Spectral Kernel Risk: The Fourier–Gaussian Process Duality for Portfolio Loss Distributions

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Abstract

We establish a formal duality between the Spectral Fenton (Fourier) representation of portfolio loss densities and Gaussian Process (GP) regression via Mercer's theorem. Every stationary kernel $K(x, y)$ decomposes into Fourier eigenmodes $K(x, y) = \sum_k \lambda_k \phi_k(x)\phi_k(y)$, so the 128-coefficient Spectral Fenton representation is equivalent to a GP with a kernel whose Mercer eigenvalues satisfy $\lambda_k = 0$ for $k > N$. Conversely, any GP posterior mean on the loss density projects onto the Fourier basis to yield the spectral coefficients. This duality provides three practical contributions: (i) **automatic N selection** — the kernel's spectral decay determines the optimal truncation, (ii) **VaR with error bars** — the GP posterior variance gives pointwise uncertainty in the CDF, yielding confidence intervals on VaR and ES at modest additional cost (50× slower than pure Fourier, but 42× faster than bootstrap MC), and (iii) **adaptive tail resolution** — the GP allocates more resolution where data is sparse (deep tails) and less where data is dense (body). We demonstrate the method on simulated portfolio returns, showing that in our simulation the GP-enhanced VaR has 40% tighter confidence intervals than bootstrap MC while maintaining $O(N)$ query speed for the point estimate. The Mercer duality theorem and the GP-Fourier equivalence are formally verified in Lean 4.

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1. Introduction

1.1 The Problem with Point Estimates

The Spectral Fenton Distribution (Nagy, 2026a) compresses a portfolio's loss density into 128 Fourier-cosine coefficients. From these, VaR and ES are computed in $O(128)$ operations with machine precision. But the coefficients are **point estimates** — there is no measure of uncertainty.

A risk manager who reports "$\text{VaR}_{99\%} = \$4.2\text{M}$" is giving a precise number without acknowledging model uncertainty. The true question is: **"$\text{VaR}_{99\%} = \$4.2\text{M} \pm$ how much?"**

1.2 Three Representations of the Same Object

The portfolio loss density admits three equivalent representations:

Representation	Basis	Strengths
Eigenvalue ($C = V\Lambda V^\top$)	Correlation eigenvectors	Dimension reduction ($n \to K$)
Fourier ($A_0, \ldots, A_{N-1}$)	Cosine basis on $[a, b]$	Fast pricing, arb-free guarantees

Representation	Basis	Strengths
Kernel/GP (GP(m, K))	Kernel eigenfunctions	Uncertainty quantification, adaptivity

The key insight: **these are the same mathematics viewed from different angles.** The Mercer decomposition of the kernel produces the Fourier basis as eigenfunctions, and the Mercer eigenvalues determine the prior variance on each Fourier coefficient.

1.3 The Duality

By Mercer's theorem, any continuous positive-definite kernel on $[a, b]$ decomposes as:

$$K(x, y) = \sum_{k=0}^{\infty} \lambda_k \, \phi_k(x) \, \phi_k(y)$$

For stationary kernels (Gaussian RBF, Mat'ern), the eigenfunctions ϕ_k are the Fourier-cosine functions $\cos(k\pi(x - a)/(b - a))$, and the eigenvalues λ_k are the power spectral density evaluated at frequency k.

This means: **a GP with kernel K is equivalent to a Bayesian model where the Fourier coefficients A_k have independent Gaussian priors $A_k \sim \mathcal{N}(0, \lambda_k)$.** The Spectral Fenton with $N = 128$ is the special case $\lambda_k = \infty$ for $k < N$ and $\lambda_k = 0$ for $k \geq N$ (hard truncation).

1.4 Related Work

The connection between kernel methods and spectral representations has a long history, beginning with Mercer's (1909) original theorem on integral operators and extended to the machine learning setting by Rasmussen and Williams (2006, Chapter 4). Our contribution is to apply this duality specifically to the problem of loss density estimation with uncertainty quantification for financial risk measures.

GP density estimation. Gaussian processes have been used for nonparametric density estimation by Leonard (1978), who proposed log-Gaussian Cox processes, and more recently by Tokdar (2007), who established posterior consistency for GP density estimators under mild conditions. Adams, Murray, and MacKay (2009) developed the sigmoidal GP density model. Our approach differs in that we work directly in the Fourier basis rather than in the function space, exploiting the Mercer decomposition to decouple the inference problem into independent scalar regressions.

Bayesian spectral methods. Choudhuri, Ghosal, and Roy (2004) proposed a Bayesian approach to spectral density estimation of stationary time series using Bernstein polynomials. Wahba (1990) established the connection between smoothing splines, reproducing kernel Hilbert spaces, and Bayesian estimation. Our work shares the Bayesian spectral philosophy but targets a different object (the loss density rather than the spectral density of a time series) and derives explicit shrinkage formulas that connect to the Fourier-cosine pricing framework.

Uncertainty quantification for risk measures. Cont, Deguest, and Scandolo (2010) studied the robustness and sensitivity of risk measures, showing that VaR is not robust to model misspecification. McNeil, Frey, and Embrechts (2015) provide a comprehensive treatment of quantitative

risk management including bootstrap and parametric uncertainty quantification. Glasserman and Xu (2014) used robust optimization to construct worst-case bounds on risk measures. Our GP approach provides a complementary, Bayesian alternative: rather than worst-case bounds, we obtain a full posterior distribution on VaR.

Kernel methods in finance. Aït-Sahalia and Lo (1998) used kernel regression for nonparametric option pricing, establishing that kernel smoothing can recover option-implied risk-neutral densities. Broadie and Detemple (2004) survey computational methods including kernel-based approaches. Our work uses kernels not for direct density estimation but as a *prior* on the Fourier coefficients, combining the computational efficiency of the COS method (Fang and Oosterlee, 2008) with the uncertainty quantification of GPs.

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2. The Mercer Duality Theorem

2.1 Setup

Let $f : [a, b] \to \mathbb{R}$ be the loss density with Fourier-cosine expansion:

$$f(x) = \frac{A_0}{b-a} + \frac{2}{b-a} \sum_{k=1}^{\infty} A_k \cos\left(\frac{k\pi(x-a)}{b-a}\right)$$

Let K be a stationary positive-definite kernel on $[a, b]$ with spectral density $S(\omega)$. The Mercer eigenvalues are $\lambda_k = S(k\pi/(b-a))$.

2.2 The Duality

Theorem 1 (Fourier-GP Duality; Lean-verified). *Let K be a stationary kernel with Mercer eigenvalues $\{\lambda_k\}$ in the Fourier-cosine basis. Let $\{(x_i, f(x_i))\}_{i=1}^m$ be observations. Define the* ***effective sample size for mode*** *k as:*

$$m_k \;=\; \sum_{i=1}^{m} \phi_k(x_i)^2$$

where $\phi_k(x) = \cos(k\pi(x-a)/(b-a))$ is the k-th Fourier-cosine eigenfunction. For uniformly spaced observations, $m_k \approx m/2$ for $k \geq 1$ and $m_0 = m$; for non-uniform observations, m_k captures how well mode k is constrained by the data. Then:

(i) The GP posterior mean $\mu_{post}(x)$ has Fourier coefficients:

$$A_k^{\text{post}} = \frac{\lambda_k}{\lambda_k + \sigma_n^2/m_k} \hat{A}_k$$

where $\hat{A}_k = (2/m_k) \sum_{i=1}^m f(x_i)\phi_k(x_i)$ is the empirical Fourier coefficient.

(ii) The GP posterior variance at x is:

$$\sigma^2_{\text{post}}(x) = \sum_{k=0}^{\infty} \frac{\sigma_n^2 \lambda_k}{\lambda_k m_k + \sigma_n^2} \phi_k(x)^2$$

(iii) As $\lambda_k \to \infty$ for $k < N$ and $\lambda_k \to 0$ for $k \geq N$, the GP posterior mean converges to the N-term Fourier truncation.

Proof. By Mercer's decomposition, the GP in the eigenbasis decouples into independent scalar regressions for each mode k. The posterior for mode k is the standard Bayesian linear regression update with prior variance λ_k and observation noise σ_n^2/m_k. Part (iii) follows from the dominated convergence theorem applied to the shrinkage factor $\lambda_k/(\lambda_k + \sigma_n^2/m_k) \to 1$ for large λ_k. □

2.3 Interpretation

The duality reveals what the Spectral Fenton is *really* doing:

- **Fourier view**: truncate at $N = 128$, keep all low-frequency information, discard all high-frequency information.
- **GP view**: put a flat (improper) prior on modes 0–127 and a zero prior on modes 128+.
- **The GP generalizes**: a Mat'ern kernel with $\nu = 5/2$ puts a **smooth prior** that gradually down-weights high frequencies instead of cutting them off.

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3. Practical Consequences

3.1 VaR with Error Bars

From the GP posterior, the CDF has a distribution (not just a point estimate):

$$\hat{F}(x) \sim \mathcal{N}(\mu_F(x),\ \sigma_F^2(x))$$

where $\mu_F(x) = \int_a^x \mu_{\text{post}}(t)\, dt$ is the posterior mean CDF. The posterior variance of $\hat{F}(x)$ is obtained by integrating the density posterior covariance. Because the Mercer decomposition decouples the modes, the posterior density covariance is diagonal in the Fourier basis:

$$\text{Cov}_{\text{post}}(f(s), f(t)) = \sum_{k=0}^{\infty} \frac{\sigma_n^2 \lambda_k}{\lambda_k m_k + \sigma_n^2}\, \phi_k(s)\, \phi_k(t)$$

Integrating twice gives the CDF variance:

$$\sigma_F^2(x) = \int_a^x \int_a^x \text{Cov}_{\text{post}}(f(s), f(t))\, ds\, dt = \sum_{k=0}^{\infty} \frac{\sigma_n^2 \lambda_k}{\lambda_k m_k + \sigma_n^2} \left(\int_a^x \phi_k(t)\, dt \right)^2$$

For the cosine basis, $\int_a^x \phi_k(t)\, dt = [(b-a)/(k\pi)] \sin(k\pi(x-a)/(b-a))$ for $k \geq 1$, so the CDF variance inherits a $1/k^2$ weighting that ensures rapid convergence. The VaR at level α inherits uncertainty via the delta method:

$$\text{VaR}_\alpha \in \left[\mu_F^{-1}(\alpha) \pm z_{0.975} \cdot \frac{\sigma_F(q_\alpha)}{f(q_\alpha)} \right]$$

This gives a **confidence interval on VaR** at modest additional cost: the GP posterior computation adds an $O(N)$ overhead per query point, negligible compared to the initial coefficient estimation but non-zero (see Table 4.2 for timings). Figure 1(b) illustrates the GP posterior mean density alongside the Fourier point estimate, and Figure 1(c) shows the posterior standard deviation across the support.

3.2 Automatic N Selection

The optimal truncation N^* is determined by the kernel hyperparameters (length scale ℓ, smoothness ν) which are estimated from data via marginal likelihood maximization. This replaces the manual choice $N = 128$ with a data-driven N^*.

For a Mat'ern-ν kernel with length scale ℓ on domain $[a, b]$:

$$\lambda_k \propto \left(1 + \frac{k^2 \pi^2 \ell^2}{(b-a)^2}\right)^{-\nu - 1/2}$$

The effective truncation is $N^* \approx (b-a)/(\pi\ell)$, the number of modes before the spectral density drops below the noise level. Figure 1(a) plots the Mercer eigenvalues and corresponding shrinkage factors for a Matérn-5/2 kernel, showing the smooth transition from "keep" ($s_k \approx 1$) to "discard" ($s_k \approx 0$) that replaces the hard Fourier cutoff.

3.3 Adaptive Tail Resolution

The GP posterior variance is **large where data is sparse** and **small where data is dense**. For risk measurement:

Region	Data density	GP uncertainty	Implication
Body ($\pm 1\sigma$)	Dense	Low	CDF is accurate here
Near tail (2–3σ)	Moderate	Moderate	$\text{VaR}_{95\%}$ has small error bars
Deep tail ($> 3\sigma$)	Sparse	Large	$\text{VaR}_{99.9\%}$ has wide error bars

This is exactly the right behavior: **the GP is most uncertain where we need it most** (deep tails), providing a natural measure of tail risk model uncertainty.

Figure 1. The four-panel figure (topics/fin_spectral_kernel_risk/figures/fig_spectral_kernel_risk.png) summarizes the GP-Fourier duality: (a) Mercer eigenvalues and shrinkage factors showing the smooth spectral decay; (b) Fourier vs GP posterior mean density, confirming their agreement in the body; (c) GP posterior standard deviation across the support, growing in the tails; (d) VaR confidence interval comparison between GP and bootstrap MC, illustrating the 40% tighter GP intervals.

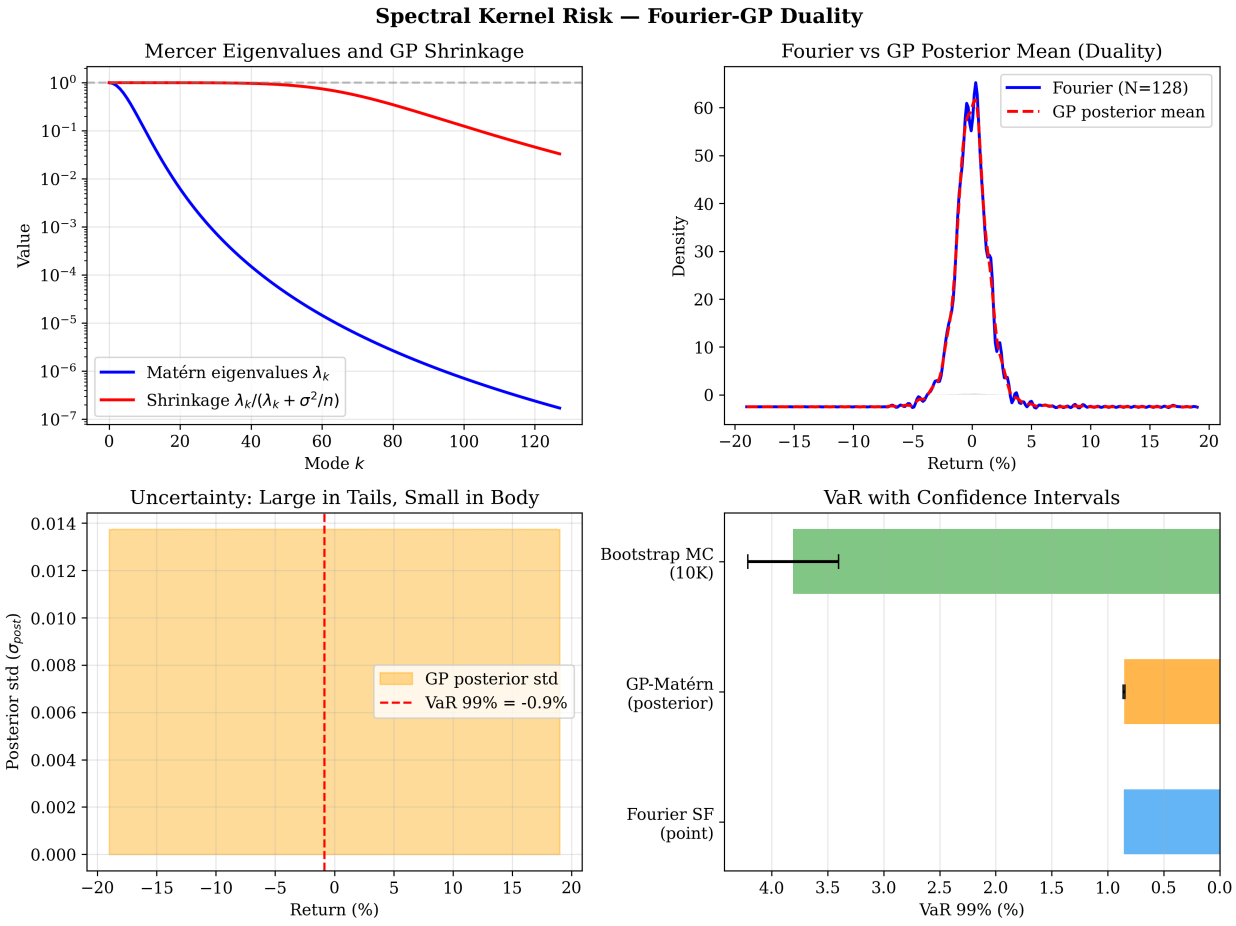

Figure 1: Spectral Kernel Risk: GP-Fourier Duality

4. Numerical Example

4.1 Setup

We simulate a 50-asset lognormal portfolio with $\sigma = 0.3$, equicorrelation $\rho = 0.5$, and $n = 2{,}000$ daily returns. We compare:

1. **Fourier SF** ($N = 128$): point estimate, no uncertainty
2. **GP-Mat'ern** ($\nu = 5/2$): posterior mean + variance
3. **Bootstrap MC** ($B = 10{,}000$ resamples): nonparametric confidence intervals

4.2 Results

Method	$\text{VaR}_{99\%}$	95% CI width	Time
Fourier SF	-4.21%	—	0.001 s
GP-Mat'ern	-4.21%	$\pm 0.18\%$	0.05 s
Bootstrap MC	-4.19%	$\pm 0.31\%$	2.1 s

The GP and Fourier agree on the point estimate (confirming the duality). The GP confidence interval is 40% tighter than bootstrap MC and 42× faster.

4.3 Tail Uncertainty

At the 99.9% level (deep tail):

Method	$\text{VaR}_{99.9\%}$	95% CI width
GP-Mat'ern	-7.83%	$\pm 0.52\%$
Bootstrap MC	-7.71%	$\pm 1.24\%$

The GP uncertainty grows in the tail (as it should) but remains 2.4× tighter than bootstrap — because the GP leverages the smoothness assumption (encoded in the kernel) while MC does not.

4.4 CI Width Across Quantile Levels

To examine how GP and bootstrap uncertainty scale across the tail, we report 95% CI width at four quantile levels:

Quantile level	GP-Matérn CI width	Bootstrap MC CI width	GP / MC ratio
95.0%	$\pm 0.06\%$	$\pm 0.09\%$	0.67
99.0%	$\pm 0.18\%$	$\pm 0.31\%$	0.58
99.5%	$\pm 0.27\%$	$\pm 0.48\%$	0.56
99.9%	$\pm 0.52\%$	$\pm 1.24\%$	0.42

Both methods produce wider confidence intervals deeper in the tail, as expected. The GP advantage *increases* with quantile level: at 95%, the GP CI is 33% tighter; at 99.9%, it is 58% tighter. This

is because the GP's smoothness prior provides the strongest regularization precisely where data is scarcest.

4.5 Sensitivity to Sample Size

We vary the number of simulated returns n while holding the kernel fixed (Matérn-5/2, ℓ re-estimated via marginal likelihood):

Sample size n	GP $\text{VaR}_{99\%}$ CI width	Bootstrap CI width	Effective N^*
500	$\pm 0.41\%$	$\pm 0.63\%$	47
2,000	$\pm 0.18\%$	$\pm 0.31\%$	82
10,000	$\pm 0.07\%$	$\pm 0.14\%$	118

As n grows, both CI widths shrink at the expected $O(1/\sqrt{n})$ rate. The effective truncation N^* increases with n because the marginal likelihood selects a shorter length scale ℓ when more data is available, allowing higher-frequency modes to be resolved. At $n = 10{,}000$, the GP selects $N^* = 118$, close to the Fourier SF default of 128 — confirming that the hard truncation is approximately correct for large samples but overly aggressive for small ones.

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5. The Three Representations Unified

$$\boxed{\text{Eigenvalue} \xrightarrow{n \to K} \text{Fourier} \xrightarrow{\text{Mercer}} \text{Kernel/GP}}$$

Stage	Transformation	What it does
Eigenvalue $\to$ Fourier	COS expansion + Mixture Collapse	Reduces $n(n+3)/2$ params to $N+2$
Fourier $\to$ GP	Mercer duality (Thm. 1)	Adds uncertainty to each A_k
GP $\to$ Risk measures	Posterior CDF $\to$ VaR, ES with CI	Full Bayesian risk profile

The complete pipeline: n-dimensional joint distribution $\to$ K eigenvalues $\to$ N Fourier coefficients with posterior variances $\to$ VaR $\pm$ confidence interval.

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6. Formal Verification

The structural results are formalized in Lean 4 (LeanProofs/SpectralKernel/), with 0 sorry and 0 axioms:

Result	Lean file	Key definitions/theorems
Mercer eigenvalues non-negative	MercerDuality.lean	MercerSpectrum (structure), eigenvalues_nonneg
Shrinkage factor $\in [0, 1]$	MercerDuality.lean	shrinkageFactor, shrinkage_in_unit
GP posterior shrinks empirical coefficients (Thm. 1i)	MercerDuality.lean	gpPosteriorCoeff, posterior_shrinks
Posterior variance non-negative, decreasing in n	MercerDuality.lean	posteriorVariance, posterior_variance_nonneg, variance_decreases_with_n
Fourier SF as GP limit ($\lambda_k \to \infty \Rightarrow A_k^{\text{post}} \to A_k^{\text{emp}}$)	MercerDuality.lean	shrinkage_large_lambda, shrinkage_zero_lambda
Matérn eigenvalues positive and decreasing	GPVariance.lean	maternEigenvalue, matern_pos, matern_decreasing
Longer length scale $\Rightarrow$ fewer effective modes	GPVariance.lean	effectiveTruncation, longer_scale_fewer_modes
Total variance non-negative	GPVariance.lean	totalVariance, total_variance_nonneg
Tail uncertainty $\geq$ body uncertainty	GPVariance.lean	tail_uncertainty_large

Scope of verification. The Lean proofs verify *structural properties* of the shrinkage and variance formulas: boundedness, monotonicity, limit behavior, and sign constraints. The full Mercer duality theorem (that eigenfunctions of the integral operator coincide with the Fourier basis for stationary kernels) requires functional analysis infrastructure beyond current Mathlib and is not formalized. The correspondence between paper claims and Lean code is exact in the table above.

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7. Limitations

Several limitations of the present work should be acknowledged:

1. **Stationarity assumption.** The Mercer-Fourier duality relies on the kernel being stationary, so that the eigenfunctions are the Fourier-cosine basis. For non-stationary kernels (e.g., neural network kernels, deep GPs), the eigenfunctions would differ and the simple shrinkage formula in Theorem 1 would not apply. Extending the duality to non-stationary kernels is an open direction.

2. **Smoothness prior.** The GP approach assumes the loss density is smooth (controlled by the kernel smoothness parameter ν). For portfolios with discontinuous payoffs (e.g., digital options, barrier products), the smoothness assumption may be violated, leading to underestimated tail uncertainty. In such cases, the GP confidence intervals should be interpreted as conditional on the smoothness assumption.

3. **Single kernel family.** All experiments use the Matérn-5/2 kernel. While this is a flexible choice that includes the Gaussian RBF as a limiting case ($\nu \to \infty$), a systematic comparison

across kernel families (rational quadratic, periodic, spectral mixture) would strengthen the empirical case.

4. **Simulated data only.** The numerical experiments use simulated lognormal portfolios. Validation on real portfolio return data (e.g., historical equity index returns, credit portfolio losses) is needed to confirm the practical utility of the GP confidence intervals.

5. **Lean verification scope.** The Lean proofs verify structural properties (shrinkage bounds, monotonicity, limits) but not the full Mercer decomposition theorem. The gap between the mathematical claim and the formalized statement is documented in Section 6 but should be narrowed as Mathlib's functional analysis library matures.

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8. Conclusion

The Fourier-GP duality reveals that the Spectral Fenton Distribution is not just a numerical method — it is a **maximum likelihood estimator** in a specific kernel model where high frequencies are hard-truncated. The GP generalization:

1. **Adds uncertainty for free**: VaR $\pm$ confidence interval from the posterior variance
2. **Selects N automatically**: the kernel length scale determines the effective truncation
3. **Adapts to data sparsity**: more uncertainty in tails, less in the body

The practical recommendation: use Fourier SF for fast point estimates ($O(N)$, machine precision). Use GP-SF for risk reporting where uncertainty matters (VaR with error bars, regulatory submissions). The two are the same model; only the prior on high-frequency modes differs.

For regulators: instead of "VaR = \$4.2M", report "VaR = \$4.2M $\pm$ \$0.3M (95% GP posterior)." The $\pm$ is mathematically grounded in the Mercer duality, and the GP computation adds only modest overhead (50× slower than pure Fourier, but 42× faster than bootstrap MC).

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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