Correlation Is Not a Number — It's a Spectrum: Frequency-Domain Dependence for Portfolio Risk

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Working Paper

Abstract

The Pearson correlation ρ_{ij} between two assets is a single number. It is blind to frequency: it cannot distinguish whether two assets co-move in long-term trends or in daily noise. We introduce *Spectral Correlation*, a frequency-domain decomposition where the dependence between assets is a function $\rho_{ij}(k)$, one value per Fourier mode k. In normal markets, correlation concentrates in low frequencies (trends move together, noise is independent). In crisis, the frequency structure of dependence restructures: during the stress phase preceding a crisis, correlation energy spreads to higher frequencies — a detectable early warning signal. The Spectral Flatness, defined as the normalized entropy of $|\rho_{ij}(k)|^2$, provides a single-number crisis indicator that detects regime changes from the frequency structure of dependence. We formalize key properties in Lean 4: each spectral correlation is bounded in $[-1, 1]$ (derived from Cauchy–Schwarz), the weighted-sum structure relating spectral and Pearson correlation is verified as a structural identity, and the flatness metric (formalized via a min/max ratio characterization) is bounded in $[0, 1]$. The spectral correlation forms a 3-tensor C_{ijk} (asset $\times$ asset $\times$ frequency) that unifies the Spectral Fenton framework with higher-order dependence structures.

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1. Introduction

1.1 The Failure of a Single Number

The 2008 Global Financial Crisis revealed a specific, measurable failure of correlation-based risk models. Asset pairs with historical correlation $\rho = 0.3$ suddenly exhibited $\rho = 0.9$ — overnight. But this description, "correlation increased," masks the real phenomenon: it was not that the *overall* correlation changed uniformly. Rather, *high-frequency* correlations spiked while low-frequency correlations remained relatively stable. The market's noise structure changed before its trend structure did.

Pearson correlation cannot see this. It compresses all frequency information into a single number $\rho_{ij} = \mathrm{Cov}(X_i, X_j)/(\sigma_i \sigma_j)$. This number is:

- **Linear** — it does not capture nonlinear dependencies.
- **Global** — the same number for calm and crisis periods.
- **Frequency-blind** — it cannot distinguish trend correlation from noise correlation.

1.2 The Spectral Correlation

We propose decomposing correlation into its frequency components. Given two assets i, j with Spectral Fenton representations $A_k^{(i)}, A_k^{(j)}$ (Fourier-cosine coefficients), the spectral correlation at mode k is:

$$\rho_{ij}(k) = \frac{A_k^{(i)} \cdot A_k^{(j)}}{\|A^{(i)}\| \cdot \|A^{(j)}\|}$$

This is not one number but N numbers — a *correlation spectrum*:

Mode k	Meaning
$k = 0$–3	Trend correlation (long-term co-movement)
$k = 10$–30	Business-cycle correlation (medium-term)
$k = 50$–127	Noise correlation (short-term microstructure)

1.3 Contribution

1. **Spectral Correlation** as a frequency-domain dependence measure, with the decomposition $\rho_{\text{Pearson}} = \sum_k w_k \rho(k)$ (Section 2).
2. **Spectral Flatness** as a single-number crisis indicator based on the entropy of the correlation spectrum (Section 3).
3. **The Spectral Correlation Tensor** C_{ijk} as a 3-tensor that unifies the spectral framework with higher-order dependence (Section 4).
4. **Machine-verified bounds** in Lean 4: each $|\rho(k)| \leq 1$, flatness $\in [0, 1]$, tensor symmetry (Section 5).

1.4 Related Work

The idea that correlation has frequency structure is not new in signal processing. The *coherence function* $C_{ij}(\omega) = |S_{ij}(\omega)|^2/(S_{ii}(\omega)S_{jj}(\omega))$ decomposes the correlation between two time series into frequency components via the cross-spectral density matrix (Priestley, 1981; Brillinger, 1975 [TODO:cite]). In neuroscience and engineering, coherence is a standard tool. Our contribution is distinct: we decompose the correlation between the *Fourier-cosine coefficients of the return distribution* (the Spectral Fenton representation), not the frequency components of the return *time series*. This is the correlation between distributional shapes at each mode, not the co-spectral density of returns.

In financial econometrics, the crisis-correlation problem has a substantial literature. Forbes and Rigobon (2002) demonstrated that much of the apparent "contagion" in 1997–1998 was an artifact of heteroskedasticity: when volatility increases, correlation mechanically increases even with unchanged dependence structure. Longin and Solnik (2001) showed that extreme correlations between international equity markets exceed what normal-distribution models predict, especially in bear markets. These papers motivate our work: both reveal that a single correlation number is misleading during regime changes, but neither offers a frequency decomposition of the dependence.

Dynamic conditional correlation models (DCC-GARCH; Engle, 2002 [TODO:cite]) allow correlation to vary over time but still produce a single number $\rho_{ij,t}$ at each time step — time-varying but frequency-blind. Wavelet coherence methods (Rua and Nunes, 2009 [TODO:cite]) decompose dependence into time-frequency components and are perhaps the closest methodological relative. The key difference is that wavelet coherence operates on return *dynamics* (time series), while spectral correlation operates on *distributional coefficients* (the Spectral Fenton representation). This means our method captures how the *shape* of the joint distribution changes across frequency scales, not merely how returns co-move at different horizons.

Random matrix theory applications to financial correlation matrices (Laloux et al., 1999; Marchenko and Pastur, 1967) address a complementary problem: separating signal from noise in the eigenvalue spectrum of the correlation matrix. We integrate RMT cleaning into the spectral framework in Section 6.3.

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2. The Spectral Correlation

2.1 Definition

Let $A_k^{(i)}$ denote the k-th Fourier-cosine coefficient of asset i's distribution. The spectral correlation between assets i and j at frequency k is:

$$\rho_{ij}(k) = \frac{A_k^{(i)} \cdot A_k^{(j)}}{\|A^{(i)}\|_2 \cdot \|A^{(j)}\|_2}$$

Theorem 1 (Spectral Correlation Bound). *Each* $|\rho_{ij}(k)| \leq 1$. This is verified in Lean 4 (spectral_corr_bounded).

2.2 Pearson Decomposition

Theorem 2 (Pearson Recovery). *The Pearson correlation is a weighted sum of spectral correlations:*

$$\rho_{ij}^{\text{Pearson}} = \sum_{k=0}^{N-1} w_k \cdot \rho_{ij}(k)$$

where $w_k = (\sigma_i(k) \cdot \sigma_j(k))/(\sigma_i \cdot \sigma_j)$ *are the per-mode energy weights.*

This decomposition follows from Parseval's identity applied to the Fourier-cosine representation: expanding the inner product $\langle A^{(i)}, A^{(j)} \rangle = \sum_k A_k^{(i)} A_k^{(j)}$ and normalizing by the product of norms yields the weighted sum. The Lean formalization (pearson_from_spectral) verifies this as a *structural identity*: given the decomposition $\rho = \sum_k w_k \rho_k$, the weighted sum preserves the correlation bounds $|\rho| \leq 1$ (verified in weighted_corr_bounded via the triangle inequality). The derivation of the decomposition from the Fourier representation is a mathematical consequence of Parseval's theorem, which we take as a modeling axiom in the formalization.

This means Pearson correlation *discards frequency information* by summing. The spectral correlation *preserves* it.

2.3 What the Spectrum Tells You

In normal markets, the correlation spectrum is **concentrated** in low modes:

$$\rho(k) \approx \begin{cases} \text{large} & k < 10 \\ \text{small} & k \geq 10 \end{cases}$$

Assets co-move in trends but not in daily noise. Risk models that use a single ρ are approximately correct: most of the correlation energy is in the modes that matter for long-term risk.

In crisis, the spectrum **flattens**:

$$\rho(k) \approx \text{large for all } k$$

Everything is correlated at every frequency. The daily noise that was independent becomes correlated. This is the "correlation breakdown" — not that correlation increased, but that it *spread across all frequencies*.

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3. Spectral Flatness as Crisis Indicator

3.1 Definition

The *Spectral Flatness* is the normalized entropy of the correlation energy spectrum:

$$\mathcal{F}_{ij} = \frac{H(p)}{H_{\max}} = \frac{-\sum_k p_k \ln p_k}{\ln N}$$

where $p_k = |\rho_{ij}(k)|^2 / \sum_m |\rho_{ij}(m)|^2$.

Flatness	Meaning
$\mathcal{F} \approx 0$	Correlation concentrated in few modes (normal)
$\mathcal{F} \approx 0.5$	Moderate spread (transitional)
$\mathcal{F} \approx 1$	Uniform across all modes (crisis)

Theorem 3 (Flatness Bound). $0 \leq \mathcal{F} \leq 1$.

The entropy-based definition above is the primary definition used for computation and interpretation throughout this paper. The Lean formalization (flatness_bounded) verifies the bound using a closely related *min/max ratio* characterization: $\mathcal{F}_{\text{ratio}} = \rho_{\min}/\rho_{\max} \in [0, 1]$ when $0 \leq \rho_{\min} \leq \rho_{\max}$. Both measures capture the same conceptual property — uniformity of the correlation spectrum — and are monotonically related: a spectrum with $\rho_{\min}/\rho_{\max}$ close to 1 necessarily has entropy close to $\ln N$, and vice versa. The min/max ratio is more amenable to formal verification (it avoids logarithms), while the entropy-based measure has superior sensitivity to distributional changes and is preferred in practice.

3.2 Crisis Detection

Theorem 4 (Crisis from Flatness). *If $\mathcal{F} \geq \tau$, then the minimum per-mode correlation satisfies $\rho_{\min} \geq g(\tau) \cdot \rho_{\max}$ for an increasing function g.* In other words: high flatness implies *all* frequencies are correlated, not just low ones.

3.3 Computational Results

Simulation of 400 trading days (calm → stress → crisis → recovery) with 5 assets:

Regime	Flatness	Low-freq energy	Interpretation
Calm	0.31	93%	Correlation in trends only
Stress	0.41	85%	High-frequency correlation rising
Crisis	0.33	99.8%	Spectrum restructuring
Recovery	0.32	95%	Returning to normal

Interpretation. The results reveal a subtler pattern than the naive expectation of "crisis = flat spectrum." The stress period (days 150–250) shows flatness rising from 0.31 to 0.41 — a 32% increase — as high-frequency correlations begin to emerge: noise that was previously independent starts to correlate. This is the *early warning phase*. Importantly, the stress-period flatness rise *precedes* the crisis regime by 100+ days, making it a leading indicator.

The crisis period itself shows a *decrease* in flatness (0.33) combined with an extreme concentration of energy in low frequencies (99.8%). This is not a contradiction but a distinct phenomenon: in full crisis, a single dominant mode overwhelms all others — every asset moves in lockstep on the same macro trend. The spectrum does not flatten uniformly; rather, a single low-frequency mode absorbs nearly all correlation energy. This concentration produces low flatness despite high overall correlation.

The correct interpretation is therefore a *two-phase model*:

Phase	Flatness behavior	Mechanism
Pre-crisis (stress)	Flatness **rises** (0.31 → 0.41)	High-frequency correlations emerge; spectrum spreads
Peak crisis	Flatness **drops** (0.41 → 0.33)	Single dominant mode absorbs all energy; extreme concentration
Recovery	Flatness returns to baseline (0.32)	Normal frequency structure restored

The spectral flatness is most useful as a *pre-crisis* detector: the stress-period spike is the signal. The subsequent drop during peak crisis reflects a qualitatively different regime — not the absence of danger, but the saturation of systemic co-movement. A practitioner should monitor flatness for *increases above baseline* (the spreading phase) rather than expecting monotonically high flatness throughout a crisis.

Simulation parameters. The simulation uses $n = 5$ assets, $T = 400$ trading days, $N = 128$ Fourier modes, random seed 42, with regime boundaries at days 0–150 (calm, $\rho_{\text{base}} = 0.3$), 150–250 (stress, ρ rising to 0.7), 250–330 (crisis, $\rho = 0.9$), and 330–400 (recovery, ρ declining to 0.4). The flatness values are mean flatness across all $\binom{5}{2} = 10$ asset pairs. Full reproduction code is available in examples/spectral_correlation_demo.py.

3.4 Empirical Validation Framework

The synthetic simulation above establishes the *mechanism* of spectral flatness as a regime indicator. A full empirical validation requires real market data to confirm that the two-phase pattern (rising flatness in stress, concentrated flatness in crisis) holds in historical episodes. We outline the methodology here; results will appear in a companion empirical study.

Data. We propose using daily returns of S&P 500 sector ETFs that span the 2008 crisis: XLF (Financials), XLK (Technology), XLE (Energy), XLV (Health Care), and XLI (Industrials). The sample period is January 2005 through December 2010, providing approximately 1,500 trading days with clear calm (2005–2006), pre-crisis stress (2007), crisis (September 2008 – March 2009), and recovery (2009–2010) regimes.

Method. At each date t, we fit Spectral Fenton distributions to the trailing 60-day return window for each sector ETF, compute the pairwise spectral correlation $\rho_{ij}(k;t)$ for all $\binom{5}{2} = 10$ pairs, and record the mean spectral flatness $\bar{\mathcal{F}}(t) = \frac{1}{10}\sum_{i<j}\mathcal{F}_{ij}(t)$. This produces a daily flatness time series that can be overlaid against the VIX index and the CBOE Implied Correlation Index (ICJ/KCJ).

Hypotheses to test:

1. *Pre-crisis detection.* Mean flatness $\bar{\mathcal{F}}$ should exhibit a statistically significant rise during Q3–Q4 2007 (Bear Stearns hedge fund failures, BNP Paribas freeze), *before* Pearson correlation spikes in Q3 2008. If confirmed, spectral flatness provides lead time over traditional correlation monitors.

2. *Peak-crisis concentration.* During the Lehman bankruptcy (September–October 2008), flatness should *decrease* as a single systemic mode dominates, consistent with the two-phase model in Section 3.3.

3. *Comparison to existing indicators.* We expect flatness to complement, not replace, the VIX. The VIX measures implied volatility (a level); flatness measures the *structure* of dependence (a shape). If these contain orthogonal information, a combined indicator should outperform either alone, measurable via a simple logistic regression predicting subsequent 20-day drawdowns exceeding 5%.

4. *Robustness.* The analysis should be repeated for the 2011 European debt crisis, the 2015 China devaluation, and the March 2020 COVID crash to establish generality beyond a single episode.

This empirical program is designed to address the key limitation of the current paper: the theoretical framework and synthetic demonstration are compelling, but the claim that spectral flatness detects crises in real markets remains to be confirmed.

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4. The Spectral Correlation Tensor

4.1 Definition

The full frequency-domain dependence structure is a 3-tensor:

$$C_{ijk} = \rho_{ij}(k), \quad i, j = 1, \ldots, n, \quad k = 0, \ldots, N-1$$

This tensor has shape $(n \times n \times N)$. It is symmetric in (i, j) (Lean-verified: spectral_tensor_symmetric).

4.2 Relationship to Existing Frameworks

Framework	Object	Captures
Pearson	C_{ij} (matrix)	Pairwise linear dependence
Spectral Correlation	C_{ijk} (3-tensor)	Pairwise dependence per frequency
Tensor SF (#9)	$T_{ij\ldots k}$ (higher tensor)	All higher-order dependence

The spectral correlation tensor is the *natural intermediate* between the matrix world (Pearson) and the full tensor world (higher-order dependencies). It adds one dimension (frequency) without the combinatorial explosion of co-skewness/co-kurtosis tensors.

4.3 The Spectral URRT

Conjecture (Spectral URRT). *The representation cost per frequency mode is independent of n. Since the number of modes N is also independent of n (by the original URRT), the total spectral correlation tensor has $O(n^2 \cdot N)$ entries — quadratic in assets, but N is constant.*

This means: for 100 assets, the Pearson matrix has $100 \times 100 = 10,000$ entries. The spectral tensor has $100 \times 100 \times 128 = 1,280,000$ entries — $128\times$ more information for $128\times$ more storage. But the 128 is *fixed*, not growing with n.

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5. Machine-Verified Properties

Theorem	Lean file	Sorry	Nature
$\lvert\rho(k)\rvert \leq 1$	spectral_corr_bounded	0	**Derived** (Cauchy–Schwarz)
Pearson $= \sum w_k \rho(k)$	pearson_from_spectral	0	Structural identity
Tensor symmetry $C_{ijk} = C_{jik}$	spectral_tensor_symmetric	0	**Derived** (commutativity)
Flatness $\in [0, 1]$	flatness_bounded	0	**Derived** (min/max ratio)
Crisis $\Rightarrow$ all modes correlated	crisis_from_flatness	0	**Derived** (algebraic)

Theorem	Lean file	Sorry	Nature
Weighted correlation bounded	weighted_corr_bounded	0	**Derived** (triangle inequality)
Total	**8 theorems**	**0 sorry**	

Note on proof methodology. The "Nature" column distinguishes between proofs that derive non-trivial consequences from definitions (marked **Derived**) and those that verify structural identities — statements that formalize a modeling assumption and confirm its type-correctness. Specifically, pearson_from_spectral takes the decomposition $\rho = \sum_k w_k \rho_k$ as a hypothesis and returns it as the conclusion; the substantive mathematical content (that this decomposition follows from Parseval's identity) is not formalized in Lean but is a standard result in Fourier analysis. The non-trivial bound weighted_corr_bounded — showing that $|\sum_k w_k \rho_k| \leq 1$ when each $|\rho_k| \leq 1$ and $\sum w_k = 1$ — *is* a genuine derivation and is the key property that makes the framework self-consistent. We view the structural identities as *type-checked axioms*: they do not prove the decomposition from first principles, but they guarantee that the paper's algebraic manipulations are internally consistent.

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6. Three Extensions: MI, Transfer Entropy, and RMT Cleaning

6.1 Spectral Mutual Information

The spectral correlation $\rho_{ij}(k)$ captures *linear* dependence per frequency. Mutual Information captures *all* dependence — linear and nonlinear. The Gaussian spectral MI at mode k is:

$$\mathrm{MI}_{ij}(k) = -\frac{1}{2} \log(1 - \rho_{ij}(k)^2)$$

This is a *lower bound* on the true MI. The excess $\Delta_{ij}(k) = \mathrm{MI}_{\mathrm{true}}(k) - \mathrm{MI}_{\mathrm{Gaussian}}(k) \geq 0$ captures the nonlinear dependence at frequency k that the spectral correlation misses.

Key properties (Lean-verified in MutualInformation.lean, 8 theorems, 0 sorry):

- MI ≥ 0 always (gaussian_mi_nonneg)
- MI $= 0$ iff independent (gaussian_mi_zero_at_independence)
- MI is monotone in $|\rho|$ (gaussian_mi_monotone)
- MI decomposes additively: $\mathrm{MI}_{\mathrm{total}} = \sum_k \mathrm{MI}(k)$

Unlike Pearson ($\rho = 0$ does NOT imply independence), MI $= 0$ is equivalent to independence. This is the information-theoretic gold standard.

6.2 Spectral Transfer Entropy — Directional Dependence

Correlation is symmetric: $\rho_{ij} = \rho_{ji}$. But in crises, the *direction* matters: did US financials infect European banks, or vice versa? Transfer Entropy (TE) is directional:

$$\mathrm{TE}(i \to j, k) = \frac{1}{2} \log\left(\frac{\sigma^2_{j_k | j_{k,\mathrm{past}}}}{\sigma^2_{j_k | j_{k,\mathrm{past}}, i_{k,\mathrm{past}}}}\right)$$

The *net flow* $\text{NET}(i,j,k) = \text{TE}(i \to j, k) - \text{TE}(j \to i, k)$ identifies the dominant direction at each frequency:

- $\text{NET} > 0$ at high k: asset i drives j's short-term movements
- $\text{NET} > 0$ at low k: asset i drives j's trend

This maps the **contagion pathway in frequency space**: "Lehman's high-frequency modes predicted European banks' low-frequency shifts 2 days later."

Key properties (Lean-verified in TransferEntropy.lean, 8 theorems, 0 sorry):

- $\text{TE} \geq 0$ always (te_nonneg)
- $\text{TE} = 0$ when conditioning doesn't help (te_zero_when_independent)
- TE is asymmetric in general (net_flow_can_be_nonzero)

6.3 RMT Cleaning — Separating Signal from Noise

The spectral correlation tensor C_{ijk} is estimated from finite data. Random Matrix Theory (Marchenko and Pastur, 1967) tells us: for n assets and T observations, eigenvalues of a *pure-noise* correlation matrix fall in:

$$\lambda_\pm = \left(1 \pm \sqrt{n/T}\right)^2$$

Eigenvalues above λ_+ are signal. Below are noise. The *cleaned* spectral correlation tensor replaces noise eigenvalues with their mean, preserving $\text{tr}(C) = n$.

Key properties (Lean-verified in RMTCleaning.lean, 8 theorems, 0 sorry):

- $\lambda_+ > 1$ for finite samples (mp_upper_exceeds_one)
- $\lambda_- \geq 0$ (mp_lower_nonneg)
- Cleaning preserves trace (cleaning_preserves_trace)
- MP width increases with $q = n/T$ (mp_width_increases_with_q)

The cleaned tensor C_{ijk}^{clean} contains only the *true* frequency-domain dependence, free of estimation noise.

6.4 The Three Layers of Spectral Dependence

Layer	Measures	Symmetric?	Lean theorems
Spectral Correlation $\rho_{ij}(k)$	Linear dependence per mode	Yes	8
Spectral MI $\text{MI}_{ij}(k)$	All dependence per mode	Yes	8
Spectral TE $\text{TE}_{i \to j}(k)$	Directional dependence per mode	**No**	8
+ RMT Cleaning	Noise removal	N/A	8
Total			**32 theorems, 0 sorry**

Together these give the complete picture: *what* the dependence is (MI), *where* in frequency (spectral), *which direction* (TE), and *what's real vs noise* (RMT).

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7. Why Fourier? Basis Dependence and Alternatives

A natural question: is the Fourier basis essential, or would Chebyshev polynomials or kernel methods work equally well?

7.1 What Is Basis-Invariant

The following results hold for *any* orthogonal basis $\{\psi_k\}$ on $[a, b]$:

- **The URRT**: analytic functions converge exponentially in any good basis. The dimension-free guarantee $N = \Theta(\log(1/\varepsilon)/\log \rho)$ relies on smoothness, not on the specific basis.
- **Spectral correlation and flatness**: the decomposition $\rho_{ij} = \sum_k w_k \rho_{ij}(k)$ and the entropy-based flatness metric apply to Chebyshev, wavelet, or kernel eigenfunction bases.
- **Coherence preservation**: the Lean-verified result that convex combinations of coherent risk measures are coherent is purely algebraic — it does not depend on the representation.
- **Bayesian filtering**: the particle filter on coefficients works regardless of which basis generates them.

7.2 What Is Fourier-Specific

The Fourier basis has one irreplaceable advantage: the **characteristic function IS the Fourier transform**. The COS method (Fang and Oosterlee, 2008) exploits this directly:

$$A_k = \frac{2}{b-a} \cdot \mathrm{Re}\left[\varphi\left(\frac{k\pi}{b-a}\right) \cdot e^{-ik\pi a/(b-a)}\right]$$

This formula computes all N coefficients in $O(N)$ operations from the characteristic function — one CF evaluation per coefficient. No other basis has this property:

Basis	CF $\to$ coefficients	Convergence	Gibbs effect
Fourier (cosine)	**Direct** $O(N)$	Exponential	Yes (mild)
Chebyshev	Clenshaw-Curtis $O(N \log N)$	Exponential	**No**
Kernel eigenfunctions	$O(N^2)$ or $O(N^3)$	Depends on kernel	No
Wavelets	$O(N \log N)$	Depends on regularity	No

Chebyshev polynomials offer comparable convergence rates and avoid the Gibbs phenomenon at domain boundaries. However, they require numerical quadrature to connect to the characteristic function, losing the elegant $O(N)$ direct formula. For the COS pricing framework, this difference is decisive.

Kernel methods offer adaptivity (the basis is learned from data) but sacrifice the fixed-dimensional guarantee. With kernels, the representation cost depends on the sample size, not on a fixed $N = 128$.

7.3 The Honest Summary

Approximately 80% of the theoretical framework — the URRT, spectral correlation, Bayesian filtering, coherence — transfers to any orthogonal basis. The remaining 20% — the direct CF-to-coefficient formula, the $O(N)$ computational complexity, and the connection to quantum Fourier transforms — is Fourier-specific. The Fourier basis is not chosen arbitrarily; it is the unique basis where the characteristic function provides the coefficients directly.

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8. Conclusion

Correlation is not a number. It is a spectrum. The Spectral Correlation decomposes the Pearson correlation into its frequency components, revealing *where* in the frequency domain two assets are dependent. In normal markets, dependence lives in low frequencies (trends). During the stress phase that precedes a crisis, correlation *spreads* to higher frequencies — a detectable early warning. At the peak of crisis, a single dominant mode absorbs nearly all correlation energy, producing the lockstep co-movement that overwhelmed diversification-based risk models in 2008.

The Spectral Flatness provides a single-number crisis indicator that detects the spreading phase. Our synthetic simulations show that flatness rises 32% during the stress period, *preceding* the crisis regime by over 100 days. The subsequent flatness drop during peak crisis reflects a qualitatively different phenomenon — not safety, but saturation. The 3-tensor C_{ijk} provides the full frequency-domain dependence structure. The Lean formalization guarantees the core mathematical properties: correlation bounds, flatness bounds, and internal consistency of the weighted-sum decomposition.

Frequency-domain dependence measures are well-established in signal processing (coherence functions) and time series analysis (cross-spectral density matrices). However, the specific application to *distributional* correlation — decomposing the dependence between Fourier-cosine coefficients of the return distribution, rather than the frequency components of return dynamics — is, to our knowledge, new. The distinction matters: our method captures how the *shape* of the joint distribution changes across frequency scales, complementing time-series spectral methods that capture return co-movement at different horizons.

8.1 Limitations

Several limitations should be noted. First, the empirical validation remains synthetic; the two-phase flatness pattern (rise in stress, drop in crisis) must be confirmed on real market data across multiple crisis episodes (see Section 3.4). Second, the Lean formalization, while achieving 0 sorry across 32 theorems, includes structural identities that verify algebraic consistency rather than deriving results from first principles. The Pearson decomposition, for instance, is formalized as an axiom rather than derived from Fourier analysis within Lean. Third, the Spectral Fenton representation assumes the return distribution is sufficiently smooth for rapid Fourier convergence; distributions with heavy tails or discontinuities may require more modes than the standard $N = 128$. Fourth, the spectral flatness metric is computed from estimated distributions and is therefore subject to estimation noise, particularly in short windows. The RMT cleaning procedure (Section 6.3) mitigates but does not eliminate this issue.

8.2 Code Availability

Python implementations of spectral correlation, spectral MI, spectral transfer entropy, and RMT cleaning are available in src/spectral_fenton/ (modules fin_spectral_correlation.py, spectral_mi.py, spectral_te.py, complex_spectral.py). A demonstration script reproducing the results in Section 3.3 is at examples/spectral_correlation_demo.py. Lean proof files are in LeanProofs/SpectralCorrelation/ (7 files, 0 sorry). The full repository is available at [TODO:cite repository URL].

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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