The ATM Skew Power Law: A Machine-Verified Derivation from the rBergomi Model

125 theorems, zero axioms, sub-second Rust engine — from Volterra kernel to live SPX calibration

Every exponent identity proved by machine, then deployed to price real options

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Executive Summary (Non-Technical)

Options markets exhibit a persistent asymmetry: put options cost more than equivalent calls. This asymmetry — the volatility skew — steepens as the option maturity shrinks. Classical stochastic volatility models like Heston predict a skew that flattens at short maturities, contradicting market data. Rough volatility models, where the Hurst parameter $H \approx 0.1$ replaces the classical $H = 1/2$, reproduce the observed steepening. But the derivation connecting the model definition to the skew formula passes through stochastic calculus, Malliavin derivatives, and Volterra integrals — a chain where algebraic errors are easy to make and hard to detect.

This paper presents a fully machine-verified derivation of the ATM skew power law $\psi(T) \sim C \cdot T^{H-1/2}$ from the rBergomi model, extended to hedging error bounds, pricing error analysis, and cross-model divergence under model misspecification. The derivation chain — from the Volterra kernel $K(t,s) = c_H(t-s)^{H-1/2}$ through the Malliavin derivative, the Alòs–León–Vives decomposition, the power integral rule, and the normalization step — is formalized as 125 theorems with zero axioms. Every exponent identity, every positivity condition, and every inequality is verified by the Platonic proof kernel. The stochastic calculus results (Malliavin derivative, Alòs formula) are cited as published facts; the algebraic backbone that connects them is proved.

The most practically significant results: **(1)** when a trader hedges with Heston ($H = 1/2$) but the true process has $H < 1/2$, the hedging error variance scales as T^{2H-1} — diverging at short maturities; **(2)** delta-vega hedging does not help — both errors share the exponent $2H - 1$; **(3)** the observable ATM skew predicts 75% of the hedge error variance across maturities ($R^2 = 0.75$ on real SPX data). Empirical validation using VIX data (2021–2026) yields $H = 0.35$ from the variogram and $H = 0.30$ from the skew power law — both firmly in the rough regime. Rolling estimation over 16 quarterly windows confirms H is stable (CV = 9.8%, always < 0.5). In calm markets (VIX < 20), the theoretical scaling $\mathrm{Var}(\mathrm{P\&L}) \sim T^{2H-1}$ is directly observed ($R^2 = 0.76$). A practical risk capital formula follows: short-maturity options require 1.9× more capital than the $\sqrt{T}$ rule predicts.

Beyond verification, the theoretical results are implemented as a production-grade computation engine. A Rust-native rough Heston pricer — solving the fractional Riccati equation via Adams discretization with precomputed Volterra kernel weights and recovering prices via the COS method with batch characteristic function evaluation — calibrates a full 60-quote implied volatility surface in 1.2 seconds (cold-start) or 0.85 seconds (warm-start). The calibration pipeline combines compact

differential evolution (global search) with a Gauss-Newton Levenberg-Marquardt optimizer featuring parallel Jacobian computation, achieving 0.5 bps RMSE on self-consistent data and 4–6 bps on noisy market surfaces. In a controlled head-to-head comparison against SABR-style per-slice quadratic fits on a realistic surface (45 quotes, 5 maturities, 5 bps bid-ask noise), the 8-parameter rough Heston engine wins 4:1 in-sample (5.7 vs 6.1 bps average) with half the parameters (8 vs 15), and delivers 6.5× better out-of-sample extrapolation (13.6 vs 88.5 bps at unseen maturities).

What this paper does not claim: it does not prove the stochastic calculus foundations. It does not provide a new model. It does not claim rough Heston universally outperforms SABR on per-slice fit — with dedicated per-maturity parameters, SABR can match any individual smile. It proves that the exponent arithmetic connecting the rBergomi model to the skew power law and hedging error bounds is correct, establishes a complete exponent hierarchy for all observable quantities, and shows that a production engine built on these results provides structural advantages (arbitrage freedom, extrapolation, consistent dynamics) that complement existing SABR infrastructure. The engine calibrates in under 2 seconds on commodity hardware, making it viable for intraday recalibration.

Abstract

We derive the ATM implied volatility skew power law $\psi(T) \sim C \cdot T^{H-1/2}$ from the rough Bergomi (rBergomi) model through a machine-verified chain of 125 theorems. The derivation proceeds in seven stages: (i) the Volterra kernel $K(t,s) = c_H(t-s)^{H-1/2}$ is analyzed for singularity and integrability; (ii) the power integral $\int_0^T u^{H-1/2}\,du = T^{H+1/2}/(H+1/2)$ is formalized with typed rpow expressions; (iii) the Alòs–León–Vives normalization by T yields the skew exponent identity; (iv) delta-hedging error under model misspecification scales as T^{2H-1}, diverging at short maturities; (v) the vega correction shares the same exponent, proving that delta-vega hedging cannot reduce model error; (vi) the optimal rebalancing count scales as $T^{2(1-H)}$; (vii) the observed smile curvature provides a lower bound on the hedging error coefficient. A complete exponent hierarchy is established: forward slope < hedge error < skew < 0. The formalization extends to variance swap term structure, calendar spread no-arbitrage constraints, VaR scaling, a complete exponent atlas showing all observables are affine functions of a single parameter $\alpha = H - 1/2$, the short-time smile expansion geometry, pricing error bounds under H misspecification, cross-model divergence rates, and the fractional Riccati structure of the rough Heston characteristic function. Empirical validation on real SPX/VIX data (2021–2026) yields $H = 0.35$ (variogram) and $H = 0.30$ (skew fit), and confirms that the ATM skew predicts 75% of cross-maturity hedge error variance. Rolling H estimation over 16 quarterly windows shows remarkable stability (CV = 9.8%, range $[0.30, 0.42]$, always below 1/2). Sub-period analysis separates regimes: in calm markets (VIX < 20), the model-misspecification scaling $\mathrm{Var}(\mathrm{P\&L}) \sim T^{-0.26}$ is directly observed ($R^2 = 0.76$), while in volatile markets, discrete hedging error dominates. A practical risk capital formula shows short-maturity options require 1.9× more capital than the $\sqrt{T}$ rule predicts. All 125 theorems are verified by the Platonic proof kernel with zero axioms; stochastic calculus results are cited as published facts. The theoretical framework is implemented as a production computation engine: a Rust-native rough Heston pricer solving the fractional Riccati equation via Adams discretization with precomputed Volterra kernel weights, batch characteristic function evaluation, and COS-method pricing at 11,600 surface points/second. A three-stage calibration pipeline — compact differential evolution (global search), Nelder-Mead simplex, and Gauss-Newton Levenberg-Marquardt with parallel Jacobian computation — calibrates an 8-parameter model to a 60-quote surface in 1.2 seconds (cold-start) or 0.85 seconds (warm-start). In a controlled head-to-head comparison against SABR-style per-slice quadratic fits on a realistic 45-quote surface with 5 bps noise, the 8-parameter rough Heston model wins 4:1 on in-sample fit (5.7 vs 6.1 bps RMSE) with half the parameters, and delivers 6.5× better out-of-sample extrapolation (13.6 vs 88.5 bps at unseen maturities). Live calibration to SPY options (April 2026, 146 quotes, 4 maturities) achieves 75 bps RMSE with 8 parameters, beating SABR at the 1-month tenor (101 vs 132 bps) but losing at medium/long maturities where per-slice calibration with dedicated parameters dominates. The structural advantages — arbitrage-free extrapolation, consistent dynamics, and unified Greeks — complement rather than replace existing SABR infrastructure. A SABR enhancement layer provides drop-in hedging corrections requiring only one additional parameter. Monte Carlo simulation extends to barrier, Asian, and variance swap pricing.

Keywords: rough volatility, ATM skew, Hurst exponent, rBergomi model, Volterra kernel, machine verification, power law

MSC 2020: 91G20 (Derivative securities), 60G22 (Fractional processes), 91G60 (Numerical methods)

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1. Introduction

1.1 The Problem

The implied volatility surface of equity index options displays a characteristic pattern: at-the-money (ATM) skew steepens as maturity $T \to 0$. Specifically, the ATM skew — defined as $\psi(T) = \partial\sigma_{\text{BS}}/\partial k\big|_{k=0}$ where k is log-moneyness — follows a power law

$$\psi(T) \sim C \cdot T^{H-1/2}$$

with $H \approx 0.1$, empirically measured across equity indices (Gatheral, Jaisson, and Rosenbaum, 2018). Since $H - 1/2 \approx -0.4 < 0$, the skew diverges as $T \to 0$ — a behavior that classical stochastic volatility models cannot produce. The Heston model corresponds to $H = 1/2$, giving exponent 0: a flat short-maturity skew.

This discrepancy motivated the rough volatility program (Gatheral, Jaisson, and Rosenbaum, 2018; Bayer, Friz, and Gatheral, 2016): replace the standard Brownian motion driving variance with a fractional Brownian motion of Hurst index $H < 1/2$. The resulting models — rBergomi, rough Heston — reproduce the power-law skew, but the derivation connecting model parameters to the skew formula involves:

1. The Volterra kernel $K(t, s) = c_H(t - s)^{H-1/2}$ and its integral properties.
2. Malliavin calculus to express the sensitivity of variance to the driving noise (El Euch and Rosenbaum, 2019).
3. The Alòs–León–Vives decomposition linking the Malliavin derivative to the skew (Alòs, León, and Vives, 2007).
4. A chain of exponent identities and positivity conditions connecting these results.

Steps (2) and (3) are deep stochastic calculus; the proofs are published and accepted. Step (4) is algebra and real analysis — straightforward in principle, but involving enough moving parts that errors are plausible and consequential. An incorrect exponent identity invalidates the entire derivation.

1.2 This Paper's Contribution

We provide a machine-verified derivation of the ATM skew power law from the rBergomi model definition. The contribution has four parts:

1. **The derivation chain is fully formalized.** Starting from the model parameters $(H, \eta, \rho, \sigma_0, c_H)$, the derivation proceeds through 125 theorems covering the Volterra kernel analysis, the power integral rule, the skew power law, hedging error bounds (delta, vega, rebalancing), the smile curvature predictor, variance swap term structure, calendar spread constraints, VaR scaling, the complete exponent atlas, short-time smile expansion geometry, pricing error under H misspecification, cross-model divergence, the fractional Riccati structure, moment scaling, forward variance slope, convergence rates, classical model comparisons, the Hawkes microstructure bridge, and the estimator bias debate. Zero axioms are used; stochastic calculus results are cited as facts from the referenced literature.

2. **The exponent arithmetic is independently auditable.** Each theorem has a precise statement, an explicit proof strategy (ring algebra, linear arithmetic, or nonlinear arithmetic), and a machine-checked verification. A reader who doubts any step can inspect the proof file and re-run it.

3. **The formalization reveals structural relationships and quantifies model risk.** The hedging error variance exponent $2H-1$ is identical to the smile curvature exponent (Theorem 52) — both arise from the squared skew effect. The complete exponent hierarchy (Theorem 59: forward slope < hedge error < skew < 0) provides a single diagnostic. Delta-vega hedging does not help (Theorem 66): both error terms share the same exponent, so adding vega hedging cannot reduce the model-misspecification floor.

4. **The smile curvature predicts hedging risk** (Theorems 76–80). The observed curvature coefficient C_{curv} provides a lower bound $C_{\text{hedge}} \geq (\rho^2/4) \cdot C_{\text{curv}}$ on the hedging error coefficient. A risk manager can read hedging risk directly from the options market, bypassing the Hurst parameter estimation debate entirely.

1.3 What This Paper Does Not Claim

This paper does not prove the Malliavin derivative formula (El Euch and Rosenbaum, 2019, Proposition 3.1) or the Alòs–León–Vives decomposition (Alòs, León, and Vives, 2007, Theorem 2.1). These are deep stochastic calculus results with published proofs. We cite them as facts and prove everything downstream.

This paper does not propose a new volatility model. The rBergomi model is due to Bayer, Friz, and Gatheral (2016). We formalize the existing derivation, not a new one.

This paper does not address the debate on whether volatility IS rough in the statistical sense (Cont and Das, 2022; Fukasawa, 2021). Section 5 formalizes the estimator bias analysis — the algebraic constraint on how large the bias can be — but takes no position on the empirical question.

1.4 Organization

Section 2 sets up the rBergomi model and notation. Section 3 presents the main derivation chain from model definition to skew power law. Section 4 extends to hedging error bounds (delta, vega, rebalancing), the smile curvature predictor, variance swap term structure, calendar spread constraints, VaR scaling, the complete exponent atlas, short-time smile geometry, pricing error under H misspecification, cross-model divergence, and the fractional Riccati structure. Section 5 provides empirical validation on real market data, including temporal stability analysis, sub-period robustness tests, and a practical risk capital formula. Section 6 addresses the roughness debate. Section 7 describes the formalization architecture. Section 8 discusses implications, including the model risk measure and the Latent framework connection. Section 9 describes the production RoughVol engine (implementation, optimization, calibration, controlled SABR comparison, and live-market benchmarks).

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2. Setup: The rBergomi Model

2.1 Model Definition

The rough Bergomi model (Bayer, Friz, and Gatheral, 2016) specifies the spot variance process as

$$\sigma^2(t) = \xi_0 \exp\Big(\eta W_t^H - \tfrac{1}{2}\eta^2 t^{2H}\Big)$$

where W^H is a fractional Brownian motion with Hurst index $H \in (0, 1/2)$, $\eta > 0$ is the vol-of-vol, and $\xi_0 > 0$ is the initial forward variance. The log-variance is

$$\log \sigma^2(t) = \log \xi_0 + \eta W_t^H - \tfrac{1}{2}\eta^2 t^{2H}$$

The fractional Brownian motion has the Volterra representation

$$W_t^H = \int_0^t K(t,s)\, dW_s, \qquad K(t,s) = c_H (t-s)^{H-1/2}$$

where W is a standard Brownian motion, $c_H > 0$ is a normalization constant ensuring $\mathrm{Var}(W_t^H) = t^{2H}$, and the kernel exponent $H - 1/2$ governs the singularity structure.

2.2 Model Parameters

We work with the following standing hypotheses:

Parameter	Symbol	Constraint	Role
Hurst index	H	$0 < H < 1/2$	Roughness parameter
Vol-of-vol	η	$\eta > 0$	Amplitude of variance fluctuations
Spot volatility	σ_0	$\sigma_0 > 0$	$\sigma_0 = \sqrt{\xi_0}$
Kernel constant	c_H	$c_H > 0$	Volterra normalization

Parameter	Symbol	Constraint	Role
Leverage	ρ	$-1 \leq \rho < 0$	Correlation between price and variance
Maturity	T	$T > 0$	Option expiry

The constraint $H < 1/2$ is the "rough regime." The classical Heston model corresponds to $H = 1/2$, where the fractional Brownian motion degenerates to a standard Brownian motion and the kernel becomes a constant.

2.3 Notation

We write $\mathrm{rpow}(x, r) = x^r$ for the real power function, well-defined for $x > 0$ and arbitrary $r \in \mathbb{R}$. The set integral $\int_0^T f(u)\,du$ is denoted $\mathrm{SetIntegral}(f, 0, T)$ in the formalization.

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3. The Derivation Chain

This section presents the main derivation: from the rBergomi model definition to the ATM skew power law $\psi(T) \sim C \cdot T^{H-1/2}$. The chain has five links, each corresponding to a group of verified theorems.

3.1 Link 1: Kernel Exponent Analysis (Theorems 6–10)

The Volterra kernel $K(t, s) = c_H (t - s)^{H-1/2}$ has exponent $H - 1/2$. Two properties are essential:

Theorem 6 (Singular kernel). *If* $H < 1/2$*, then* $H - 1/2 < 0$*.* The kernel diverges as $s \to t$.

Theorem 7 (Integrable kernel). *If* $H > 0$*, then* $H - 1/2 > -1$*.* The singularity is integrable.

Together: the kernel is singular but integrable, a hallmark of rough processes. This is why fractional Brownian motion with $H < 1/2$ has rougher sample paths than standard Brownian motion but remains well-defined as a stochastic integral.

The integral exponent $H + 1/2$ is positive (Theorem 8) and sublinear (Theorem 9: $H + 1/2 < 1$ when $H < 1/2$). The rough regime also implies $2H < 1$ (Theorem 10), which will be needed for the curvature and forward variance analysis.

3.2 Link 2: The Volterra Integral (Theorems 11–14)

The power integral

$$\int_0^T u^{H-1/2}\,du = \frac{T^{H+1/2}}{H + 1/2}$$

is the classical calculus step connecting the kernel to the skew. We formalize it with typed rpow expressions and verify its prerequisites:

Theorem 11. $H > 0 \implies H - 1/2 > -1$*.* The exponent satisfies the power rule's integrability condition.

Theorem 12. $(H - 1/2) + 1 = H + 1/2$. The exponent shift identity connecting kernel exponent to integral exponent.

Theorem 13. $T > 0 \implies \mathit{rpow}(T, H + 1/2) > 0$. The numerator is positive, via the rpow_pos_of_pos property.

Theorem 14. $H > 0, T > 0 \implies T^{H+1/2}/(H + 1/2) > 0$. The full integral result is positive, combining numerator positivity with denominator positivity ($H + 1/2 > 0$).

The integral rule itself is cited as a fact (classical calculus). What is proved is that the prerequisites are met and the result is well-defined and positive — conditions that are easy to assume implicitly but can fail at boundary cases.

3.3 Link 3: The Stochastic Calculus Bridge (Facts)

Two results from the literature connect the Volterra integral to the implied volatility surface:

Fact (Malliavin derivative). *In the rBergomi model, the Malliavin derivative of the log-variance satisfies* $D_s(\log \sigma_t^2) = \eta \cdot c_H \cdot (t - s)^{H-1/2}$.

This is El Euch and Rosenbaum (2019), Proposition 3.1. It expresses the infinitesimal sensitivity of variance at time t to the noise at time s — the causal influence of W_s on σ_t^2.

Fact (Alòs–León–Vives decomposition). *The ATM skew satisfies* $\psi(T) = \frac{\rho}{2\sigma_0 T} \cdot E\left[\int_0^T D_t\sigma_t^2 \cdot g(t,T)\, dt\right]$, *where* $g(t,T)$ *is a weight function with leading order* $g(t,T) \sim T - t$.

This is Alòs, León, and Vives (2007), Theorem 2.1. The key feature is the division by T in front of the integral — this normalization step is what creates the skew exponent.

3.4 Link 4: The Exponent Algebra (Theorems 1–5, 15–17)

Combining the Volterra integral with the Alòs normalization:

$$\psi(T) \sim \text{const} \cdot \frac{T^{H+1/2}}{T} = \text{const} \cdot T^{(H+1/2)-1}$$

The exponent identity $(H+1/2)-1 = H-1/2$ (Theorem 1, proved by ring algebra) gives the skew power law.

Theorem 15 (Skew power law). *The skew exponent equals* $H - 1/2$. This is the central result, derived from the integral exponent fact and the Alòs normalization fact via linear arithmetic.

Theorem 16 (Skew divergence). *In the rough regime* ($H < 1/2$)*, the skew exponent is negative.* Since $T^{H-1/2} \to \infty$ as $T \to 0$ when $H - 1/2 < 0$, the ATM skew diverges at short maturities.

Theorem 17 (Skew constant finite). $H + 1/2 > 0$, *so the proportionality constant in the power law is finite.*

3.5 Link 5: Classical Model Comparison (Theorems 18–20)

Theorem 18 (Rough steeper than Heston). $H - 1/2 < 0$ *in the rough regime, while the Heston exponent is* 0. Rough models produce steeper short-maturity skew than classical models.

Theorem 19 (Monotonicity). $H_1 < H_2 \implies H_1 - 1/2 < H_2 - 1/2$. Smaller Hurst index means steeper skew. Rougher volatility implies more extreme short-maturity behavior.

Theorem 20 (Empirical case). $H = 1/10 \implies H - 1/2 = -2/5$. The empirically estimated $H \approx 0.1$ (Gatheral, Jaisson, and Rosenbaum, 2018) predicts a skew exponent of -0.4, consistent with market observations.

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4. Extended Results

4.1 Smile Curvature (Theorems 21–23)

The second-order term in the implied volatility expansion — the smile curvature — scales as T^{2H-1} (Fukasawa, 2011).

Theorem 21 (Curvature power law). *The curvature exponent is* $2H - 1$*.*

Theorem 22 (Curvature negative). $2H - 1 < 0$ *in the rough regime.* The smile flattens at short maturities — but less aggressively than the skew steepens.

Theorem 23 (Curvature = 2 × skew). *The curvature exponent* $2H - 1$ *equals twice the skew exponent* $H - 1/2$*.* This structural relationship is a consequence of the quadratic nature of the smile expansion and is verified through the full chain of intermediate facts.

4.2 Moment Scaling (Theorems 35–41)

The n-th moment of the log-variance process scales as

$$E\big[(\log \sigma^2(t))^{2n}\big] \sim C_n \cdot t^{2Hn}$$

because $W_t^H \sim \mathcal{N}(0, t^{2H})$ and Gaussian moments scale with the variance exponent.

Theorem 36 (Sublinear scaling). *If* $H < 1/2$ *and* $n > 0$*, then* $2Hn < n$*.* Moments grow sublinearly in the moment order — a signature of rough processes. In a classical model ($H = 1/2$), $2Hn = n$: linear scaling.

Theorem 39 (Constant moment step). $2H(n+1) - 2Hn = 2H$. Consecutive moment exponents differ by exactly $2H$. This means the moment sequence $\{2Hn\}_{n \geq 1}$ is an arithmetic progression with common difference $2H$ — a simple structural fact, but one that constrains any statistical estimator of H from moment data.

Theorem 41 (Kurtosis T-independent). $4H - 2 \cdot 2H = 0$. The kurtosis exponent vanishes: the log-normal kurtosis does not depend on T, regardless of H. This is because the fourth-moment exponent $4H$ is exactly twice the variance exponent $2H$, and kurtosis is the ratio of fourth moment to squared variance.

4.3 Forward Variance Slope (Theorems 42–47)

The forward variance curve $\xi(T) = E[\sigma_T^2 \mid \mathcal{F}_0]$ satisfies

$$\left.\frac{d\xi}{dT}\right|_{T\to 0} \sim T^{2H-2}$$

Theorem 43 (Slope singular). *If $H < 1/2$, then $2H - 2 < -1$.* The forward variance slope diverges faster than $1/T$ at short maturities — an infinite initial slope.

Theorem 45 (Rougher than Heston). *If $H < 1/2$, the slope exponent $2H - 2$ is strictly less than the Heston slope exponent $2(1/2) - 2 = -1$.* Rough models produce wilder forward variance curves near $T = 0$ than classical models.

Theorem 47 (Empirical case). $H = 1/10 \implies 2H - 2 = -9/5$. An exponent of -1.8 — nearly twice as singular as Heston's -1.

4.4 Discretization Convergence (Theorems 24–26)

Hybrid simulation schemes for the rBergomi model (Bayer, Friz, and Gatheral, 2016, Theorem 4.3) converge at rate $O(N^{-(H+1/2)})$.

Theorem 24. $H + 1/2 > 1/2$. The convergence rate is better than Monte Carlo's $O(N^{-1/2})$.

Theorem 25. $H + 1/2 < 1$. The convergence rate is slower than Heston's $O(N^{-1})$.

Theorem 26 (Deficit identity). $H+1/2 = 1-(1/2-H)$. The convergence deficit from roughness is exactly $1/2 - H$. The rougher the process, the slower the simulation converges — a quantitative cost of rough volatility.

4.5 Hawkes Microstructure Bridge (Theorems 27–30)

El Euch and Rosenbaum (2018) showed that near-critical Hawkes processes converge in scaling limit to rough volatility models. The Hawkes kernel power-law exponent α maps to the Hurst parameter via $H = \alpha - 1/2$.

Theorem 27. $\alpha > 1/2 \implies H = \alpha - 1/2 > 0$. The mapped Hurst parameter is positive.

Theorem 28. $\alpha < 1 \implies H < 1/2$. The mapped Hurst parameter is in the rough regime.

Theorem 29 (Near-critical is subcritical). *If $n = 1 - \varepsilon$ with $\varepsilon > 0$, then $n < 1$.* The branching ratio is strictly subcritical.

These theorems verify that the Hawkes-to-rBergomi correspondence preserves the rough regime constraints. The microstructure foundation provides an economic mechanism for roughness: near-critical self-exciting order flow generates rough volatility in the scaling limit.

4.6 Hedging Error Bounds (Theorems 48–62)

This is the central practical result: when a trader uses a misspecified model (Heston, $H = 1/2$) to hedge an option on an underlying whose true volatility is rough ($H < 1/2$), the hedging error has a quantifiable lower bound that diverges at short maturities.

4.6.1 The Delta Mismatch

The ATM delta correction — the difference between the true (rough) delta and the Black-Scholes delta — scales with the same exponent as the skew (El Euch, Fukasawa, and Rosenbaum, 2018; Fukasawa, 2011):

$$\Delta_{\text{rough}}(T) - \Delta_{\text{BS}}(T) \sim C_\Delta \cdot T^{H-1/2}$$

The Heston delta correction exponent is 0 (since $H = 1/2$ gives $1/2 - 1/2 = 0$). The mismatch between the rough delta and the Heston delta therefore scales as $T^{H-1/2}$.

Theorem 48 (Delta correction negative). *In the rough regime, the delta correction exponent* $H - 1/2 < 0$*.* The correction grows as maturity shrinks.

4.6.2 The Hedging Error Variance

The P&L variance from delta-hedging with a misspecified model scales as the square of the delta mismatch:

$$\text{Var}(\text{P\&L}_{\text{hedge}}) \sim C \cdot T^{2(H-1/2)} = C \cdot T^{2H-1}$$

Theorem 49 (Hedging error = curvature exponent). *The hedging error variance exponent is* $2H - 1$*.* This is identical to the smile curvature exponent — a structural coincidence that reveals a deep connection: the curvature of the smile and the variance of the hedging error are governed by the same quantity.

Theorem 50 (Divergence). *If* $H < 1/2$*, the hedging error exponent* $2H - 1 < 0$*.* The hedging error diverges as $T \to 0$. Short-dated options are the hardest to hedge with the wrong model.

Theorem 52 (Structural identity). *The hedging error exponent equals the curvature exponent.* This is not a coincidence: both arise from the squared skew exponent. The curvature measures the second-order shape of the smile; the hedging error measures the second-order consequence of the first-order delta mismatch.

4.6.3 The Exponent Hierarchy

The rough regime produces a strict ordering of singularity exponents:

$$\underbrace{2H-2}_{\text{fwd slope}} < \underbrace{2H-1}_{\text{hedge error}} < \underbrace{H-1/2}_{\text{skew}} < 0$$

Theorem 59 (Hierarchy). *For* $0 < H < 1/2$*:* $2H - 2 < 2H - 1$*.* The forward variance slope diverges faster than the hedging error, which diverges faster than the skew (in absolute terms, the skew exponent is less negative).

Theorem 58 (Hedge error more singular than skew). *If* $H < 1/2$*, then* $2H - 1 < H - 1/2$*.* The hedging error is more singular than the skew — a practitioner who sees a manageable skew may underestimate how bad the hedging error will be at the same maturity.

4.6.4 Practical Quantification

Theorem 56 (Empirical case). $H = 1/10 \implies 2H - 1 = -4/5$. The hedging error exponent is -0.8. For a 1-week option ($T = 1/52$ year) versus $T = 1$ year, $\mathrm{Var}(\mathrm{P\&L}_{\mathrm{week}})/\mathrm{Var}(\mathrm{P\&L}_{\mathrm{year}}) \approx (1/52)^{2H-1} = (1/52)^{-4/5} = 52^{4/5} \approx 28$, so short-maturity hedge-error variance is about $28\times$ **larger** than at one year (equivalently, the one-year variance is about 1/28 of the one-week variance).

Theorem 61 (Magnification exponent). $1 - 2 \cdot (1/10) = 4/5$. The magnification exponent $|2H - 1| = 4/5$ governs how quickly the hedge error grows as maturity shrinks. At $H = 0.1$, every halving of maturity increases the hedge error by a factor of $2^{0.8} \approx 1.74$.

Theorem 62 (Model risk measure). *The gap between rough and Heston hedge error exponents is* $2H - 1$*.* This quantity — the model risk measure — is directly computable from the estimated Hurst parameter and tells a risk manager exactly how much model risk they carry by using a classical model.

4.7 Vega Hedging Error (Theorems 63–67)

The vega correction — the mismatch between rough and Heston vega — involves the integrated forward variance, which scales as T^{2H-1}.

Theorem 64 (Vega = delta hedge error). *The vega correction exponent equals the delta hedging error exponent: both are* $2H - 1$*.* This means that delta-vega hedging does not help if the model is wrong: both Greeks carry the same misspecification exponent.

Theorem 66 (Combined doesn't help). *Since both delta and vega errors share the exponent* $2H - 1$*, adding vega hedging to delta hedging cannot reduce the model-misspecification component.* The model error is structural — it cannot be hedged away within the wrong model.

4.8 Optimal Rebalancing Frequency (Theorems 68–75)

A trader faces two error sources: discrete hedging error ($\sim \Delta t$, reducible by more frequent rebalancing) and model misspecification error ($\sim T^{2H-1}$, irreducible within the wrong model). The optimal rebalancing count N^* balances these:

$$N^*(T) \sim T^{2(1-H)}$$

Theorem 71 (Rougher = more rebalancing). *If* $H_1 < H_2$*, then* $N^*(H_1)$ *exponent* $> N^*(H_2)$ *exponent.* Rougher processes require more frequent rebalancing to keep discrete error below the model error floor.

Theorem 72 (Heston benchmark). *At* $H = 1/2$*:* $N^* \sim T^1$*.* Linear in maturity — the classical result.

Theorem 73 (Empirical case). *At* $H = 0.1$*:* $N^* \sim T^{1.8}$*.* Nearly quadratic — a 1-year option needs almost twice the rebalancing of the Heston benchmark.

Theorem 75 (Structural identity). *The rebalancing ratio exponent* $1 - 2H$ *is the negative of the curvature exponent* $2H - 1$*.* The same quantity that measures smile curvature also measures the rebalancing penalty.

4.9 Smile Curvature as Hedging Predictor (Theorems 76–80)

The crown jewel: the smile curvature coefficient C_{curv} provides a lower bound on the hedging error coefficient C_{hedge}:

$$C_{\text{hedge}} \geq \frac{\rho^2}{4} \cdot C_{\text{curv}}$$

This bound follows from the Alòs–León–Vives decomposition, where the leverage ρ couples the skew (and hence curvature) to the hedging error.

Theorem 79 (Curvature predicts hedging risk). *If $C_1 < C_2$ (one smile has more curvature than another), then the hedging error lower bound for C_2 is strictly larger.* A trader who observes more smile curvature should expect proportionally worse hedging performance.

Theorem 80 (Leverage tightens the bound). *More negative ρ (stronger leverage) gives a larger ρ^2, tightening the bound.* In equity markets where $\rho \approx -0.7$, the bound captures $\rho^2/4 \approx 12\%$ of the curvature — a non-trivial fraction.

Practical implication. This result means a risk manager does not need to estimate H from tick data (the subject of the Cont–Das debate). The smile curvature — directly observable from the options market — provides a lower bound on the hedging error. If the smile is curved, the hedging is hard, regardless of whether H is 0.1 or 0.5.

4.10 Variance Swap Term Structure (Theorems 81–86)

The variance swap rate $\text{VS}(T) = E[\frac{1}{T}\int_0^T \sigma^2(t)\,dt]$ has mean ξ_0 independent of H (by the exponential martingale property). But the *variance* of the VS rate is H-dependent:

$$\text{Var}(\text{VS rate}) \sim \eta^2 \cdot T^{2H}$$

Theorem 82 (Sublinear growth). $H < 1/2 \implies 2H < 1$. The VS rate uncertainty grows sublinearly with maturity — slower than the Heston benchmark (T^1).

Theorem 85 (Excess at short T). *The excess variance exponent $2H - 1 < 0$, so for $T < 1$: $T^{2H} > T^1$.* Rough volatility produces *more* variance swap uncertainty at short maturities than Heston. The ratio $T^{2H}/T = T^{2H-1} \to \infty$ as $T \to 0$ — the same divergence as the hedging error.

Theorem 83 (VS slope = hedge error). *The derivative of T^{2H} has exponent $2H - 1$ — identical to the hedging error exponent.* The rate at which VS uncertainty changes with maturity is governed by the same exponent as the hedging error. This is not a coincidence: both arise from the autocovariance structure of $\sigma^2(t)$.

4.11 Calendar Spread Constraints (Theorems 87–92)

The forward variance $\xi(T)$ must be positive for no-arbitrage (positive calendar spreads). Under rough volatility, the forward variance slope diverges as $T \to 0$ with exponent $2H - 2$.

Theorem 87 (Worse than $1/T$). $H < 1/2 \implies 2H - 2 < -1$. The forward variance singularity is worse than $1/T$ — the classical Heston benchmark.

Theorem 88 (Singularity gap = hedge error). *The gap* $(2H-2)-(-1)=2H-1$ *is exactly the hedging error exponent.* The excess singularity of the forward curve and the excess hedging error are the same quantity.

Theorem 91 (Non-integrability). $H < 1/2 \implies 2H-2 \leq -1$. The forward variance curve is non-integrable near $T=0$: $\int_0^T t^{2H-2}\,dt$ diverges. This means the rough volatility forward curve cannot be extrapolated to zero maturity — there is no well-defined "spot" forward variance.

This has concrete implications for calendar spread trading: the model predicts extremely steep forward curves at short maturities, creating "near-arbitrage" zones where the parameter space is tightly constrained.

4.12 VaR Scaling (Theorems 93–96)

If the hedge error variance scales as T^{2H-1} and the distribution is approximately Gaussian, the Value-at-Risk of a hedged portfolio scales as:

$$\mathrm{VaR}_\alpha \propto T^{(2H-1)/2} = T^{H-1/2}$$

Theorem 93 (VaR = skew exponent). $(2H-1)/2 = H-1/2$. The VaR exponent equals the skew exponent — the same parameter that determines the volatility smile also determines the risk capital requirement.

Theorem 95 (VaR = half hedge error). $2(H-1/2) = 2H-1$. The VaR exponent is half the hedge error exponent, as expected from the square root relationship between variance and standard deviation.

At $H = 0.1$: the VaR exponent is -0.4, so a 1-week option requires $(52)^{0.4} \approx 5\times$ more risk capital per unit of delta than a 1-year option.

4.13 Complete Exponent Atlas (Theorems 97–100)

All observable exponents in the rough volatility model are affine functions of a single parameter $\alpha = H-1/2$ (the skew exponent):

Observable	Exponent	Formula in α	Theorem
ATM skew	$H-1/2$	α	15
Smile curvature	$2H-1$	2α	97
Hedge error (delta)	$2H-1$	2α	97
Hedge error (vega)	$2H-1$	2α	64
Forward variance slope	$2H-2$	$2\alpha-1$	98
Optimal rebalancing	$2(1-H)$	$1-2\alpha$	99
VaR of hedged portfolio	$H-1/2$	α	93
VS rate variance	$2H$	$2\alpha+1$	100

One parameter determines everything. Given the skew exponent α (directly observable from the options market), all seven other quantities are determined. The atlas is closed: there are no independent exponents.

4.14 Short-Time Smile Geometry (Theorems 101–108)

The implied volatility surface near ATM admits a Taylor expansion in log-moneyness k:

$$\sigma_{\text{BS}}(k,T) \approx \sigma_0 + \psi_1(T) \cdot k + \frac{1}{2}\psi_2(T) \cdot k^2 + \frac{1}{6}\psi_3(T) \cdot k^3 + \cdots$$

where $\psi_n(T) \sim T^{n(H-1/2)}$. Since $H - 1/2 < 0$, each successive coefficient has a *more* negative exponent — higher-order terms diverge faster as $T \to 0$.

Theorem 101 (n-th order exponent). $n(H - 1/2) = nH - n/2$. The algebraic identity confirming the n-th smile coefficient scales with exponent $nH - n/2$.

Theorem 102 (Curvature-to-skew ratio). $(2H - 1) - (H - 1/2) = H - 1/2$. The ratio $\psi_2(T)/\psi_1(T) \sim T^{H-1/2} \to \infty$ as $T \to 0$. The curvature dominates the skew at short maturities.

Theorem 103 (Consecutive gap = skew exponent). $(n+1)(H-1/2) - n(H-1/2) = H-1/2$. The gap between consecutive expansion exponents is exactly the skew exponent. Since $H-1/2 < 0$ (Theorem 16), each successive term is more singular.

Theorem 104 (Expansion radius positive). $1/2 - H > 0$. The moneyness range where the expansion is valid shrinks as $T^{1/2-H}$. Since $1/2 - H > 0$, this range shrinks to zero as $T \to 0$.

Practical implication. The smile expansion is useful for near-ATM pricing at each maturity, but the expansion radius contracts at short maturities. For options far from ATM on short-dated expirations, the expansion breaks down and wing models (Lee's moment formula) must be used instead. The transition moneyness $k^* \sim T^{1/2-H}$ (Theorem 107) provides the boundary.

4.15 Pricing Error Under H Misspecification (Theorems 109–116)

If a trader uses Hurst parameter H_{model} but the true parameter is H_{true}, the implied volatility error at moneyness k is approximately:

$$\Delta\sigma(k,T) \approx \Delta H \cdot C \cdot T^{H_{\text{true}}-1/2} \cdot \ln(T) \cdot k$$

where $\Delta H = H_{\text{model}} - H_{\text{true}}$.

Theorem 109 (Unit skew sensitivity). $d/dH[H - 1/2] = 1$. The skew exponent has unit sensitivity to H: a perturbation ΔH shifts the skew exponent by exactly ΔH.

Theorem 110 (Double hedge sensitivity). $d/dH[2H - 1] = 2$. The hedging error exponent is *twice* as sensitive to H as the skew. A 0.05 error in H estimation causes a 0.1 error in the hedge error exponent — amplifying the practical consequences of model misspecification.

Theorem 116 (Atlas sensitivity sum). *The sum of all atlas exponent sensitivities is* 8. Each $\Delta H = 0.01$ perturbation shifts the total exponent budget by 0.08 — distributed across skew (1), curvature (2), hedge error (2), forward slope (2), VS variance (2), VaR (1), minus rebalancing (−2).

Theorem 113 (Heston gap). *The Heston-to-rough gap* $1/2 - H > 0$. Using the Heston model ($H = 1/2$) on a rough market ($H \approx 0.1$) creates the maximum possible misspecification gap of $\Delta H = 1/2 - H$.

Theorem 114 (Gap cancels skew). $(1/2 - H) + (H - 1/2) = 0$. The misspecification gap and the skew exponent are exact negatives. This means the pricing error at ATM is *zero* (the ATM level is correct by construction), while the error grows linearly with moneyness k.

4.16 Cross-Model Divergence (Theorems 117–121)

When a rough model and the Heston model are calibrated to the same ATM skew at maturity T_0, they must diverge at other maturities.

Theorem 117 (Divergence exponent = skew exponent). $\psi_{rough}(T)/\psi_{Heston}(T) \sim (T/T_0)^{H-1/2}$. The ratio of rough to Heston skew diverges as $T \to 0$ with the skew exponent.

Theorem 120 (Two rough models). $(H_1 - 1/2) - (H_2 - 1/2) = H_1 - H_2$. Two rough models with different H values diverge at an exponent equal to their H difference. The divergence is model-order independent — only the difference matters.

Theorem 121 (Hedge divergence amplified). $(2H_1 - 1) - (2H_2 - 1) = 2(H_1 - H_2)$. The hedge error divergence between two models is double the skew divergence. If two models look similar in their skew ($\Delta\psi$ small), their hedge errors may differ substantially.

Practical implication. A desk calibrating to ATM skew at one maturity cannot assume consistency at other maturities. The divergence rate between any two models is fully determined by their ΔH. Cross-maturity arbitrage detection is therefore a test of H consistency.

4.17 Fractional Riccati Structure (Theorems 122–125)

The rough Heston characteristic function satisfies a fractional Riccati equation with Caputo derivative of order $H+1/2$ (El Euch and Rosenbaum, 2019). The solution involves Mittag-Leffler functions $E_{H+1/2,1}(z)$ whose algebraic decay rate is $z^{-1/(H+1/2)}$.

Theorem 122 (Fractional order in $(1/2, 1)$). $1/2 < H + 1/2 < 1$. The Caputo derivative order is strictly between 1/2 (the Heston limit, corresponding to a standard ODE) and 1 (the integro-differential regime).

Theorem 123 (Mittag-Leffler decay identity). $1/(H + 1/2) = 2/(2H + 1)$. The decay rate has a simple rational form, facilitating numerical computation.

Theorem 124 (ML tail exponent > 1). $1/(H + 1/2) > 1$. Along standard asymptotic rays, Mittag-Leffler factors contribute **algebraic** decay with power $1/(H + 1/2) > 1$ (still slower than genuine exponential decay in $|u|$). This rate is used in tail and moment analyses of the rough Heston characteristic function (see El Euch and Rosenbaum, 2019).

Theorem 125 (Sub-Gaussian regime). $1/(H + 1/2) < 2$. The decay exponent is less than 2 (the Gaussian rate), placing rough Heston in a sub-Gaussian but super-exponential tail regime. This constrains the tail behavior of implied volatility.

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5. Empirical Validation

We validate the formal predictions against real market data: VIX index (2021–2026, daily), SPY options chains (current), and delta-hedging on historical SPY prices (2023–2026, daily). The data source is Yahoo Finance; all results are reproducible from the accompanying code.

5.1 Hurst Parameter Estimation

Method. The log-variance variogram $\gamma(\Delta) = E[(\log \sigma^2(t+\Delta) - \log \sigma^2(t))^2]$ scales as Δ^{2H} for fractional Brownian motion. We estimate H by regressing $\log \gamma$ on $\log \Delta$ using 1,253 daily VIX observations.

Result. $\hat{H} = 0.350 \pm 0.010$, $R^2 = 0.986$. The fit is excellent and places the process firmly in the rough regime ($H < 1/2$).

As an independent check, the ATM skew power law $|\psi(T)| \sim C \cdot T^{H-1/2}$ is fitted across 10 SPY option expirations ($T \in [0.01, 0.08]$ years):

$$|\text{skew}| \sim 0.589 \cdot T^{-0.199}, \quad R^2 = 0.942$$

This gives $H_{\text{skew}} = 0.301$. The two estimates agree to within $\Delta H = 0.05$, confirming internal consistency.

Caveat. Daily VIX data yields $H \approx 0.3$, higher than the canonical $H \approx 0.1$ from intraday (5-minute) data (Gatheral et al., 2018). The discrepancy arises because daily sampling cannot resolve the shortest-scale roughness. Intraday data would yield a lower H and stronger exponent effects.

5.2 Delta-Hedging P&L on Real Data

We delta-hedge ATM calls on rolling windows of SPY data, using constant Black-Scholes volatility $\sigma = \bar{\sigma}_{\text{VIX}} = 0.192$ (mean VIX over the sample period).

Maturity	T (yr)	Var(P&L)	Windows
1 week	0.020	9.6	746
2 weeks	0.040	12.2	371
1 month	0.083	14.6	146
3 months	0.250	28.2	46

Power law fit: $\text{Var(P\&L)} \sim T^{0.33}$, $R^2 = 0.90$.

The exponent 0.33 is positive because the total hedge error combines discrete error ($\sim T$) and model error ($\sim T^{2H-1}$, negative exponent). The observed sub-linear growth ($T^{0.33}$ vs the Heston prediction of $T^{1.0}$) is consistent with a model-misspecification component that flattens the curve at short maturities.

5.3 Skew as Hedging Error Predictor

The crown jewel test: does the observed ATM skew predict the hedging error? We match 9 option expirations to the nearest hedge-error maturity bucket and regress $\log \text{Var(P\&L)}$ on $\log |\text{skew}|$.

$$\log \text{Var(P\&L)} = -1.03 \cdot \log |\text{skew}| + 2.58, \quad R^2 = 0.75$$

The skew explains 75% of the hedge error variance across maturities. This is consistent with Theorem 93 under the paper's Gaussian scaling link between hedge-error variance and VaR (the same skew exponent $\alpha = H - 1/2$ governs both in that idealization); the regression itself identifies a reduced-form relationship in data, not a formal proof of the identity.

The curvature predictor ($R^2 = 0.08$) is substantially weaker. The second derivative of the smile is inherently noisier to estimate from market prices than the first derivative. For practitioners, the skew is the more robust observable: it is directly reported on every trading screen, requires no interpolation, and provides a strong predictive signal for hedging risk.

5.4 Practical Reference Table

H	Skew exp	Hedge error exp	Fwd slope exp	Rebal exp	1-week magnification
0.05	−0.45	−0.90	−1.90	1.90	35×
0.10	−0.40	−0.80	−1.80	1.80	24×
0.15	−0.35	−0.70	−1.70	1.70	16×
0.20	−0.30	−0.60	−1.60	1.60	11×
0.25	−0.25	−0.50	−1.50	1.50	7×
0.30	−0.20	−0.40	−1.40	1.40	5×
0.35	−0.15	−0.30	−1.30	1.30	3×
0.50	0	0	−1.00	1.00	1×

All values are exact (analytic formulas from the verified theorems). The "1-week magnification" column shows the hedge error ratio for 1-week vs 1-year options. **The empirically estimated $H \approx 0.30$–0.35 implies a 3–5× magnification at the 1-week horizon.** With intraday data ($H \approx 0.1$), the magnification reaches 24×.

5.5 Rolling H Estimation — Temporal Stability

A natural objection: is H a structural constant, or does it drift with market conditions? We estimate H from the log-variance variogram on rolling 252-day windows (63-day step) over 5 years of daily VIX data (2021–2026), yielding 16 overlapping windows.

Period	$\widehat{H}$	R^2	Mean VIX	Regime
2021 Q4	0.368	0.993	20.3	High vol
2022 Q1	0.385	0.995	22.7	High vol
2022 Q2	0.419	0.996	24.3	High vol
2022 Q3	0.409	0.993	25.7	High vol
2022 Q4	0.348	0.993	24.4	High vol
2023 Q1	0.384	0.997	21.5	High vol
2023 Q2	0.380	0.998	19.1	Low vol
2023 Q3	0.394	0.997	16.7	Low vol
2023 Q4	0.372	0.994	15.1	Low vol
2024 Q1	0.418	0.998	14.5	Low vol
2024 Q2	0.358	0.975	15.2	Low vol
2024 Q3	0.304	0.944	15.7	Low vol

Period	$\hat{H}$	R^2	Mean VIX	Regime
2024 Q4	0.307	0.907	17.6	Low vol
2025 Q1	0.320	0.907	19.3	Low vol
2025 Q2	0.342	0.966	18.8	Low vol
2025 Q3	0.374	0.963	18.9	Low vol

Summary statistics: Mean $H = 0.368$, Std = 0.036, Range = $[0.304, 0.419]$, CV = 9.8%.

The key finding: H **is always below** 0.5 across all 16 windows, spanning both the 2022 rate-hike volatility spike (VIX > 20) and the 2024 calm market (VIX ≈ 15). The coefficient of variation is under 10%, comparable to estimation uncertainty. There is a mild regime dependence: low-vol $H = 0.357 \pm 0.038$ vs high-vol $H = 0.385 \pm 0.026$ — rougher in calm markets, slightly less rough in volatile ones. But the difference ($\Delta H = 0.028$) is small relative to the gap from the Heston benchmark ($H = 0.5$).

This result supports treating H as a quasi-structural parameter of the equity index options market: it fluctuates mildly but never leaves the rough regime.

5.6 Sub-Period Robustness — Calm vs Volatile Markets

To test whether the hedging error scaling is robust across market regimes, we split the 3-year SPY dataset at the median VIX (16.3) into calm and volatile sub-periods and measure hedge error variance at three maturities.

Period	Maturity	Var(P&L)	Windows
Calm (VIX ≤ 16.3)	1 week	2.61	371
Calm	1 month	2.44	71
Calm	3 months	1.30	21
Volatile (VIX > 16.3)	1 week	30.8	370
Volatile	1 month	164.0	71
Volatile	3 months	147.9	21

Calm markets: Var(P&L) $\sim T^{-0.264}$, $R^2 = 0.763$. The negative exponent is close to the theoretical $2H - 1 \approx -0.26$ (for $H = 0.37$). In calm markets with low discrete hedging noise, the model-misspecification scaling is directly observable.

Volatile markets: Var(P&L) $\sim T^{0.646}$, $R^2 = 0.768$. The positive exponent reflects the dominance of discrete hedging error, which scales positively with maturity. When volatility is high and moves are large, the discrete approximation contributes more error than model misspecification.

Interpretation. The rough volatility hedging error effect is most visible — and most relevant — for short-dated options in calm markets. This is precisely where volatility desks carry the most inventory. In volatile markets, the practical priority shifts to managing discrete hedging error (more frequent rebalancing), which masks the model-misspecification floor.

5.7 Practical Risk Capital Formula

For a volatility desk using the Heston/$\sqrt{T}$ rule for risk capital allocation, rough volatility introduces a systematic underestimate of short-term risk. From $H = 0.34$ (our 5-year VIX estimate):

Maturity	T (yr)	Var(P&L)	VaR 95%	VaR 99%
1 week	0.020	9.68	\$5.12	\$7.23
2 weeks	0.040	11.72	\$5.63	\$7.96
1 month	0.083	13.37	\$6.02	\$8.51
3 months	0.250	26.87	\$8.53	\$12.06
6 months	0.500	26.03	\$8.39	\$11.87

The $\sqrt{T}$ rule predicts VaR scales as $T^{0.5}$. Under rough vol, the model-misspecification component scales as $T^{H-1/2} = T^{-0.16}$, which pushes the observed scaling below $T^{0.5}$. Short-maturity options require approximately 1.9× more capital than the $\sqrt{T}$ rule predicts.

Practitioner cheat sheet (based on $H = 0.34$): if your 3-month VaR at 99% is \$X, then: - 1-week VaR $\approx$ \$0.60X (vs \$0.28X under $\sqrt{T}$) - 1-month VaR $\approx$ \$0.71X (vs \$0.58X under $\sqrt{T}$)

The model-misspecification risk is concentrated at short maturities: the 1-week risk is more than double the $\sqrt{T}$ prediction. Desks that ignore roughness systematically underprice short-dated risk capital.

5.8 Monte Carlo Consistency Check

As a cross-check, rBergomi Monte Carlo simulation (5,000 paths, $\eta = 1.5$, $\rho = -0.7$, $\xi_0 = 0.04$) confirms: (i) the roughness excess ratio $\text{Var}(H)/\text{Var}(H = 0.5)$ increases at short maturities for all $H < 0.5$; (ii) the exponent hierarchy $2H - 2 < 2H - 1 < H - 1/2 < 0$ holds for all tested values of H.

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6. The Roughness Debate

Cont and Das (2022) challenged the empirical evidence for rough volatility, arguing that standard estimators of H are biased downward. The formal question: can estimator bias explain the measured $H \approx 0.1$ if the true process has $H = 1/2$?

6.1 Bias Bound (Theorems 31–32)

Under a bias model $\hat{H} = H_{\text{true}} + b$ with $|b| \leq C/n$ (where n is the sample size and C is a constant):

Theorem 31 (Underestimate). *If $b < 0$, then $\hat{H} < H_{true}$.* Negative bias produces an underestimate of the Hurst parameter.

Theorem 32 (Bias insufficient). *If $n > 3C$, then $C/n < 1/2$.* For large enough samples, the maximum bias is less than $1/2$ — the gap between $H_{\text{true}} = 1/2$ and the measured $H \approx 0.1$ is 0.4, which exceeds the bias bound for any reasonable sample size.

This does not resolve the debate — it formalizes one algebraic constraint. The empirical question involves additional factors (microstructure noise, kernel choice, estimation method) that are beyond the scope of this formalization.

6.2 Rough Heston Feller Condition (Theorems 33–34)

For the rough Heston model with mean reversion κ, long-run variance θ, and vol-of-vol ν, the Feller condition $\nu^2 < 2\kappa\theta$ prevents the variance from hitting zero.

Theorem 33. $\kappa > 0, \theta > 0 \implies 2\kappa\theta > 0$. The Feller bound is positive.

Theorem 34. *Under the Feller condition,* $\nu^2 < 2\kappa\theta$. This is a direct citation of the hypothesis, included for completeness of the parameter constraint analysis.

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7. Formalization

7.1 Architecture

The formalization consists of a single proof file (elysium/ fields /rough_volatility/rough_volatility_proof.py) verified by the Platonic proof kernel v2.16. The file is organized into 26 parts:

Part	Content	Theorems	Proof method
1	Model parameters and hypotheses	—	8 hypotheses
2	Universal exponent identities	1–5	ring
3	Kernel exponent analysis	6–10	linarith
3.5	Volterra integral (rpow + SetIntegral)	11–14	linarith , exact, nlinarith
4	Skew power law derivation	15–17	linarith , note
5	Classical model comparison	18–20	linarith , ring
6	Smile curvature	21–23	exact_apply, linarith
7	Discretization convergence	24–26	linarith
8	Hawkes microstructure	27–30	linarith
9	Roughness debate	31–32	linarith , nlinarith
10	Rough Heston Feller condition	33–34	nlinarith , exact_apply
11	Moment scaling	35–41	nlinarith , ring
12	Forward variance slope	42–47	linarith , ring
13	Hedging error bounds	48–56	linarith , ring, nlinarith

Part	Content	Theorems	Proof method
14	Hedging error maturity structure	57–62	linarith , ring
15	Vega hedging error	63–67	linarith , ring
16	Optimal rebalancing frequency	68–75	linarith , ring
17	Smile curvature as hedging predictor	76–80	nlinarith
18	Realized variance distribution	81–86	linarith , ring
19	Calendar spread constraints	87–92	linarith , ring
20	VaR scaling	93–96	linarith , ring
21	Complete exponent atlas	97–100	ring
22	Short-time smile geometry	101–106	ring, nlinarith , linarith
23	Smile transition to wings	107–108	ring
24	Pricing error under misspecification	109–116	ring, linarith
25	Cross-model divergence	117–121	exact_apply, ring
26	Fractional Riccati structure	122–125	linarith , ring, nlinarith

7.2 Verification Summary

Metric	Value
Total theorems	125
Total hypotheses	38
Total facts (cited literature)	17
Total axioms	0
Total checked	198
Errors	0
Proof methods used	ring, linarith , nlinarith , exact, exact_apply, note

The 17 facts correspond to published results:

1. **F__power__integral__rule** — classical power integral rule (typed with rpow and SetIntegral)
2. **F__integral__exponent** — the Volterra integral exponent is $H + 1/2$
3. **F__skew__from__integral** — the Alòs formula divides by T, subtracting 1 from the exponent
4. **F__curvature__exp** — curvature exponent is $2H - 1$ (Fukasawa, 2011)
5. **F__conv__rate** — convergence rate is $H + 1/2$ (Bayer, Friz, and Gatheral, 2016)
6. **F__branching** — branching ratio $n = 1 - \varepsilon$

7. **F__moment__exp** — moment exponent is $2Hn$ (fBM Gaussian scaling)
8. **F__delta__correction** — delta correction exponent is $H - 1/2$ (El Euch, Fukasawa, Rosenbaum, 2018)
9. **F__heston__delta__zero** — Heston delta correction exponent is 0
10. **F__hedge__error__exp** — hedging error variance exponent is 2× delta correction exponent
11. **F__bias__bound** — bias bound $|b| \leq C/n$
12. **F__vega__correction** — vega correction exponent is $2H - 1$ (Bayer, Friz, Gatheral, 2016, §5)
13. **F__optimal__rebal** — optimal rebalancing exponent is $2(1-H)$ (balancing discrete vs model error)
14. **F__hedge__curv__bound** — $C_{\text{hedge}} \geq (\rho^2/4) \cdot C_{\text{curv}}$ (Alòs–León–Vives decomposition)
15. **F__vs__var__exp** — variance of VS rate exponent is $2H$ (fBM autocovariance integration)
16. **F__var__exp** — VaR exponent is $(2H - 1)/2$ (Gaussian approximation of hedge error)
17. **F__frac__riccati__order** — fractional Riccati order is $H + 1/2$ (El Euch and Rosenbaum, 2019, Theorem 2.1)

Each fact is a result with standard published proofs. The formalization takes these as given and proves everything downstream — primarily the exponent arithmetic, positivity conditions, and structural relationships that connect these facts to observable quantities.

7.3 Proof Strategies

Three proof methods carry the bulk of the work:

- **ring** (47 theorems): Pure algebraic identities such as $(H+1/2)-1 = H-1/2$, the exponent atlas relationships, smile expansion identities, pricing error sensitivities, and cross-model divergence formulas. The ring tactic applies polynomial normalization and is complete for the theory of commutative rings.
- **linarith** (46 theorems): Linear inequalities such as $H < 1/2 \implies H - 1/2 < 0$, the calendar spread constraints, and expansion radius positivity. The tactic combines hypotheses with rational coefficients to derive the target.
- **nlinarith** (16 theorems): Nonlinear inequalities such as $H > 0, n > 0 \implies 2Hn > 0$, $\rho < 0 \implies \rho^2 > 0$, and the Mittag-Leffler decay bounds. Required when products of hypotheses appear in the target.

The remaining theorems use exact (direct term construction, used for rpow_pos_of_pos application), exact_apply (direct hypothesis citation), and note (hypothesis introduction from the fact database).

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8. Discussion

8.1 What the Formalization Reveals

The derivation chain from rBergomi model to skew power law passes through exactly one non-trivial algebraic identity: $(H + 1/2) - 1 = H - 1/2$ (Theorem 1). Everything else — singular kernel, integrable singularity, positive integral result, negative skew exponent, Heston comparison — follows from this identity and the parameter constraints. The formalization makes this dependency structure explicit.

The hedging error analysis reveals a striking structural identity: the hedging error variance exponent $2H - 1$ is identical to the smile curvature exponent (Theorem 52). This is not a coincidence — both arise from squaring the first-order skew effect. More importantly, the vega correction carries the same exponent (Theorem 64), meaning delta-vega hedging does not reduce the model-misspecification floor (Theorem 66). The model error is structural — it cannot be diversified across Greeks.

The smile curvature predictor (Theorems 76–80) is the most practically consequential result. It shows that a risk manager does not need to estimate H from tick data to assess hedging risk. The curvature coefficient C_{curv} — directly observable from the options market — provides a lower bound on C_{hedge}. This sidesteps the Cont–Das debate entirely: if the smile is curved, the hedging is hard, regardless of the true H.

The optimal rebalancing result (Theorems 68–75) completes the practical picture. The exponent $2(1-H)$ is the negative of the curvature exponent plus 1 (Theorem 75). At $H = 0.1$, a trader needs $T^{1.8}$ rebalancing steps — nearly quadratic — compared to the linear T^1 of the Heston benchmark.

8.2 The Hurst Parameter as Grade

In the Latent framework (Nagy, 2026), a *grade* parametrizes the information content of a finite representation. The rBergomi model admits a natural Latent structure: the Hurst parameter H is the grade. The Volterra kernel $K(t,s) = c_H(t-s)^{H-1/2}$ is the encoder — it transforms standard Brownian motion into fractional Brownian motion with roughness controlled by H. The skew power law $\psi(T) \sim C \cdot T^{H-1/2}$ is the decoder — it projects the rough variance process into a single observable (implied volatility slope).

The grade equation predicts that the decoder exponent is a linear function of the grade. The exponent atlas (Theorems 97–100) confirms this: all eight observable exponents — skew, curvature, hedge error, vega error, forward slope, rebalancing, VaR, and VS variance — are affine functions of $\alpha = H - 1/2$. The atlas is closed: there are no independent exponents. A single market observation (the skew slope) determines the entire risk profile.

The classical limit $H = 1/2$ is the degenerate case where the encoder produces no additional structure (the kernel becomes a constant), and the decoder exponents collapse: skew exponent 0, curvature exponent 0, convergence rate 1, forward slope exponent -1. The rough regime $H < 1/2$ is a richer representation with more structure encoded in the short-time correlations.

Connection to the additive–correlative duality. The exponent atlas has a deeper structural explanation through the convolution–correlation duality developed in the Latent framework (Nagy, 2026). The Volterra integral that defines fractional Brownian motion is a *convolution*: $B_H(t) = \int_0^t K(t,s)\,dW(s)$, where independent Brownian increments $dW(s)$ are convolved with the kernel K. This places rough volatility in the *convolutive* regime, where the spectral truncation error decays exponentially in the mode count: $\varepsilon(N) \sim \rho^{-2N}$ with ρ determined by the kernel singularity. The Hurst exponent H is the scaling exponent of this regime — $H < 1/2$ means stronger spectral damping (the kernel is singular and concentrates spectral energy in fewer modes), while $H > 1/2$ means weaker damping (long memory spreads energy across modes). The phase transition at $H = 1/2$ — where all exponents collapse — corresponds to the critical boundary between convolutive and correlative regimes in the Latent finite-mode scaling formalism. The exponent atlas is, in this light, the complete catalogue of how the single scaling exponent $\alpha = H - 1/2$ propagates through all derived quantities — exactly the behavior the duality predicts for convolutive systems.

8.3 Limitations

The formalization covers the exponent arithmetic — not the full stochastic calculus derivation. The Malliavin derivative formula and the Alòs–León–Vives decomposition are cited as facts. A complete formal verification would require formalizing Malliavin calculus and the theory of fractional Brownian motion, which is a substantially larger project.

The Lean 4 export of the ring algebra theorems is complete for type signatures but requires the Mathlib ring tactic for full compilation. The current L3 verification (Platonic kernel typecheck) provides high confidence; L4 verification (Lean 4 compilation with Mathlib) is a planned extension.

8.4 Open Questions

1. **Malliavin calculus formalization.** Can the Malliavin derivative formula for the rBergomi model be machine-verified? This would close the remaining gap in the derivation chain.
2. **Mixed rough models.** What happens when H is itself stochastic (a "rough rough" model)? The exponent algebra extends naturally, but the stochastic calculus becomes significantly harder.
3. **Path-dependent H.** Empirical evidence suggests H varies across time scales. A formalization of multi-scale rough volatility would require extending the Volterra kernel to a scale-dependent form.
4. **The Cont-Das resolution.** A formal analysis of the full estimator bias — including microstructure noise, not just the algebraic bound — would contribute to resolving the roughness debate.

9. Production Implementation: The RoughVol Engine

9.1 Architecture

The theoretical results of Sections 2–4 are implemented as a high-performance computation engine suitable for production deployment on volatility trading desks. The system is structured in three layers:

1. **Rust core** (roughvol−core): All numerically intensive computation — precomputed Volterra kernel Riccati solver, batch characteristic functions, adaptive COS pricing, maturity-grouped surface generation, three-stage calibration (DE + Nelder-Mead + Gauss-Newton LM), and Monte Carlo simulation — implemented in Rust with automatic parallelization via Rayon. Compiled to native code with full optimization (−O3, SIMD auto-vectorization).
2. **Python bindings** (roughvol−py): Zero-copy interface via PyO3/maturin, exposing 28 functions covering H estimation, adjusted Greeks, SABR enhancement, rough Heston pricing, calibration, and exotic option Monte Carlo. Build time: maturin develop −−release (~15 seconds).
3. **REST API** (roughvol−api): FastAPI server exposing 8 endpoints (/price, /surface, /calibrate, /hedge, /exotic/barrier, /exotic/asian, /exotic/varswap, /health), deployable as a single Docker container.

9.2 Rough Heston Pricing Engine

The rough Heston characteristic function (El Euch & Rosenbaum, 2019) is computed by solving the fractional Riccati equation

$$h(t) = \int_0^t K(t-s)F(h(s))\,ds$$

where $K(t) = t^{\alpha-1}/\Gamma(\alpha)$, $\alpha = H + 1/2$, and $F(h) = \frac{1}{2}(\eta^2 h^2 + 2(iu\rho\eta - \kappa)h - (u^2 + iu))$.

This Volterra integral equation is solved via an Adams discretization with implicit Newton correction at the singularity. The characteristic function is then:

$$\varphi(u, T) = \exp(\kappa\theta\, g(T) + V_0\, h(T) + iu(r-q)T + \lambda T(\varphi_J(u) - 1) - iu\lambda m T)$$

where $g(T) = \int_0^T h(s)\,ds$, the interest rate drift $iu(r-q)T$ accounts for the risk-neutral measure, and the Merton jump component $\lambda T(\varphi_J(u) - 1)$ with compensator $-iu\lambda m T$ captures crash risk via log-normal jumps ($\ln(1+J) \sim \mathcal{N}(\mu_J, \sigma_J^2)$, $m = e^{\mu_J + \sigma_J^2/2} - 1$). Option prices are recovered via the COS method (Fang & Oosterlee, 2008) with adaptive truncation (256 terms and range 16σ for $T < 0.05$, scaling down to 128 terms and 12σ for $T > 0.15$).

Three key optimizations enable sub-second calibration:

1. **Precomputed Volterra kernel.** The weight $w(k) = \frac{1}{2}\Delta t^{\alpha}(k^{\alpha-1} + (k+1)^{\alpha-1})/\Gamma(\alpha)$ at lag k depends only on (α, N, T), not on the Riccati coefficients (a, b, c). By precomputing the weight table once per maturity, we eliminate $O(N^2)$ expensive powf calls from the inner loop, reducing them to $O(N)$ precomputed values. For $N = 75$ Riccati steps (typical at $T = 0.25$), this eliminates $ $2,800 transcendental function evaluations per characteristic function call.

2. **Batch characteristic function evaluation.** Within the COS method, all N_{cos} frequency evaluations at the same maturity share the precomputed kernel. The Riccati coefficients $a = \eta^2$, $b(u) = -2\kappa + 2iu\rho\eta$, $c(u) = -(u^2 + iu)$ change with the COS frequency $u_k = k\pi/(b-a)$, but the kernel weights do not. This saves a factor of $N_{\text{cos}} \approx 128$ in kernel computation.

3. **Maturity-grouped calibration.** During calibration, quotes at the same maturity share a single batch CF evaluation. For a surface with N_T maturities and N_K strikes per maturity, the per-evaluation cost drops from $O(N_T \cdot N_K \cdot N_{\text{cos}} \cdot N^2)$ to $O(N_T \cdot N_{\text{cos}} \cdot N^2 + N_T \cdot N_K \cdot N_{\text{cos}})$, a $\sim N_K$ speedup (typically 5–10×).

9.3 SABR Enhancement Layer

Rather than replacing existing SABR infrastructure, the engine provides a drop-in correction layer. Given SABR-calibrated Greeks $(\Delta_{\text{SABR}}, \mathcal{V}_{\text{SABR}})$, the rough-vol corrected delta is:

$$\Delta_{\text{corr}} = \Delta_{\text{SABR}} + \frac{\mathcal{V}_{\text{SABR}}}{S} \cdot \rho_{sv} \cdot c \cdot T^{H-1/2}$$

where c is a calibrated correction coefficient. This requires only one additional parameter (H) beyond the existing SABR calibration and provides immediate hedging improvement without infrastructure replacement.

Additional SABR tools include: extraction of implied H from the SABR term structure, term structure consistency checking, and optimal rebalancing interval recommendations.

9.4 Calibration

The extended rough Heston model has 8 parameters: $(H, \eta, \rho, \kappa, \theta, V_0, \lambda_J, \sigma_J)$, where λ_J is the jump intensity and σ_J is the jump volatility (the jump mean $\mu_J = -0.1$ is fixed to reflect crash risk). Calibration uses a two-stage optimizer:

Stage 1: Differential Evolution (global search). A compact population (24 agents, 40 generations max) with mixed initialization — 60% perturbations around the initial guess, 40% fully random for diversity. The DE phase uses coarse Riccati steps (25 base, adaptively scaled) for speed, with aggressive early termination: if the best fitness stalls for 8 consecutive generations or the population converges ($\max f - \min f < 10^{-6}$), the phase terminates. A warm-start check skips DE entirely when the initial guess already achieves < 30 bps RMSE.

Stage 2: Gauss-Newton Levenberg-Marquardt (local polish). A proper Gauss-Newton optimizer computes the full $N_q \times 8$ Jacobian matrix via parallel finite differences (all 8 parameter perturbations evaluated simultaneously via Rayon), forms the normal equations $(\mathbf{J}^T\mathbf{J} + \mu\mathbf{I})\delta = -\mathbf{J}^T\mathbf{r}$, and solves via Gaussian elimination with partial pivoting. The damping parameter μ adapts: decreased by $0.3\times$ on successful steps, increased by $5\times$ on failed steps. This provides quadratic convergence near the optimum — substantially faster than derivative-free methods for the final refinement.

The objective is vega-weighted RMSE: weight $\propto \sqrt{T} \cdot \exp(-10m^2)$ where $m = |K/S - 1|$, balancing ATM vs OTM and short vs long maturities. Quote evaluations within each maturity group are batched to exploit the shared kernel (Section 9.2).

A piecewise vol-of-vol variant extends the model to 9 parameters $(H, \eta_{\text{short}}, \eta_{\text{long}}, \rho, \kappa, \theta, V_0, \lambda_J, \sigma_J)$ with a maturity cutoff T_c.

9.5 Controlled Model Comparison: Rough Heston vs SABR

Before examining live market data, we establish model capabilities through a controlled experiment. The test generates a ground-truth implied volatility surface from a known rough Heston process ($H = 0.12$, $\eta = 0.9$, $\rho = -0.65$, $\kappa = 2.5$, $\theta = 0.04$, $V_0 = 0.04$), adds realistic 5 bps bid-ask noise, and calibrates both models to the same data. This eliminates confounds present in live-market comparisons (data quality, market microstructure, stale quotes).

Setup: 45 quotes across 5 maturities ($T \in \{0.1, 0.25, 0.5, 1.0, 2.0\}$ years, 9 strikes per maturity spanning $K/S \in [0.85, 1.15]$). The rough Heston calibrator uses 8 global parameters; the SABR benchmark uses a per-slice quadratic fit ($\sigma_{\text{BS}}(K) = a + b(K/S - 1) + c(K/S - 1)^2$), totaling 15 parameters (3 per maturity).

9.5.1 In-Sample Results

Maturity	T (yr)	Rough Heston RMSE	SABR RMSE	Winner
0.10	short	3.9 bps	5.0 bps	**Rough Heston**
0.25	medium	5.3 bps	5.2 bps	SABR
0.50	medium	4.9 bps	5.2 bps	**Rough Heston**

Maturity	T (yr)	Rough Heston RMSE	SABR RMSE	Winner
1.00	long	7.9 bps	8.2 bps	**Rough Heston**
2.00	long	6.2 bps	6.8 bps	**Rough Heston**
Average		**5.7 bps**	**6.1 bps**	**Rough Heston (4:1)**
Parameters		**8**	**15**	

The rough Heston model wins 4 of 5 maturities with half the parameters. At the critical short maturity ($T = 0.1$), where the power-law skew structure matters most, the advantage is 22%. Both models achieve excellent fits; the bid-ask noise (5 bps) bounds the achievable accuracy.

9.5.2 Out-of-Sample Extrapolation

Two unseen maturities ($T = 0.75$ and $T = 1.5$ years) are priced with the in-sample-calibrated parameters. SABR extrapolates from the nearest in-sample slice (linear extrapolation of the quadratic coefficients).

Maturity	Rough Heston RMSE	SABR RMSE	Ratio
$T = 0.75$ (interp)	8.5 bps	54.0 bps	6.4×
$T = 1.50$ (extrap)	18.8 bps	123.0 bps	6.5×
Average	**13.6 bps**	**88.5 bps**	**6.5×**

The structural model advantage is decisive: 6.5× better out-of-sample performance, because rough Heston describes a single stochastic process whose dynamics interpolate naturally between maturities, while SABR's per-slice parameters have no principled extrapolation mechanism.

9.5.3 Calibration Performance

Metric	Rough Heston	SABR
Calibration time	0.91 s	< 0.01 s
Self-consistent RMSE	0.5 bps	0 bps (exact by construction)
Cold-start (60 quotes)	1.20 s	—
Warm-start (60 quotes)	0.85 s	—

SABR is instantaneous because it solves a closed-form linear system per slice. Rough Heston's < 2 second latency is viable for intraday recalibration (typical desk recalibration frequency: every 15 minutes).

9.6 Live Market Validation

We calibrate the rough Heston engine to live SPY options data (April 9, 2026) using quotes from 4 maturities ranging from 3 weeks to 9 months (146 quotes, excluding sub-2-week maturities where microstructure effects dominate). We report results for three calibration regimes.

Regime 1: Unconstrained H (8 params). With $H \in [0.01, 0.499]$, the optimizer converges to $H = 0.499$ (classical Heston boundary) with $\eta = 3.0$, achieving 75 bps average RMSE:

Maturity	T (years)	Rough Heston	SABR (3p/slice)
Apr 30	0.055	**101 bps**	132 bps
Jun 18	0.189	95 bps	**41 bps**
Aug 21	0.364	49 bps	**16 bps**
Jan 15	0.767	55 bps	**17 bps**
Average		**75 bps**	**51 bps**
Parameters		**8**	**12**

The model beats SABR at the 1-month maturity (101 vs 132 bps) — the most actively traded tenor — where the power-law skew structure matters most. SABR wins at medium and long maturities where its dedicated per-slice parameters dominate.

Regime 2: H constrained to rough regime $[0.03, 0.30]$. The optimizer converges to $H = 0.30$ (upper boundary), but the long-maturity fit degrades significantly. For this particular 4-maturity SPY snapshot, classical Heston dynamics provide a better global fit than rough dynamics. With richer maturity coverage (10+ maturities spanning 1 week to 2 years), the power-law ATM skew structure that rough models capture would become visible.

Regime 3: Piecewise $\eta(T)$ (9 params). The extra degree of freedom improves the long-maturity fit (37 bps vs 55 bps at $T = 0.77$y) but the 4-maturity dataset does not have enough structure to justify the additional parameter.

Assessment. Per-slice, SABR achieves lower RMSE (51 vs 75 bps) on this 4-maturity SPY data. This is expected: per-maturity calibration with dedicated parameters always wins on in-sample fit when data is sparse. The controlled comparison (Section 9.5) demonstrates that on richer surfaces, rough Heston matches or beats SABR in-sample while providing 6.5× better extrapolation. The structural advantages — arbitrage-free interpolation, consistent cross-maturity Greeks, dynamic hedging of path-dependent products — are independent of the in-sample fit gap.

Practical recommendation. For desks with existing SABR infrastructure, the SABR enhancement layer (Section 9.3) provides immediate hedging improvement with one additional parameter. For desks building new pricing infrastructure or trading exotic products, the rough Heston engine provides a dynamically consistent framework with sub-second calibration.

9.7 Monte Carlo Engine for Exotics

Path-dependent payoffs (barriers, Asians, variance swaps) are priced via Monte Carlo simulation of the rough Heston variance process:

$$V(t) = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[\kappa(\theta - V(s))\, ds + \eta\sqrt{V(s)}\, dW_2(s)\right]$$

The Volterra integral is discretized via Euler–Maruyama with pre-generated correlated Brownian increments ($\langle dW_1, dW_2 \rangle = \rho\, dt$). Parallelization across paths via Rayon provides near-linear scaling with CPU cores.

9.8 Performance

Operation	Latency	Throughput
Single option price	15 ms	65/sec
Full surface (186 pts)	16 ms	11,600 pts/sec
Adjusted Greeks	0.02 ms	50,000/sec
Calibration cold-start (60 quotes, DE+NM+LM)	1.2 s	—
Calibration warm-start (60 quotes, NM+LM)	0.85 s	—
Calibration cold-start (146 quotes, DE+NM+LM)	< 5 s	—
MC exotic (50K paths)	4 s	—

The three orders-of-magnitude speedup from the previous version (7 minutes → 1.2 seconds) comes from the optimizations described in Section 9.2: precomputed Volterra kernel weights eliminate $O(N^2)$ transcendental function calls, batch characteristic function evaluation amortizes the kernel across COS frequencies, and maturity-grouped calibration shares the batch CF across strikes. The Gauss-Newton LM stage (Section 9.4) provides quadratic convergence in the final polish, typically requiring 3–5 iterations to reach machine precision.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, formalization, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Complete Theorem List

For reference, all 125 theorems and their proof methods:

#	Name	Statement	Method
1	skew_exp_identity	$(h+1/2)-1 = h-1/2$	ring
2	curvature_skew_relation	$2(h-1/2) = 2h-1$	ring
3	skew_half_curvature	$h-1/2 = (2h-1)/2$	ring
4	heston_exp_zero	$1/2-1/2 = 0$	ring
5	conv_exp_identity	$h+1/2 = 1-(1/2-h)$	ring
6	kernel_singular	$H < 1/2 \to H-1/2 < 0$	linarith
7	kernel_integrable	$H > 0 \to H-1/2 > -1$	linarith
8	integral_exp_pos	$H > 0 \to H+1/2 > 0$	linarith
9	integral_exp_sublinear	$H < 1/2 \to H+1/2 < 1$	linarith
10	two_H_lt_one	$H < 1/2 \to 2H < 1$	linarith
11	volterra_alpha_gt_neg_one	$H > 0 \to H-1/2 > -1$	linarith
12	volterra_exp_shift	$(H-1/2)+1 = H+1/2$	ring
13	volterra_numerator_pos	$T > 0 \to T^{H+1/2} > 0$	exact
14	volterra_result_pos	$H > 0, T > 0 \to T^{H+1/2}/(H+1/2) > 0$	nlinarith
15	skew_power_law	$\text{se} = H-1/2$	linarith
16	skew_exp_negative	$\text{se} < 0$	linarith
17	skew_constant_finite	$H+1/2 > 0$	linarith
18	rough_steeper_than_heston	$H-1/2 < 0$	linarith
19	skew_monotone_in_H	$h_1 < h_2 \to \text{skew}(h_1) < \text{skew}(h_2)$	linarith
20	empirical_exponent	$1/10-1/2 = -2/5$	ring

#	Name	Statement	Method
21	curvature_power_law	$\text{ce} = 2H - 1$	exact_apply
22	curvature_negative	$\text{ce} < 0$	linarith
23	curvature_is_double_skew	$\text{ce} = 2 \cdot \text{se}$	linarith
24	conv_better_than_mc	$H + 1/2 > 1/2$	linarith
25	conv_slower_than_heston	$H + 1/2 < 1$	linarith
26	conv_deficit	$\text{cr} = 1 - (1/2 - H)$	linarith
27	hawkes_H_positive	$\alpha > 1/2 \to H > 0$	linarith
28	hawkes_H_rough	$\alpha < 1 \to H < 1/2$	linarith
29	near_critical_subcritical	$n = 1 - \varepsilon, \varepsilon > 0 \to n < 1$	linarith
30	branching_positive	$0 < \varepsilon < 1 \to 1 - \varepsilon > 0$	linarith
31	neg_bias_underestimate	$b < 0 \to \hat{H} < H_{\text{true}}$	linarith
32	bias_insufficient	$n > 3C \to C/n < 1/2$	nlinarith
33	feller_lhs_positive	$\kappa > 0, \theta > 0 \to 2\kappa\theta > 0$	nlinarith
34	feller_bound	$\nu^2 < 2\kappa\theta$	exact_apply
35	moment_exp_positive	$H > 0, n > 0 \to 2Hn > 0$	nlinarith
36	moment_exp_sublinear	$H < 1/2, n > 0 \to 2Hn < n$	nlinarith
37	variance_scaling_exp	$2H \cdot 1 = 2H$	ring
38	fourth_moment_exp	$2H \cdot 2 = 4H$	ring
39	moment_step_constant	$2H(n+1) - 2Hn = 2H$	ring
40	moment_ordering	$n_1 < n_2, H > 0 \to 2Hn_1 < 2Hn_2$	nlinarith
41	kurtosis_exp_zero	$4H - 2 \cdot 2H = 0$	ring
42	fwd_slope_exp	$2H - 2 = (2H - 1) - 1$	ring
43	fwd_slope_singular	$H < 1/2 \to 2H - 2 < -1$	linarith
44	fwd_slope_negative	$H < 1 \to 2H - 2 < 0$	linarith
45	fwd_slope_rougher_than_heston	$H < 1/2 \to$ slope rough $<$ slope Heston	linarith
46	fwd_slope_monotone	$h_1 < h_2 \to \text{slope}(h_1) < \text{slope}(h_2)$	linarith
47	empirical_slope_exp	$2(1/10) - 2 = -9/5$	ring
48	delta_correction_negative	$H < 1/2 \to$ delta correction < 0	linarith
49	hedge_error_is_curvature	hedge error exp $= 2H - 1$	linarith
50	hedge_error_diverges	$H < 1/2 \to$ hedge error exp < 0	linarith
51	hedge_error_double_skew	hedge error exp $= 2 \cdot$ skew exp	linarith
52	hedge_error_equals_curvature	hedge error exp $=$ curvature exp	linarith

#	Name	Statement	Method
53	hedge_error_monotone	$h_1 < h_2 \to \text{error}(h_1) < \text{error}(h_2)$	linarith
54	heston_hedge_error_zero	$2(1/2) - 1 = 0$	ring
55	rough_hedge_worse_than_heston	$H < 1/2 \to 2H - 1 < 0$	linarith
56	empirical_hedge_error_exp	$2(1/10) - 1 = -4/5$	ring
57	hedge_error_less_singular_than_2nd	$2H - 2 < 2H - 1$	linarith
58	hedge_error_more_singular_than_skew	$H < 1/2 \to 2H - 1 < H - 1/2$	linarith
59	exponent_hierarchy	$2H - 2 < 2H - 1$ (full hierarchy)	linarith
60	hedge_error_ratio_exp	$2(H - 1/2) = 2H - 1$	ring
61	magnification_exponent	$1 - 2(1/10) = 4/5$	ring
62	model_risk_measure	$(2H - 1) - 0 = 2H - 1$	ring
63	vega_correction_is_curvature	vega correction $= 2H - 1 =$ curvature	linarith
64	vega_equals_delta_hedge	vega correction = delta hedge error exp	linarith
65	vega_correction_negative	$H < 1/2 \to$ vega correction < 0	linarith
66	delta_vega_same_exponent	delta error exp = vega error exp	linarith
67	empirical_vega_error_exp	$2(1/10) - 1 = -4/5$	ring
68	rebal_exp_identity	$2(1 - H) = 2 - 2H$	ring
69	rebal_exp_gt_one	$H < 1/2 \to 2(1 - H) > 1$	linarith
70	rebal_exp_lt_two	$H > 0 \to 2(1 - H) < 2$	linarith
71	rebal_monotone	$h_1 < h_2 \to \text{rebal}(h_1) > \text{rebal}(h_2)$	linarith
72	heston_rebal_linear	$2(1 - 1/2) = 1$	ring
73	empirical_rebal_exp	$2(1 - 1/10) = 9/5$	ring
74	rebal_ratio_exp	rebal ratio exp $= 1 - 2H$	ring
75	rebal_ratio_neg_curvature	$1 - 2H = -(2H - 1)$	ring
76	rho_sq_positive	$\rho < 0 \to \rho^2 > 0$	nlinarith
77	rho_sq_quarter_pos	$\rho < 0 \to \rho^2/4 > 0$	nlinarith
78	hedge_bound_positive	$(\rho^2/4)C_{\text{curv}} > 0$	nlinarith
79	curvature_predicts_hedge	$C_1 < C_2 \to \text{bound}(C_1) < \text{bound}(C_2)$	nlinarith
80	leverage_tightens_bound	$\rho_1 < \rho_2 < 0 \to \rho_1^2 > \rho_2^2$	nlinarith
81	vs_var_positive	$H > 0 \to$ Var(VS) exp > 0	linarith
82	vs_var_sublinear	$H < 1/2 \to 2H < 1$	linarith
83	vs_slope_is_hedge_error	VS slope exp $= 2H - 1 =$ hedge exp	ring

#	Name	Statement	Method
84	vs_sd_exp	SD(VS rate) exp = H	ring
85	vs_excess_rough	$2H - 1 < 0$ [rough > Heston at short T]	linarith
86	empirical_vs_var_exp	$2(1/10) = 1/5$	ring
87	fwd_var_worse_than_inv_T	$H < 1/2 \to 2H - 2 < -1$	linarith
88	singularity_gap_is_hedge	$(2H - 2) - (-1) = 2H - 1$	ring
89	calendar_tightens_with_roughness	$h_1 < h_2 \to$ singularity$(h_1) <$ singularity(h_2)	linarith
90	empirical_singularity_exp	$2(1/10) - 2 = -9/5$	ring
91	fwd_curve_non_integrable	$H < 1/2 \to 2H - 2 \leq -1$	linarith
92	non_integrability_gap	$(2H - 2) - (-1) < 0$	linarith
93	var_exp_is_skew	$(2H - 1)/2 = H - 1/2 =$ skew exp	ring
94	var_exp_negative	$H < 1/2 \to$ VaR exp < 0	linarith
95	var_half_hedge	$2(H - 1/2) = 2H - 1$ [$2 \cdot$ VaR = hedge]	ring
96	empirical_var_exp	$(2(1/10) - 1)/2 = -2/5$	ring
97	atlas_curvature_from_skew	$2(\alpha + 1/2) - 1 = 2\alpha$	ring
98	atlas_fwd_from_skew	$2(\alpha + 1/2) - 2 = 2\alpha - 1$	ring
99	atlas_rebal_from_skew	$1 - 2\alpha = -(2\alpha - 1)$	ring
100	atlas_vs_from_skew	$2(\alpha + 1/2) = 2\alpha + 1$	ring
101	smile_nth_exp	$n(h - 1/2) = nh - n/2$	ring
102	curv_skew_ratio_exp	$(2h - 1) - (h - 1/2) = h - 1/2$	ring
103	smile_consecutive_gap	$(n + 1)(h - 1/2) - n(h - 1/2) = h - 1/2$	ring
104	expansion_radius_pos	$1/2 - H > 0$	linarith
105	expansion_radius_neg_skew	$1/2 - h = -(h - 1/2)$	ring
106	cubic_smile_exp	$3(h - 1/2) = 3h - 3/2$	ring
107	transition_moneyness_exp	$1/2 - h = -(h - 1/2)$	ring
108	radius_monotone_identity	$(1/2 - h_1) - (1/2 - h_2) = h_2 - h_1$	ring
109	skew_H_sensitivity	$(h + \Delta h - 1/2) - (h - 1/2) = \Delta h$	ring
110	hedge_H_sensitivity	$(2(h + \Delta h) - 1) - (2h - 1) = 2\Delta h$	ring
111	hedge_sens_double_skew_sens	$2\Delta h = 2\Delta h$	ring
112	pricing_error_grows_with_dH	$\Delta H_1 < \Delta H_2$	linarith
113	heston_misspec_gap	$1/2 - H > 0$	linarith
114	gap_cancels_skew	$(1/2 - h) + (h - 1/2) = 0$	ring

#	Name	Statement	Method
115	fwd_slope_H_sensitivity	$(2(h + \Delta h) - 2) - (2h - 2) = 2\Delta h$	ring
116	atlas_sensitivity_sum	$1 + 2 + 2 + 2 + (-2) + 2 + 1 = 8$	ring
117	cross_model_divergence_exp	$s = H - 1/2$	exact_apply
118	divergence_short_T_amplifies	$s < 0$	exact_apply
119	divergence_monotone_identity	$(h_1 - 1/2) - (h_2 - 1/2) = h_1 - h_2$	ring
120	two_rough_divergence	$(h_1 - 1/2) - (h_2 - 1/2) = h_1 - h_2$	ring
121	hedge_divergence_double_skew	$(2h_1 - 1) - (2h_2 - 1) = 2(h_1 - h_2)$	ring
122	frac_order_in_interval	$1/2 < H + 1/2 < 1$	linarith
123	ml_decay_identity	$1/(h + 1/2) = 2/(2h + 1)$	ring
124	ml_decay_gt_one	$1/(H + 1/2) > 1$	nlinarith
125	ml_decay_lt_two	$1/(H + 1/2) < 2$	nlinarith