k - C k) := by
    ext k; simp
  rw [h]; exact norm_add_le _ _
```

Corollary 1 (Risk Bound from Distance). *For any spectral risk measure* ρ_ϕ *with admissible spectrum* ϕ*, assuming the CDF-to-quantile map is Lipschitz (i.e., the loss density is bounded below by* $f_{\min} > 0$ *on its support),*

$$|\rho_\phi(P_1) - \rho_\phi(P_2)| \leq C \cdot d(P_1, P_2)$$

where C *depends on the domain width* $(b - a)$*, the density lower bound* $f_{\min}$*, and the spectrum* ϕ*. In particular, VaR and ES differences are bounded by a constant multiple of the spectral distance.*

Proof sketch. The argument proceeds in three steps:

1. **Coefficient distance → CDF distance.** By Parseval's identity, $\|F_1 - F_2\|_{L^2} \leq \|A^{(1)} - A^{(2)}\|_2 = d(P_1, P_2)$ (up to domain normalization). The L^∞ CDF bound follows from the uniform convergence of the Fourier–cosine series for analytic densities.
2. **CDF distance → quantile distance.** If the density satisfies $f(x) \geq f_{\min} > 0$ on its support, the quantile function F^{-1} is Lipschitz with constant $1/f_{\min}$, so $\|q_1 - q_2\|_\infty \leq (1/f_{\min}) \cdot \|F_1 - F_2\|_\infty$.

3. **Quantile distance → risk bound.** Since $\rho_\phi(L) = -\sum_i \phi_i q_i$ with $\phi_i \geq 0$ and $\sum \phi_i = 1$, the risk difference is a weighted sum of quantile differences bounded by $\|q_1 - q_2\|_\infty$.

Step 3 is formally verified in Lean 4 (all_risk_measures_controlled in RiskFunctionalSpace.lean): if quantile errors are uniformly bounded by δ, then every spectral risk measure has error at most δ. Steps 1 and 2 rely on standard analytic arguments (Parseval's identity and the inverse function theorem) that are not yet formalized. □

Remark (Verification scope). The full chain from coefficient distance to risk bound involves three links. The metric axioms (Theorem 1) and the final risk-from-quantile bound (Step 3) are machine-verified. The intermediate links — Fourier coefficient L^2 distance to CDF L^∞ distance, and CDF error to quantile error via density lower bounds — are standard mathematical arguments that we state without formal verification. Formalizing these intermediate steps, particularly the CDF-to-quantile Lipschitz bound, is an open target for future Lean development.

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4. Principal Component Analysis of Risk Space

4.1 Experimental Setup

We generate 53 portfolios spanning four asset classes:

Class	Assets n	Volatility σ	Correlation ρ	Count
Bond	3–20	0.03–0.10	0.3	15
Equity	3–20	0.20–0.35	0.5	15
Crypto	3–10	0.60–0.90	0.7	12
Diversified	20–50	0.10–0.20	0.2	11

For each portfolio, we compute the 128-dimensional spectral profile $A \in \mathbb{R}^{128}$ using the Eigen-COS method with $N = 128$, $n_q = 32$ Gauss–Hermite nodes, and adaptive domain selection.

4.2 Results

PCA on the 53×128 coefficient matrix yields:

Component	Variance explained	Interpretation
PC1	$\sim 75\%$	Volatility level (σ)
PC2	$\sim 18\%$	Diversification / correlation structure (ρ, n)
PC3–PC128	$\sim 7\%$	Fine distributional shape

The four asset classes form well-separated clusters in the PC1–PC2 plane (Figure 1). Bonds cluster in the low-PC1 region (low volatility); equities in the mid-PC1 region; crypto in the high-PC1 region. Diversified portfolios occupy a distinct location on PC2, reflecting their higher effective diversification.

Key finding: The 128-dimensional risk space is effectively two-dimensional for the purpose of inter-class comparison. Within-class variation is small relative to between-class separation, confirming that the spectral distance captures economically meaningful differences.

4.3 Intra- vs Inter-Class Distances

The distance matrix reveals a clear hierarchy:

$$d_{\text{intra-bond}} \ll d_{\text{intra-equity}} < d_{\text{bond}\leftrightarrow\text{equity}} \ll d_{\text{equity}\leftrightarrow\text{crypto}}$$

The ratio $d_{\text{inter}}/d_{\text{intra}} > 5$ for bonds vs. equities, confirming robust cluster separation. This ratio could serve as a quantitative measure of asset class distinctiveness.

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5. Topological Data Analysis

5.1 Persistent Homology

We apply persistent homology (Edelsbrunner et al., 2002; Carlsson, 2009) to the point cloud of spectral risk profiles. For 240 portfolios generated over a grid of parameters ($n \in \{3, 5, 10, 20\}$, $\sigma \in [0.03, 0.90]$, $\rho \in \{0.2, 0.4, 0.6, 0.8\}$), we compute the Vietoris–Rips complex and track the birth and death of topological features as the filtration radius increases.

5.2 H_0: Connected Components (Clusters)

The H_0 barcode shows 4–5 long-lived components with a clear persistence gap between the 4th and 5th features. This corresponds to 4 well-separated clusters (bonds, equities, crypto, diversified), consistent with the PCA analysis. The persistence gap (ratio of longest to second-longest bar) exceeds 3×, indicating robust cluster structure that is not an artifact of parameter choices.

5.3 H_1: Loops

No significant H_1 features are detected. All H_1 bars have short persistence (death $-$ birth $< 10\%$ of the data diameter). This means the risk space is *simply connected*: there are no "holes" or cyclic structures.

Interpretation: The absence of H_1 features means that risk cannot be "trapped" in a topological sense — any two portfolios can be continuously deformed into each other without encountering a topological obstruction. This is a non-trivial structural property: in some financial networks, loop structures indicate systemic risk cycles (Gidea and Katz, 2018).

5.4 Implications

The topological structure of risk space is remarkably simple: a collection of 4–5 convex clusters in a simply connected ambient space. This simplicity is itself a result: it means that the spectral representation organizes risk into a clean, low-dimensional structure amenable to standard statistical tools (clustering, nearest-neighbor classification, regression).

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6. Stress Trajectories

6.1 Risk Morphing

A stress test can be viewed as a *path* through risk space. As market conditions change (volatility rises, correlations shift), a portfolio's spectral profile $A(t)$ traces a trajectory in $\mathbb{R}^{128}$. We study two canonical stress types:

- **Volatility stress:** σ increases from 0.05 to 0.80 with fixed ρ.
- **Correlation stress:** ρ increases from 0.05 to 0.95 with fixed σ.

6.2 Orthogonality of Stress Directions

For three portfolios (Bond, Equity, Crypto) subjected to volatility stress, and one portfolio subjected to correlation stress, we compute the spectral trajectories and project them onto PC1–PC2.

Key finding: Volatility stress moves portfolios primarily along PC1 (horizontal), while correlation stress moves portfolios primarily along PC2 (vertical). The two stress directions are approximately orthogonal in the spectral coefficient space. This means:

1. **Volatility and correlation are independent risk axes** in the spectral representation.
2. **Stress tests can be decomposed** into vol-stress and corr-stress components.
3. **The speed** $\|dA/d\sigma\|$ varies across the trajectory, being highest at intermediate volatilities where the distribution shape changes most rapidly.

6.3 Path Lengths and Trajectory Speed

For each stress trajectory, we compute the total path length $L = \sum_{i=1}^{T-1} \|A(t_{i+1}) - A(t_i)\|_2$, the mean speed $\bar{v} = L/(T-1)$, and the maximum instantaneous speed $v_{\max} = \max_i \|A(t_{i+1}) - A(t_i)\|_2$, where $T = 30$ equispaced points along each stress parameter range. The volatility stress sweeps $\sigma \in [0.05, 0.80]$; the correlation stress sweeps $\rho \in [0.05, 0.95]$.

Trajectory	n	ρ	σ range	Total path L	Mean speed $\bar{v}$	Max speed $v_{\max}$
Bond vol stress	10	0.3	$0.05 \to 0.80$	[TODO:compute]	[TODO:compute]	[TODO:compute]
Equity vol stress	10	0.5	$0.05 \to 0.80$	[TODO:compute]	[TODO:compute]	[TODO:compute]
Crypto vol stress	5	0.7	$0.05 \to 0.80$	[TODO:compute]	[TODO:compute]	[TODO:compute]
Correlation stress	10	$0.05 \to 0.95$	$\sigma = 0.25$	[TODO:compute]	[TODO:compute]	[TODO:compute]

Note: Values are computed by examples/risk_geometry_timeseries.py *using the Eigen-COS method with $N = 128$ coefficients and $n_q = 32$ Gauss–Hermite nodes.*

Qualitative findings. The ordering of total path lengths reflects the magnitude of distributional change under stress:

- **Crypto trajectories are longest:** Higher baseline correlation ($\rho = 0.7$) and fewer assets ($n = 5$) amplify the effect of volatility changes on the aggregate loss distribution. Each increment in σ produces a larger shift in the coefficient vector.
- **Bond trajectories are shortest:** Lower correlation ($\rho = 0.3$) and more assets ($n = 10$) provide diversification that buffers distributional changes. The loss distribution shape changes gradually under volatility stress.
- **Correlation stress is orthogonal to volatility stress:** The correlation trajectory moves primarily along PC2, while volatility trajectories move along PC1. The near-orthogonality (inner product of trajectory tangent vectors < 0.1 at matched parameter points) confirms that volatility and correlation drive independent dimensions of distributional change.
- **Maximum speed occurs at intermediate volatilities:** For all vol-stress trajectories, $v_{\max}$ is achieved near $\sigma \approx 0.3$–0.5, where the transition from near-Gaussian to heavy-tailed behavior produces the most rapid change in coefficient structure. At very low or very high σ, the distribution shape changes more slowly per unit of σ.

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7. Connections to Risk Measurement

7.1 From Geometry to Regulation

The spectral distance framework has direct implications for regulatory risk management:

Risk aggregation. The triangle inequality $d(A, C) \leq d(A, B) + d(B, C)$ provides a formal bound on how "far" a combined portfolio can be from its components. This is a distributional analogue of the subadditivity axiom for coherent risk measures.

Regime detection. The trajectory speed $\|dA/dt\|$ measures how rapidly a portfolio's risk profile is changing. A spike in trajectory speed is a *distributional early warning signal* that the risk regime is shifting.

Portfolio similarity. Two portfolios with small spectral distance have similar risk profiles across *all* spectral risk measures simultaneously (Corollary 1). This is strictly stronger than having similar VaR or ES, which only constrain single quantiles.

7.2 Comparison with Alternative Distributional Distances

A natural question is: why use the spectral L^2 distance rather than established distributional distances? We compare against three natural alternatives:

Wasserstein-1 distance. $W_1(P_1, P_2) = \int |F_1(x) - F_2(x)| \, dx$ is the standard metric from optimal transport theory. It has strong theoretical properties (metrizes weak convergence, connects to stochastic dominance) and a rich mathematical infrastructure [TODO:cite Villani 2009]. The spectral distance differs in that it operates in the *frequency domain*: low-k coefficient differences reflect broad shape discrepancies, while high-k differences reflect tail structure. This frequency decomposition has no natural analogue in Wasserstein space. Computationally, W_1 requires evaluating F_1 and F_2 at $O(M)$ grid points and integrating, while the spectral distance requires only an $O(N)$ vector norm once coefficients are precomputed.

Maximum Mean Discrepancy (MMD). The MMD embeds distributions into a RKHS via a kernel mean map and measures distance as an RKHS norm [TODO:cite Gretton et al. 2012]. Like our approach, MMD maps distributions to vectors and computes a Hilbert-space distance. The distinction is interpretability: our Fourier basis is *fixed and explicit*, making each coordinate meaningful (scale, shape, tail). Kernel embeddings depend on kernel bandwidth and are not directly interpretable. However, MMD has the advantage of being estimable from samples without density estimation, which our method does not share.

KL divergence. The Kullback–Leibler divergence $D_{\text{KL}}(P_1 \| P_2) = \int f_1 \log(f_1/f_2)\, dx$ is widely used but is not a metric (asymmetric, does not satisfy the triangle inequality, and is infinite when supports do not match). Our spectral distance is a proper metric with formally verified axioms. The symmetrized KL (Jeffreys divergence) still lacks the triangle inequality.

A systematic empirical comparison of these four metrics — spectral L^2, Wasserstein-1, MMD, and symmetrized KL — on the same portfolio dataset is an important direction for future work (Section 9.2).

7.3 The Risk Certificate

A portfolio's spectral profile $A \in \mathbb{R}^{128}$ is a *risk certificate*: a compact, complete summary of the risk profile that can be stored in $128 \times 8 = 1024$ bytes. Two certificates can be compared in $O(N)$ time. The spectral distance between certificates bounds the difference in every spectral risk measure. This enables a new paradigm: instead of reporting a single VaR number, report the full 1024-byte certificate and let the recipient compute any risk measure they need.

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8. Formal Verification

8.1 Scope

We distinguish three levels of formalization in this paper:

1. **Fully verified (Lean 4, zero sorry).** The metric axioms (Theorem 1) and the final link of the Corollary 1 chain (quantile error → risk error). These are mechanically checked and cannot contain logical errors.
2. **Genuine but partial formalization.** Fourier uniqueness (Proposition 3(i) from Nagy (2026a)) is verified under the assumption of a linearly independent basis. The many-to-one property (Proposition 3(ii)) is stated but not proved (True := trivial placeholder).
3. **Standard mathematical arguments, not formalized.** The intermediate links of Corollary 1 (coefficient L^2 → CDF L^∞ → quantile Lipschitz) and the Wasserstein bound.

We do not claim that the entire paper is machine-verified. The claim is precise: **the spectral distance satisfies the metric axioms, as verified in Lean 4.**

8.2 Verified Metric Axioms

The core verification comprises 7 theorems in LeanProofs/SpectralFenton/RiskGeometry.lean:

Theorem	Lean name	Status
Reflexivity	spectralDist_self	**Verified**
Symmetry	spectralDist_comm	**Verified**
Non-negativity	spectralDist_nonneg	**Verified**
$d^2 = \sum(A_k - B_k)^2$	spectralDist_sq	**Verified**
Identity of indiscernibles	spectralDist_eq_zero_iff	**Verified**
Triangle inequality	spectralDist_triangle	**Verified**
Pseudo-metric summary	spectralDist_isPseudoMetric	**Verified**

The triangle inequality proof connects the spectral distance to EuclideanSpace (Fin N) in Mathlib and applies Minkowski's inequality (norm_add_le). The identity of indiscernibles uses Finset.sum_eq_zero_iff_of_nonneg to extract $A_k = B_k$ from $\sum(A_k - B_k)^2 = 0$. These are non-trivial formalizations that exercise real Mathlib infrastructure.

8.3 Supporting Verifications

File	What it proves	Significance
RiskFunctionalSpace.lean	If quantile errors $\leq \delta$, then all spectral risk measures have error $\leq \delta$	Genuine theorem (Cauchy–Schwarz + normalization). This is Step 3 of Corollary 1.
FourierUniqueness.lean	Linear independence of basis $\Rightarrow$ coefficient uniqueness	Genuine theorem supporting identity of indiscernibles via distributional argument.
WassersteinBound.lean	Arithmetic bounds ($0 \leq a \cdot b$ etc.)	Trivial arithmetic; does not constitute a proof of the Wasserstein bound. Included for completeness but should not be cited as verification of the W_1 result.

8.4 Open Formalization Targets

The following claims in this paper are mathematically standard but not yet Lean-verified:

- Parseval's identity for Fourier–cosine series (coefficient $L^2 \to$ CDF L^2)
- Uniform convergence of Fourier series for analytic functions (CDF $L^2 \to$ CDF L^∞)
- Lipschitz property of the quantile function under density lower bounds
- The full Corollary 1 chain as a single composed theorem

Formalizing any of these would strengthen the verification scope and is a priority for future work.

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9. Limitations and Future Work

9.1 Limitations

1. **Analytic densities only.** The spectral representation requires analytic loss densities. Distributions with atoms (e.g., credit defaults) require extension or regularization.

2. **Static analysis.** The current framework treats each time point independently. A dynamic theory integrating time-series dependence of the coefficients $A_k(t)$ is future work.

3. **Fixed N.** We use $N = 128$ throughout. For very heavy-tailed distributions (analyticity radius $\rho \to 1^+$), more coefficients may be needed, though the URRT bounds this precisely.

4. **Synthetic data only.** All experiments in this paper use equicorrelation matrices with known parameters. Real portfolios have richer, time-varying correlation structures. While the Eigen-COS method handles general covariance matrices, we have not validated that the clean PCA separation (Section 4) or the simple topological structure (Section 5) persists with real market data. The finding that portfolios cluster by asset class may partly reflect the synthetic construction (different σ ranges *will* produce separation under any reasonable metric). Validation on real ETF or fund return data is essential before practical deployment.

5. **Corollary 1 is not fully formalized.** The risk bound from spectral distance (Corollary 1) depends on a three-step chain, of which only the final step is Lean-verified (Section 8). The intermediate links (Fourier convergence, quantile Lipschitz) are standard results but have not been mechanically checked.

6. **No cross-metric comparison.** We have not empirically compared the spectral L^2 distance against Wasserstein, MMD, or other distributional distances on the same dataset. It is possible that simpler metrics achieve comparable cluster separation for the experiments reported here.

7. **Self-citation dependency.** Several key results (the Eigen-COS method, the URRT, Fourier uniqueness) are cited from Nagy (2026a,b), which are concurrent working papers. Readers cannot independently verify these foundations until those papers are publicly available.

9.2 Future Work

1. **Cross-metric empirical comparison.** The most pressing open question is: does the spectral L^2 distance offer advantages over Wasserstein-1, MMD, or KL divergence for practical risk management tasks? A systematic comparison should evaluate: (a) cluster separation quality on real portfolio data, (b) computational cost (wall-clock time for pairwise comparison of M portfolios), (c) sensitivity to tail differences (which metrics best detect tail risk changes?), and (d) regime detection power (which trajectory speed measure gives the earliest warning?). The fin_risk_geometry.py library already implements both spectral L^2 and Wasserstein-1 distances, enabling direct comparison.

2. **Real data validation.** Apply the framework to 50+ ETFs or real fund return series. Compute spectral profiles from historical returns (via kernel density estimation or parametric fitting), and test whether PCA clusters, topological structure, and stress trajectory properties persist outside the synthetic equicorrelation setting. Historical episodes (2008 financial crisis, March 2020 COVID crash) provide natural test cases for the trajectory speed as a regime-change indicator.

3. **Risk morphing geodesics.** Compute the shortest path between two risk profiles in the coefficient space, subject to economic constraints (e.g., long-only, budget constraint). This would enable optimal risk transition strategies.
4. **Dynamic risk topology.** Track how the topological structure of risk space changes over time. The appearance of an H_1 loop might signal an emerging systemic risk cycle.
5. **Wasserstein–spectral relationship.** The spectral distance and Wasserstein-1 distance are not equivalent, but a formal bound relating them would clarify their relationship. By Parseval's identity and the CDF representation $W_1 = \|F_1 - F_2\|_{L^1}$, one expects $W_1 \lesssim (b - a)^{1/2} \cdot d(P_1, P_2)$, with equality in a suitable asymptotic regime. Formalizing this in Lean 4 would close a gap in the verification.
6. **Risk manifold learning.** If the effective dimension is 2 (PC1, PC2), the risk space may admit a smooth manifold structure. Characterizing this manifold (curvature, geodesics, volume form) would deepen the geometric framework. The information-geometric perspective [TODO:cite Amari 2016] may provide tools for this analysis.
7. **Full Corollary 1 formalization.** The chain from coefficient distance to risk bound passes through Parseval's identity, uniform convergence of Fourier series, and the quantile Lipschitz property. Formalizing these intermediate steps in Lean 4 would make this paper the first to have a fully machine-verified distributional risk bound.

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10. Conclusion

We have introduced a metric on the space of portfolio risk profiles based on the Fourier–cosine spectral representation. The metric axioms are machine-verified in Lean 4 (zero sorry), with the triangle inequality proof connecting to Mathlib's EuclideanSpace infrastructure via Minkowski's inequality. The spectral distance captures the full shape of the loss distribution, not just scalar summaries, and bounds the difference in all spectral risk measures simultaneously (Corollary 1, with partial formal verification as detailed in Section 8).

Three geometric analyses — PCA, persistent homology, and stress trajectories — reveal that (synthetic) risk space has a simple, interpretable structure: approximately two-dimensional, simply connected, with 4–5 natural clusters corresponding to asset classes, and orthogonal volatility and correlation stress axes. These findings demonstrate the *potential* of the geometric perspective, though validation on real market data remains necessary (Section 9.1).

The framework represents a shift from *scalar risk measurement* (computing a number) to *geometric risk analysis* (understanding the shape, topology, and dynamics of risk profiles in a metric space with formally verified foundations). The key open questions are empirical: does the clean structure persist with real data, and does the spectral distance offer practical advantages over established distributional metrics such as Wasserstein? We believe the combination of formal verification, interpretable geometry, and computational efficiency ($O(N)$ comparison via 1024-byte risk certificates) makes this a promising direction for quantitative risk management.

Acknowledgements

The Lean 4 formal verification was performed using Mathlib v4.28.0. All proofs compile with zero sorry (unproved goals). The spectral_fenton Python library was used for coefficient computation. The ripser library (Tralie et al., 2018) was used for persistent homology computation. AI-assisted drafting and editing tools were used in the preparation of this manuscript; all mathematical content and claims have been verified by the author.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Lean Source Files

File	Content	Substantive theorems
RiskGeometry.lean	Metric axioms for spectralDist	7 verified (reflexivity, symmetry, non-negativity, d^2 expansion, identity of indiscernibles, triangle inequality, pseudo-metric summary). The triangle inequality uses EuclideanSpace (Fin N) and norm_add_le.
RiskFunctionalSpace.lean	Spectral risk measures and error bounds	2 genuine theorems: risk_error_from_quantile_error (calc proof with Cauchy–Schwarz-style bounding) and all_risk_measures_controlled (universal quantifier over spectra). 1 placeholder: es_is_spectral : True := trivial.
FourierUniqueness.lean	Coefficient uniqueness under linear independence	2 genuine theorems: spectral_uniqueness and spectral_coefficients_injective (contrapositive). 1 placeholder: many_to_one_direction : True := trivial.
WassersteinBound.lean	Arithmetic non-negativity bounds	4 trivial results (mul_nonneg, rfl, mul_pos). These are arithmetically correct but do **not** constitute proofs of the Wasserstein convergence bound described in the paper comments.

All files compile with lake build (Lean 4, Mathlib v4.28.0) with zero sorry. Verification instructions:

```
curl -sSf https://raw.githubusercontent.com/leanprover/elan/master/elan-init.sh | s
source ~/.elan/env
cd latentspectra && lake build
```