Topological Obstructions in Market Dynamics

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Draft

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Reader-Friendly Subtitle

When globally consistent arbitrage-free dynamics are mathematically impossible.

Technical Strapline

Obstruction theorems linking global topological invariants to pricing-flow consistency failure.

Executive Summary (Non-Technical)

Some market models look valid locally but fail when extended globally across state space. This paper argues that part of this failure is topological, not just statistical or numerical.

The key contribution is an obstruction framework: under specific global structures, no model can satisfy all desired arbitrage-free consistency constraints at once.

This gives a practical diagnostic benefit: instead of endless recalibration, we can identify when the model class is structurally impossible for the target domain.

The paper does not claim all markets are topologically obstructed. It identifies explicit conditions under which obstructions emerge and when they vanish.

Abstract

We introduce a topological perspective on market dynamics and prove obstruction results: under specific global state-space constraints, no arbitrage-free dynamics satisfying given regularity and observability axioms can exist. The framework links homological invariants to pricing-flow consistency and identifies when local model validity cannot be extended globally.

1. Problem

Financial modelers routinely build locally valid models — arbitrage-free dynamics on a patch of state space — but discover inconsistencies when extending globally. For instance, a vol surface that is arbitrage-free in any local slice may still violate global no-arbitrage when extrapolated across the full term structure.

We argue that some of these failures are not numerical artifacts but topological obstructions: the global state space has nontrivial topology that prevents any locally consistent pricing flow from extending consistently. This is analogous to the Hairy Ball Theorem preventing a smooth nonvanishing vector field on S^2.

2. Setup

2.1 Market State Manifold

Let M be the state manifold of market observables (e.g., the surface of implied volatilities, or the joint term-structure of rates). We assume M is a smooth compact manifold.

2.2 Pricing Flow

A pricing dynamic is a flow on M generated by a vector field or, equivalently, represented by a 1-form $\omega \in \Omega^1(M)$.

Definition 1 (Local Arbitrage-Free). A pricing 1-form ω is locally arbitrage-free if $d\omega = 0$ (closed) on each contractible open set.

Definition 2 (Global Arbitrage-Free). ω is globally arbitrage-free if it is exact: $\omega = d\phi$ for some pricing potential $\phi : M \to \mathbb{R}$.

2.3 Topological Invariants

The obstruction to global consistency is measured by the de Rham cohomology $H^1_{\mathrm{dR}}(M)$.

Definition 3 (Cohomological Obstruction). A closed 1-form ω represents a nonzero class in $H^1_{\mathrm{dR}}(M)$ if and only if ω is not exact — i.e., there exist loops γ on M with $\oint_\gamma \omega \neq 0$.

The integral $\oint_\gamma \omega$ is the arbitrage profit along the cycle γ in state space.

3. Main Theorem

Theorem Candidate 1 (Topological Arbitrage Obstruction). Let M be a compact market state manifold with $H^1_{\mathrm{dR}}(M) \neq 0$. Then:

1. For any closed pricing 1-form ω representing a nonzero cohomology class, there exists a cycle $\gamma \subset M$ with $\oint_\gamma \omega \neq 0$ (topological arbitrage).
2. No continuous deformation of ω within the closed forms can make it exact.
3. The minimal arbitrage profit along any non-trivial cycle is:

$$\pi_{\min} = \inf_{\gamma \in H_1(M)} \left| \oint_\gamma \omega \right| > 0$$

Theorem Candidate 2 (Obstruction Vanishing). If M is contractible (e.g., a convex subset of $\mathbb{R}^n$), then $H^1(M) = 0$ and every locally arbitrage-free pricing form is globally arbitrage-free.

Corollary (Torus Obstruction). If the state manifold has periodic structure (e.g., seasonal cycles in commodity markets, forming a torus T^n), then $H^1(T^n) = \mathbb{R}^n$ and there are n independent obstruction directions.

4. Proof Sketch

1. **de Rham theorem.** A closed form is exact iff its cohomology class vanishes. When $H^1 \neq 0$, closed but non-exact forms exist.
2. **Arbitrage interpretation.** $\oint_\gamma \omega =$ net pricing adjustment around cycle. If nonzero, a round-trip in state space yields nonzero profit — a topological arbitrage.
3. **Stability.** Cohomology classes are topological invariants, so the obstruction is stable under continuous perturbation. Small-parameter deformations cannot remove it.
4. **Contractibility recovery.** When M is contractible, the Poincaré lemma applies: every closed form is exact, and global consistency is automatic.

5. Empirics/Simulation

5.1 Toy Manifolds

- Circle S^1: construct a pricing form with nonzero period. Visualize topological arbitrage.
- Torus T^2: two independent obstructions, cross-term analysis.
- Contractible manifold: verify global consistency holds.

5.2 Volatility Surface Topology

- Test whether implied vol surfaces with roll-date periodicity have effective torus structure.
- Compute approximate cohomology from discrete pricing data using persistent homology.
- Report: detected cycles, estimated arbitrage profits per cycle.

5.3 Rate Term Structure

- Multi-factor term structure model.
- Check whether the factor manifold has nontrivial first homology.
- Relate detected obstructions to known basis-trade anomalies.

6. Limits

- **Manifold assumption:** real markets may not have smooth manifold structure; discrete or fractal state spaces need separate treatment.
- **Noise:** finite-sample estimation of cohomology from data is noisy; persistent homology provides robustness but not certainty.
- **Dimensionality:** high-dimensional M makes cohomology computation expensive.
- **Model scope:** the obstruction detects structural impossibility, not pricing errors from other sources.

7. Related Work

- **Geometric finance:** Ilinski (2001) — gauge theory and finance; Malaney-Weinstein on economic geometry.
- **de Rham cohomology:** Bott-Tu (1982), Warner (1983) — foundational differential topology.
- **Topological data analysis:** Carlsson (2009), Edelsbrunner-Harer (2010) — persistent homology.
- **Arbitrage theory:** our Pricing Is Allocation and Nonlinear FTAP papers — local no-arbitrage conditions that this paper shows may not extend globally.

8. Cross-Paper Connections

- **Nonlinear FTAP (paper 2):** the SNFL condition in paper 2 is a local no-arbitrage statement. This paper shows when local SNFL cannot extend globally. Specifically: if the market state manifold has $H^1 \neq 0$, then even if SNFL holds on every contractible patch, a globally consistent nonlinear pricing kernel $\{\Pi_k\}$ may not exist.
- **Phase Transitions (paper 5):** topological changes in the state manifold (e.g., a connected component splitting, or a handle forming) correspond to topological phase transitions that are distinct from the spectral phase transitions in paper 5 but can co-occur.
- **Info Geometry Bridge (paper 6):** the spectral risk geometry lives on the manifold M. Topological obstructions constrain which risk-geometric flows can exist globally.
- **Universality (paper 1):** on a state manifold with nontrivial topology, the spectral decomposition of dynamics gains additional structure from the Laplacian eigenvalue problem on M, connecting universality classes to the spectral geometry of the manifold itself.

9. Computational Persistent Homology Pipeline

9.1 From Data to Homology

Real market data is a point cloud, not a smooth manifold. Persistent homology bridges this gap.

Pipeline: 1. **Data matrix:** collect T observations of d market observables (e.g., implied vol at different strikes and tenors). This gives a point cloud $X = \{x_1, \dots, x_T\} \subset \mathbb{R}^d$.

2. **Filtration:** build the Vietoris-Rips complex $\mathrm{VR}(X, \epsilon)$ for increasing ϵ. At each scale, simplices connect points within distance ϵ.

3. **Homology computation:** for each ϵ, compute H_0 (connected components), H_1 (loops), H_2 (voids) of $\mathrm{VR}(X, \epsilon)$.

4. **Persistence diagram:** record the birth-death pairs (b_i, d_i) for each homological feature. Long-lived features (large persistence $d_i - b_i$) indicate genuine topological structure; short-lived features are noise.

5. **Obstruction detection:** a persistent H_1 feature implies a cycle in the state manifold. If a pricing 1-form has nonzero integral around this cycle, topological arbitrage exists.

9.2 Statistical Significance

Not every persistent feature is real. Bootstrap confidence: 1. Resample the point cloud $X^{(b)}$ with replacement. 2. Recompute persistence diagram for each bootstrap sample. 3. A feature is significant at level α if it appears in $> 1 - \alpha$ of bootstrap diagrams.

9.3 Arbitrage Profit Estimation

For each significant H_1 cycle γ, estimate the line integral:

$$\hat{\pi}_\gamma = \oint_\gamma \hat{\omega} = \sum_{i=1}^{n_\gamma} \hat{\omega}(x_{i+1} - x_i)$$

where $\hat{\omega}$ is the estimated pricing 1-form (e.g., from local regression on returns).

If $|\hat{\pi}_\gamma|$ is statistically significantly different from zero, the cycle represents a detectable topological arbitrage opportunity.

10. Concrete Example: Commodity Seasonal Torus

10.1 Setup

Natural gas futures exhibit strong seasonal periodicity (winter/summer). The state is characterized by: - $\theta_1 \in S^1$: calendar month (annual cycle, period 12 months) - $\theta_2 \in S^1$: storage level relative to seasonal norm (oscillates annually)

The state manifold is approximately a torus $T^2 = S^1 \times S^1$.

10.2 Homology

$H^1(T^2) = \mathbb{R}^2$, generated by two independent cycles: - γ_1: the calendar cycle (one year traversal at fixed storage). - γ_2: the storage cycle (one storage oscillation at fixed calendar time).

10.3 Topological Arbitrage

Consider a pricing 1-form ω encoding the cost-of-carry:

$$\oint_{\gamma_1} \omega = \text{annual contango/backwardation net roll cost}$$

$$\oint_{\gamma_2} \omega = \text{storage cycle net financing cost}$$

If $\oint_{\gamma_1} \omega \neq 0$, a calendar spread strategy that "goes around" the annual cycle extracts deterministic profit — this is the well-known seasonality premium in commodities.

The topological interpretation: the seasonality premium is not a statistical anomaly but a **cohomological obstruction**. It exists because the state manifold is a torus, not a contractible space. No amount of recalibration can remove it while preserving the torus structure.

10.4 Empirical Test

- Collect 20 years of monthly natural gas futures curves.
- Build point cloud in (month, storage-deviation) coordinates.
- Compute persistent homology: expect two significant H_1 features.
- Estimate $\hat{\pi}_{\gamma_1}$ and $\hat{\pi}_{\gamma_2}$ from rolling PnL of calendar-spread and storage-arbitrage strategies.
- Compare: is the estimated profit consistent with the cohomological prediction?

10.5 Design Implication

If the torus structure is genuine, then: - **No globally arbitrage-free model on T^2 exists** for natural gas futures. - Instead, models should either (a) restrict to contractible patches (local validity, no global claim), or (b) explicitly incorporate the cohomological premium as a model parameter, not a calibration error.

11. Outlook

- **Topology-aware model design:** choose model state spaces that are contractible by construction, avoiding topological arbitrage by design. When periodicity is real, use covering-space models (unwrap the torus to $\mathbb{R}^2$) rather than fighting the topology.
- **Obstruction diagnostics:** the persistent homology pipeline (Section 9) provides a pre-screening tool for model classes before calibration begins.
- **Stress testing:** use detected cycles as stress scenarios (cycle around an obstruction = worst case). The commodity torus example (Section 10) demonstrates the methodology.
- **Lean formalization:** the de Rham/Poincaré lemma direction is classical and well-suited for LeanProofs/TopologicalMarkets/ArbitrageObstruction.lean. The torus example provides an explicit computational target: $H^1(T^2) = \mathbb{R}^2$.
- **Bridge to spectral methods:** spectral decomposition on manifolds (Laplacian eigenmaps) connects topological invariants to the spectral framework of the other papers. The torus Laplacian eigenmodes are simply Fourier modes on T^2, recovering the COS basis as a special case.
- **Empirical priority:** the commodity torus test is the most concrete prediction and should be the first empirical validation target.