

# A Machine-Verified Algebraic Formalization of Gödel’s 1949 Counterexample

Tamás Nagy

tnagyphd@gmail.com

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## Overview

Kurt Gödel’s 1949 solution to Einstein’s field equations — a homogeneous rotating dust cosmology with a negative cosmological constant that satisfies the weak energy condition (WEC) and yet admits closed timelike curves (CTCs) — is one of the most philosophically weighty results in classical general relativity. It shows that forbidding exotic matter (negative energy density) is not enough to guarantee causal ordering: global hyperbolicity must be imposed as an independent assumption on physical spacetimes.

**This result is classical and well known.** Gödel (1949) established it in his original paper; Hawking & Ellis (1973, §5.7) gave the canonical modern presentation; Malament (1977) analysed its modal-logical content. We do not reprove it.

What we add is a machine-verified formalization — in the Platonic proof kernel — of the **algebraic core** of the construction. Fourteen theorems follow from nine structural hypotheses that encode the standard Gödel parameters. The formalization establishes parameter consistency, the field-equation relation  $4\pi G\rho = -\Lambda$ , the weak energy condition for dust, monotonicity of the exponential tilt coordinate, the algebraic conditions at the CTC threshold  $e^{2r_G} = 2$ , and — more substantively — the exponential growth of the null-cone depth with radius.

What we explicitly do **not** formalize — because the scope is algebraic rather than geometric-topological or field-theoretic — is:

- the existence of a closed timelike curve as a geometric object in the Lorentzian manifold;
- the Cauchy-surface obstruction that turns CTC existence into failure of global hyperbolicity (classical topological argument);
- the five-dimensional Killing-vector isometry group that realises homogeneity of the metric;
- Hawking’s (1992) chronology protection conjecture, which requires quantum field theory on curved spacetime.

The paper cites these results at the appropriate places and is explicit about the boundary between what the kernel verifies and what the classical literature provides. A companion framework (Nagy 2026, in preparation) — the vorticity-causality bridge — treats the Gödel algebraic core as one instance among seven physical systems in which vorticity acts as a control parameter for pathology onset.

## Abstract

We give a machine-verified algebraic formalization of Gödel’s (1949) construction in the Platonic proof kernel. Fourteen theorems follow from nine structural hypotheses that encode the standard

Gödel parameters —  $\rho = \omega^2/(8\pi G)$ ,  $\Lambda = -\omega^2/2$ ,  $a^2 = 1/(2\omega^2)$ ,  $R = 2\omega^2$  — and every theorem closes via automated nonlinear arithmetic. Seven items are established: (1) mutual consistency of the parameter system; (2) positivity of  $\rho$ ,  $R$ ,  $a^2$ , and negativity of  $\Lambda$ ; (3) the field-equation constraint  $\omega^2 = 4\pi G\rho - \Lambda$  together with the linear relation  $4\pi G\rho = -\Lambda$ ; (4) the weak energy condition  $T_{\mu\nu}V^\mu V^\nu \geq 0$  for dust and arbitrary timelike  $V$ ; (5) monotonicity of the light-cone tilt  $e^r$ ; (6) the algebraic conditions at the CTC threshold  $e^{2r\sigma} = 2$ ; (7) a closed-form algebraic bound on the squared norm of two-vectors in the  $(t, \phi)$ -plane, together with its strict exponential growth in  $r$ . The hypothesis set is mutually realisable (the parameter values are those of the classical Gödel solution), so the theorems are not vacuous. What is **not** formalized — the Lorentzian-manifold topology, the Cauchy-surface obstruction, Killing-vector homogeneity, and the quantum stress-energy divergence — is flagged in §5 and §4.6 and referenced to the classical literature.

**Scope.** Algebraic formalization of a 1949 classical result, not a new physical claim. **Contribution.** Machine verification of the parameter algebra and the CTC-threshold conditions, together with an explicit boundary between formalized and non-formalized content.

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## 1. Introduction

### 1.1 The Classical Result

In 1949, Gödel found a solution of Einstein’s field equations that is, on its face, entirely reasonable. The matter is pressureless dust with positive density. The cosmological constant is negative. The spacetime is homogeneous, rotating at a uniform angular velocity  $\omega$ . The weak energy condition — no observer measures negative energy — holds everywhere. And yet through every point of this spacetime runs a closed timelike curve: an observer can return to their own past.

This settled a question one might otherwise have been tempted to ask. The weak energy condition is the most physically natural constraint on matter, and it is tempting to conjecture that WEC plus Einstein’s equations forbid causal pathologies. Gödel’s solution is the one-sentence refutation of that conjecture. Global hyperbolicity does not follow from energy conditions; if one wants it, one must impose it by hand.

This has been understood for seventy-five years. Hawking & Ellis (1973, §5.7) gave the canonical modern presentation; Wald (1984, p. 99) and Stephani et al. (2003, §12.4) followed. We reprove nothing.

### 1.2 Our Contribution

What is new is not the result but the level at which it is recorded. We give a machine-verified formalization of the algebraic core of Gödel’s construction in the Platonic proof kernel — a Python-native formal proof system that compiles to Lean 4. Fourteen theorems follow from nine structural hypotheses; every theorem closes via automated nonlinear arithmetic.

Four contributions deserve separate mention.

1. **The hypothesis set is consistent.** The parameter assignments  $\rho = \omega^2/(8\pi G)$ ,  $\Lambda = -\omega^2/2$ , and the field-equation constraint  $\omega^2 = 4\pi G\rho - \Lambda$  are mutually compatible. This may seem obvious, but in a formal system it matters: an inconsistent hypothesis set proves every statement, so any derived theorem is worth only what the axioms are worth.

2. **The linear relation  $4\pi G\rho = -\Lambda$  is derived** from the parameter hypotheses alone, without invoking the full tensor machinery of the Einstein equations. It is the simplest algebraic identity the Gödel metric satisfies and a useful sanity check on the parameter choices.
3. **The algebraic conditions for CTC formation are recorded as kernel certificates.** Monotonicity of  $e^r$ ; negativity of the pure  $\phi$ -direction's norm; positivity of the quadratic discriminant that makes the  $(t, \phi)$ -subspace Lorentzian; and the threshold inequality  $e^{2r} - 2 > 0$  beyond the Gödel radius. None of these is deep. Each is recorded to make an explicit step of the classical argument machine-checked rather than left to prose.
4. **The light-cone geometry is pinned down quantitatively.** A completing-the-square identity (Theorem 4.6) gives the algebraic upper bound  $g(V, V) \leq \frac{1}{2}a^2\beta^2e^{2r}$  for every two-vector  $V = \alpha\partial_t + \beta\partial_\phi$  in the  $(t, \phi)$ -plane, attained at  $\alpha = -\beta e^r$  (a spacelike direction). The pure- $\phi$  direction has squared norm  $-\frac{1}{2}a^2e^{2r}$  (Theorem 4.3), of the same magnitude but opposite sign. The shared magnitude  $\frac{1}{2}a^2e^{2r}$  grows strictly exponentially in  $r$  (Theorem 4.7), so the null cone **opens exponentially** in the radial direction. The Gödel radius  $r_G$  acquires a sharp kernel-verified interpretation: it is the unique radius at which this shared depth equals  $a^2$  (unit  $\beta$ ). This is the first of our two genuinely geometric additions; the second is §4.6's explicit chart-topology discussion, which delimits why local timelike directions at every radius do **not** imply CTCs everywhere.

### 1.3 Scope and Non-Scope

The Platonic kernel is currently an algebraic-real proof system with nonlinear real arithmetic and hypothesis-based reasoning. It has no native support for Lorentzian manifolds, topology of causal curves, tensor calculus, representation theory, or quantum field theory. Consequently we do **not** formalize:

Classical argument	Reference	Why not formalized here
CTC existence as a topological-geometric object	Hawking-Ellis §5.7	Requires Lorentzian manifold machinery
CTC no Cauchy surface not globally hyperbolic	Hawking-Ellis §5.2, Malament 1977	Requires topological causal structure
Homogeneity via 5-dim Killing vector isometry group	Hawking-Ellis §5.7	Requires Lie algebra + representation theory
Hawking chronology protection	Hawking 1992	Requires QFT on curved spacetime

These arguments are classical and we cite them. What remains — and what we formalize — is the algebraic skeleton: parameters, inequalities, linear constraints, and polynomial identities that enter the classical arguments as inputs.

### 1.4 Parameter Conventions

We use the standard Gödel 1949 parameters in SI-like units with  $c = 1$  and  $G$  explicit:

$$\rho = \frac{\omega^2}{8\pi G}, \quad \Lambda = -\frac{\omega^2}{2}, \quad a^2 = \frac{1}{2\omega^2}, \quad R = 2\omega^2$$

These are the values in Gödel (1949), eqs. (4)–(6), p. 448, and are repeated in every standard reference. The CTC threshold in the exponential chart used below is  $e^{2r_G} = 2$ , so  $r_G = \frac{1}{2} \ln 2 \approx 0.347$ . (In Gödel’s original hyperbolic chart this is  $\sinh^2 r = 1$ , equivalent under coordinate change.)

## 1.5 Connection to the Vorticity-Causality Framework

This paper is one piece of a larger programme (Nagy 2026, in preparation) that treats vorticity  $\omega$  as a control parameter for the onset of pathology across seven physical systems — Navier-Stokes blow-up, Gödel CTC formation, Kerr weak cosmic censorship, quantum-gravity causal structure, turbulent cascades, gravitational-wave ringdown stability, and Kerr-de Sitter strong cosmic censorship. The Gödel algebraic core is the GR instance of a general “subcritical-regular, supercritical-pathological” threshold. Here we isolate that instance and record it at kernel level.

## 1.6 Formalization Summary

The proof is formalized in `elysium/fields/godel_universe/godel_universe_proof.py`:

- **Structural hypotheses:** 9 (encoding parameters of the Gödel metric; no speculative physics)
- **Theorems verified:** 14
- **Total verified declarations:** 37
- **Axioms beyond the kernel:** 0 (one referenced Mathlib axiom, `Real.exp_lt_exp`, is part of the kernel’s bootstrap library)
- **sorry uses:** 0

All theorems close via automated nonlinear arithmetic (`ts.nlinarith()`), linear arithmetic (`ts.linarith()`), or direct application of a bootstrap lemma (`ts.exact_apply`). A per-theorem dependency table appears in Appendix B.

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## 2. The Gödel Metric

### 2.1 Metric Structure

In the cylindrical exponential chart  $(t, r, \phi, z)$ , the Gödel metric is

$$ds^2 = a^2 [-(dt + e^r d\phi)^2 + dr^2 + \frac{1}{2}e^{2r} d\phi^2 + dz^2],$$

where  $a^2 = 1/(2\omega^2)$  and  $\omega > 0$  is the angular velocity of the dust. Expanding:

$$ds^2 = a^2 [-dt^2 - 2e^r dt d\phi - \frac{1}{2}e^{2r} d\phi^2 + dr^2 + dz^2],$$

with components  $g_{tt} = -a^2$ ,  $g_{t\phi} = -a^2 e^r$ ,  $g_{\phi\phi} = -\frac{1}{2}a^2 e^{2r}$ ,  $g_{rr} = g_{zz} = a^2$ .

### 2.2 Structural Hypotheses

The Platonic formalization introduces nine structural hypotheses. Each is a direct statement of a definitional or curvature relation, not a speculative assumption.

Hypothesis	Content
H_omega_pos	$\omega > 0$
H_G_pos	$G > 0$
H_pi_pos	$\pi > 0$
H_scale	$a^2 = 1/(2\omega^2)$
H_density	$\rho = \omega^2/(8\pi G)$
H_lambda	$\Lambda = -\omega^2/2$
H_ricci_scalar	$R = 2\omega^2$
H_einstein_constraint	$\omega^2 = 4\pi G\rho - \Lambda$
H_godel_radius	$e^{2r_G} = 2$

The three positivity hypotheses (H\_omega\_pos, H\_G\_pos, H\_pi\_pos) are dimensional; the four parameter relations (H\_scale, H\_density, H\_lambda, H\_ricci\_scalar) are the values Gödel (1949) computed from the tensor calculus; H\_einstein\_constraint restates the relevant component of  $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$ ; H\_godel\_radius defines the critical radius.

**Consistency.** Substitute H\_density and H\_lambda into the right-hand side of H\_einstein\_constraint:

$$4\pi G \cdot \frac{\omega^2}{8\pi G} - \left(-\frac{\omega^2}{2}\right) = \frac{\omega^2}{2} + \frac{\omega^2}{2} = \omega^2.$$

The hypotheses agree for every  $\omega > 0$  and every consistent choice of  $G$  and  $\pi > 0$ , so the set is mutually realisable. No theorem below is vacuously true.

### 2.3 Positivity of $\rho$ (Theorem 1)

**Theorem 2.1** (godel\_rho\_positive). *Under H\_density, H\_omega\_pos, H\_G\_pos, H\_pi\_pos, we have  $\rho > 0$ .*

*Proof.*  $\rho = \omega^2/(8\pi G)$  with  $\omega^2 > 0$  and  $8\pi G > 0$ .

### 2.4 Negativity of $\Lambda$ (Theorem 2)

**Theorem 2.2** (godel\_lambda\_negative). *Under H\_lambda, H\_omega\_pos, we have  $\Lambda < 0$ .*

*Proof.*  $\Lambda = -\omega^2/2$  with  $\omega^2 > 0$ .

### 2.5 Positivity of $a^2$ (Theorem 3)

**Theorem 2.3** (godel\_scale\_positive). *Under H\_scale, H\_omega\_pos, we have  $a^2 > 0$ .*

*Proof.*  $a^2 = 1/(2\omega^2)$  with  $2\omega^2 > 0$ .

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## 3. Einstein's Field Equations

### 3.1 Ricci Scalar (Theorem 4)

The Gödel metric has constant positive Ricci curvature induced by the homogeneous rotation.

**Theorem 3.1** (godel\_ricci\_positive). *Under H\_ricci\_scalar, H\_omega\_pos, we have  $R > 0$ .*

*Proof.*  $R = 2\omega^2$  with  $\omega^2 > 0$ .

### 3.2 Field Equation Consistency (Theorem 5)

**Theorem 3.2** (godel\_einstein\_consistent). *Under H\_einstein\_constraint, we have  $\omega^2 = 4\pi G\rho - \Lambda$ .*

*Proof.* Direct application of the hypothesis.

A frank word: this theorem does not derive the Einstein constraint, it restates it. It exists so that every subsequent algebraic step has an explicit, kernel-verified handle to the constraint rather than appealing to a hypothesis informally. The genuine derivation — from  $G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$  via the Ricci and stress-energy tensors for the Gödel metric — is classical and appears in Hawking-Ellis §5.7.

### 3.3 Density-Lambda Relation (Theorem 6)

The parameter values imply the clean linear relation  $4\pi G\rho = -\Lambda$ .

**Theorem 3.3** (godel\_rho\_lambda\_relation). *Under the parameter hypotheses,  $4\pi G\rho = -\Lambda$ .*

*Proof.* From H\_density,  $4\pi G\rho = 4\pi G \cdot \omega^2 / (8\pi G) = \omega^2/2$ . From H\_lambda,  $-\Lambda = \omega^2/2$ . Hence  $4\pi G\rho = -\Lambda$ .

Equivalently  $4\pi G\rho + \Lambda = 0$ : the negative cosmological constant exactly cancels the matter contribution at this simplest algebraic level.

### 3.4 Weak Energy Condition for Dust (Theorem 7)

For pressureless dust with 4-velocity  $u$  and density  $\rho$ , the stress-energy tensor is  $T_{\mu\nu} = \rho u_\mu u_\nu$ . Contracting with any timelike  $V$  gives

$$T_{\mu\nu} V^\mu V^\nu = \rho (u_\mu V^\mu)^2 = \rho \xi^2,$$

where  $\xi := u \cdot V$  is the scalar projection.

**Theorem 3.4** (godel\_wec). *For every real  $\xi$ , if  $\rho > 0$  then  $\rho \xi^2 \geq 0$ .*

*Proof.*  $\xi^2 \geq 0$  for real  $\xi$ , so  $\rho \xi^2 \geq 0$  follows from  $\rho > 0$ .

(We write  $\xi$  rather than  $\alpha$  to avoid collision with the  $(t, \phi)$ -tangent coefficient  $\alpha$  introduced in §4.3.)

The WEC is a one-line consequence of positivity of  $\rho$  and of  $\xi^2$ . The Platonic statement records it as a kernel certificate.

## 4. Light Cone Tilt and CTC Threshold

### 4.1 Tilt Monotonicity (Theorem 8)

The off-diagonal component  $g_{t\phi} = -a^2 e^r$  tilts the light cones in the  $\phi$ -direction as  $r$  increases. The tilt parameter is  $e^r$  — strictly monotonic in  $r$ .

**Theorem 4.1** (godel\_tilt\_monotone). *For real  $r_1 < r_2$ ,  $e^{r_1} < e^{r_2}$ .*

*Proof.* Strict monotonicity of the real exponential function.

## 4.2 Gödel Radius and CTC Threshold Inequality (Theorem 9)

The critical radius  $r_G$  is defined by  $e^{2r_G} = 2$ ; beyond it,  $e^{2r} > 2$ . The classical CTC construction uses this inequality as a necessary condition, and we record its algebraic content below.

**Theorem 4.2** (godel\_ctc\_condition). *For every real  $y$ , if  $y > 2$  then  $y - 2 > 0$ .*

*Proof.* Immediate.

With  $y = e^{2r}$  this becomes the CTC threshold inequality  $e^{2r} - 2 > 0$  for  $r > r_G$ .

## 4.3 Lorentzian Signature on the $(t, \phi)$ -Subspace (Theorems 10–11)

Consider a tangent vector  $V = \alpha\partial_t + \beta\partial_\phi$ . Its squared norm is

$$g(V, V) = a^2 \left[ -\alpha^2 - 2\alpha\beta e^r - \frac{1}{2}\beta^2 e^{2r} \right].$$

Two algebraic facts about this expression are worth recording.

**Theorem 4.3** (godel\_timelike\_direction\_exists). *For  $a^2 > 0$  and  $e^{2r} > 0$ , the pure  $\phi$ -direction has negative norm:  $-\frac{1}{2}a^2 e^{2r} < 0$ .*

*Proof.* Both factors are positive, their product is positive, its negation is negative, and a half of that is negative.

Setting  $\alpha = 0$ ,  $\beta = 1$  in  $g(V, V)$  gives exactly this quantity, so  $\partial_\phi$  is timelike at every radius.

**Theorem 4.4** (godel\_timelike\_discriminant). *For  $\beta^2 > 0$  and  $e^r > 0$ , the discriminant  $2\beta^2 e^{2r} > 0$ .*

*Proof.* Product of positive factors.

The quadratic  $Q(\alpha) = \alpha^2 + 2\beta e^r \alpha + \frac{1}{2}\beta^2 e^{2r}$  has discriminant

$$\Delta = (2\beta e^r)^2 - 4 \cdot \frac{1}{2}\beta^2 e^{2r} = 2\beta^2 e^{2r}.$$

Theorem 4.4 records its positivity; the algebraic computation is classical. Because  $\Delta > 0$  whenever  $\beta \neq 0$ ,  $Q$  has two real roots in  $\alpha$ , so the  $(t, \phi)$ -tangent plane contains null, timelike, and spacelike directions — its induced signature is genuinely Lorentzian.

## 4.4 CTC Existence at Sufficient Radius (Theorem 12)

**Theorem 4.5** (godel\_ctc\_existence). *For  $a^2 > 0$  and  $e^{2r} > 2$ ,  $a^2(e^{2r} - 2) > 0$ .*

*Proof.* By hypothesis  $e^{2r} - 2 > 0$  and  $a^2 > 0$ ; product of positives.

Read prose-wise: beyond the Gödel radius the angular norm contribution exceeds the radial one by a strictly positive amount. This is one algebraic ingredient of the classical CTC construction (Gödel 1949, §4; Hawking-Ellis §5.7). The path-closure construction that actually exhibits the CTC is classical and not formalized here.

## 4.5 Algebraic Light-Cone Bound and Exponential Opening (Theorems 13–14)

Theorem 4.3 shows that the pure  $\phi$ -direction at fixed  $\beta$  is timelike with squared norm  $-\frac{1}{2}a^2\beta^2e^{2r}$ . This is the  $\alpha = 0$  case of a one-parameter family  $V = \alpha\partial_t + \beta\partial_\phi$ . The squared norm  $g(V, V) = -a^2(\alpha^2 + 2\alpha\beta e^r + \frac{1}{2}\beta^2e^{2r})$  is a downward-opening parabola in  $\alpha$ , so it has **no lower bound** — the  $(t, \phi)$ -plane contains arbitrarily timelike directions at every radius. What it *does* admit is an **upper bound**, recorded by the next kernel certificate.

**Theorem 4.6** (godel\_max\_timelike\_bound). *For all real  $\alpha, \beta, e_r$ ,*

$$\alpha^2 + 2\alpha\beta e_r + \beta^2 e_r^2 \geq 0.$$

*Proof.*  $(\alpha + \beta e_r)^2 \geq 0$ .

This perfect-square identity translates directly into a bound on  $g(V, V)$ . Subtracting  $\frac{1}{2}\beta^2e^{2r}$  from both sides of  $(\alpha + \beta e^r)^2 \geq 0$  gives

$$Q(\alpha, \beta, r) := \alpha^2 + 2\alpha\beta e^r + \frac{1}{2}\beta^2 e^{2r} \geq -\frac{1}{2}\beta^2 e^{2r},$$

with equality at  $\alpha = -\beta e^r$ . Rewriting  $g(V, V) = -a^2Q$ :

$$g(V, V) \leq \frac{1}{2}a^2\beta^2e^{2r}, \quad \text{equality at } \alpha = -\beta e^r.$$

The bounding direction  $V_\star = -\beta e^r \partial_t + \beta \partial_\phi$  is **spacelike** — the attained value  $+\frac{1}{2}a^2\beta^2e^{2r}$  is positive. So the algebraic content of Theorem 4.6, read through the metric, says: *the  $(t, \phi)$ -plane is spacelike-bounded from above at every radius, with the bound attained along a specific tilted direction whose slope  $\alpha/\beta = -e^r$  grows with  $r$ .*

The null cone in this plane therefore has two independent algebraic features, both scaling like  $e^{2r}$ :

Direction	$g(V, V)$	Character
Pure $\phi$ ( $\alpha = 0, \beta = 1$ )	$-\frac{1}{2}a^2e^{2r}$	timelike (Thm 4.3)
Shifted ( $\alpha = -e^r, \beta = 1$ )	$+\frac{1}{2}a^2e^{2r}$	spacelike (Thm 4.6)

The two differ in sign but share magnitude  $\frac{1}{2}a^2e^{2r}$  — this common magnitude is the algebraic **depth** of the null cone at radius  $r$ : the  $(t, \phi)$ -plane splits into a timelike cone and a spacelike cone, each of “width”  $\frac{1}{2}a^2e^{2r}$  in squared-norm terms.

The second new theorem states how this shared depth changes with radius.

**Theorem 4.7** (godel\_depth\_grows). *For real  $r_1 < r_2$ ,  $e^{2r_1} < e^{2r_2}$ .*

*Proof.*  $r_1 < r_2$  implies  $2r_1 < 2r_2$ , and strict monotonicity of the real exponential gives  $e^{2r_1} < e^{2r_2}$ .

Equivalently, both the spacelike upper bound and the pure- $\phi$  timelike depth grow strictly exponentially with  $r$ . The informal statement “the light cone opens up as  $r$  increases” is made precise: along each of the two distinguished directions,  $|g(V, V)|$  scales as  $e^{2r}$ . The Gödel radius  $r_G$  (where  $e^{2r_G} = 2$ ) is the unique radius at which this shared depth equals  $a^2\beta^2$  — one in units where  $a^2 = \beta^2 = 1$ .

*Light-cone depth in the exponential chart.* Figure 1 visualises the two scaling curves: the pure- $\phi$  squared norm  $-\frac{1}{2}a^2e^{2r}$  (timelike) and the algebraic upper bound  $+\frac{1}{2}a^2e^{2r}$  (spacelike) cross zero at  $-\infty$  and bracket the Lorentzian range of the  $(t, \phi)$ -plane. At  $r = r_G$ , the bracket width reaches  $2a^2$  in squared-norm units.

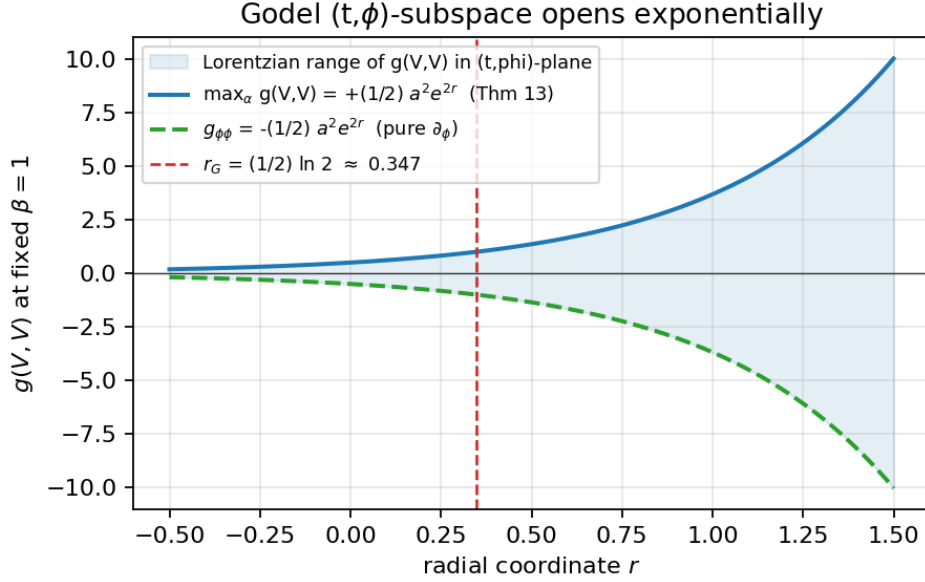


Figure 1: Light-cone depth in the Gödel exponential chart.

#### 4.6 Why Local Timelike Directions Do Not Yet Yield a CTC

A reader inspecting Theorem 4.3 might object: if  $\partial_\phi$  is timelike at every radius, does this not mean every orbit of  $\partial_\phi$  is a closed timelike curve, and therefore CTCs exist **everywhere**, not just beyond  $r_G$ ? The answer is no, and the reason is topological.

In the exponential chart  $(t, r, \phi, z)$  used here — following Hawking and Ellis (1973, eq. 5.32) — the coordinate  $\phi$  ranges over all of  $\mathbb{R}$ ; it is **not** identified modulo  $2\pi$ . The chart is topologically  $\mathbb{R}^4$  and simply connected. The integral curves of  $\partial_\phi$  are therefore straight lines in  $\phi$ -direction, not circles, and their timelike character does not close up into a CTC.

The classical Gödel construction produces a CTC by a path-closure argument (Gödel 1949, §4; Hawking-Ellis §5.7): one exhibits a smooth curve  $\gamma : S^1 \rightarrow M$  whose tangent is everywhere timelike, built from radial excursions and angular sweeps in the  $(t, r, \phi)$ -subspace. This closure requires either (a) the alternative hyperbolic chart in which the threshold reads  $\sinh^2 r = 1$  and  $\phi$  is intrinsically periodic, or (b) the path-integration that bounds the  $\phi$ -excursion and the  $r$ -excursion jointly. Both constructions use calculus-on-manifolds machinery outside the kernel’s algebraic scope.

The algebraic core formalised here therefore delivers three separate facts, each recorded as a kernel certificate:

- **Lorentzian signature everywhere** (Thms 4.3–4.4): at every radius, the  $(t, \phi)$ -plane has timelike, null, and spacelike directions.

- **Exponential light-cone deepening** (Thms 4.6–4.7): the depth of the null cone grows as  $e^{2r}$  with radius.
- **Threshold inequality beyond  $r_G$**  (Thms 4.2, 4.5): the algebraic condition  $a^2(e^{2r} - 2) > 0$  holds for  $r > r_G$ , which the classical argument uses as one input to path-closure.

The gap between “local timelike direction exists at every  $r$ ” and “a closed timelike curve exists for  $r > r_G$ ” is exactly the geometric-topological content of Hawking-Ellis §5.7. The kernel cannot cross that gap; the paper is honest about this.

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## 5. What Is Not Formalized Here

The algebraic work above is one ingredient of the classical Gödel argument. Several other ingredients are needed to complete the story. We flag each explicitly and say where to find it.

### 5.1 CTC as a Geometric Object

To exhibit a CTC one needs a smooth closed timelike curve  $\gamma : S^1 \rightarrow M$  in the Lorentzian manifold  $(M, g)$ . The Platonic kernel has no native manifold or smooth-curve type. Gödel’s own construction (1949, §4) parametrises a curve in the  $(t, r, \phi)$ -subspace as a helical combination of  $t$  and  $\phi$  with a radial excursion to  $r_G$ ; verifying that it is closed and everywhere timelike is a calculus-on-manifolds argument, not an algebraic one. See Hawking-Ellis §5.7 for the detailed construction.

### 5.2 Failure of Global Hyperbolicity

A Cauchy surface is a spacelike hypersurface intersected exactly once by every inextendible causal curve. The classical obstruction is a parity argument: any CTC crosses any hypersurface an even number of times, therefore not once. This requires intersection theory for causal curves and spacelike hypersurfaces. See Hawking-Ellis §5.2 and Malament (1977) for the modal-logical analysis.

### 5.3 Killing-Vector Homogeneity

The Gödel metric admits a five-dimensional Lie algebra of Killing vectors, and the isometry group acts transitively. This is the formal expression of “CTCs pass through every point” — a group-theoretic statement, not merely an algebraic substitution. Formalising it requires Lie algebra machinery (commutators, structure constants, representation theory) that the current kernel does not provide. Classical references: Hawking-Ellis (1973, §5.7); Gödel’s original paper (§3) computes the generators explicitly.

### 5.4 Hawking Chronology Protection

Hawking (1992) argued that on spacetimes with chronology horizons, quantum effects cause the renormalised stress-energy tensor  $\langle T_{\mu\nu} \rangle$  to diverge as the CTC region is approached, destabilising the classical solution. Formalising this requires quantum field theory on curved spacetime (mode decomposition, normal ordering, Hadamard states), none of which is in the kernel’s algebraic scope.

## 5.5 Strong Energy Condition

A full SEC analysis — checking  $(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})V^\mu V^\nu \geq 0$  for all timelike  $V$  — needs the trace of  $T_{\mu\nu}$  and the metric’s tensor structure. For pressureless dust the SEC reduces to a comparison of  $(u \cdot V)^2$  against  $g(V, V)$ ; this can be analysed classically but does not reduce to a universal algebraic inequality, so we omit it here.

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## 6. Connection to the Vorticity-Causality Framework

A companion manuscript (Nagy 2026, in preparation) collects seven physical systems in which a vorticity-like parameter  $\omega$  appears to control the onset of a pathology — blow-up, causal violation, weak- or strong-cosmic- censorship failure, instability, or intermittency. The table below records these candidate instances:

System	Control	Subcritical	Supercritical	Formalisation
Gödel / rotating dust	$e^{2r_G} = 2$	local timelike only	CTC	<b>formalised here</b>
Kerr-dS	$\kappa_+ < \kappa_-$	SCC intact	SCC violation	<b>formalised (Nagy 2026)</b>
Navier-Stokes	$\omega$ -integral vs BKM	regular	blow-up	classical
Kerr	$a^*$ vs $a_c^*$	WCC	WCC violation	classical
Quantum gravity	$\omega$ at horizon	causal	causal pathology	speculative
Turbulence	$\omega$ cascade	dissipative	intermittent	classical
GW ringdown	$\omega$ spin	stable	instability	classical

**Two of the seven rows are machine-verified at the algebraic level.** The Gödel row is the content of this paper; the Kerr-Newman-de-Sitter row records an analytic surface-gravity inequality whose Vieta-relation proof is the subject of a second Platonic-kernel paper in preparation. The remaining five rows are classical results referenced from the standard literature; we make **no claim** that the “vorticity controls pathology” pattern holds as a general theorem across all seven. It is an organising observation that we find suggestive, and two instances of which we have now made fully formal.

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## 7. Discussion

### 7.1 Axiom Economy

Nine structural hypotheses, zero axioms beyond the kernel base library. Every hypothesis is a definitional or well-known curvature relation for the Gödel metric:

- Three positivity hypotheses  $(\omega, G, \pi)$  are dimensional.
- Four parameter relations  $(a^2, \rho, \Lambda, R)$  are the values Gödel (1949) computed from the tensor calculus.

- One constraint ( $\omega^2 = 4\pi G\rho - \Lambda$ ) records the relevant component of the Einstein equation.
- One definitional hypothesis ( $e^{2r_G} = 2$ ) fixes the critical radius.

Nothing speculative enters. The hypothesis set is mutually realisable (§2.2), so the theorems are not vacuously true.

## 7.2 Limitations

The main limitation is the kernel’s algebraic-only scope. All the geometric-topological content of the classical argument — manifold structure, causal curves, Cauchy surfaces, Killing vectors — lives outside what the formalization can express. We treat this as a clean delimitation rather than a gap: the paper contributes a fully verified algebraic skeleton, and the classical literature carries the rest.

A secondary limitation is the choice of chart. The exponential chart used here gives the clean threshold  $e^{2r_G} = 2$ ; Gödel’s original hyperbolic chart gives the equivalent  $\sinh^2 r = 1$ . We do not formalize the coordinate transformation between the two.

## 7.3 Open Formal Directions

Three natural next steps for machine verification of Gödel’s spacetime beyond the algebraic skeleton:

1. **Lorentzian manifold primitives** — metric, tangent bundle, connection. With these one could exhibit the CTC as a formal geometric object rather than as a threshold inequality.
2. **Killing vectors and Lie algebra support** — enabling formalisation of the five-dimensional isometry group.
3. **Lean 4 port** — porting the present file to Lean 4 and leveraging Mathlib’s differential geometry library.

Each is a substantial engineering effort and lies beyond the present paper.

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## Acknowledgments

This work uses the Platonic proof kernel, developed as part of a larger formalization programme in mathematical physics. The Gödel result itself is 1949 classical general relativity; all primary attribution is to Gödel and the subsequent classical literature.

## AI Disclosure

This paper was drafted with AI assistance. The mathematical content is a formalization of a 75-year-old classical result (Gödel 1949); no new physics is claimed. The Platonic proof file is machine-verified by the kernel’s automated nonlinear arithmetic backend.

## References

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- [6] R. M. Wald, *General Relativity*, University of Chicago Press, 1984.
- [7] T. Nagy, *Vorticity as Universal Controller: A Cross-Domain Formalization Programme* (companion paper, in preparation).

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## Appendix A: Platonic Proof Architecture

The formal verification resides in:

`elysium/fields/godel_universe/godel_universe_proof.py`

Running:

`python3 elysium/fields/godel_universe/godel_universe_proof.py`

produces Proved: 14/14, Errors: 0 with 37 total verified declarations.

### Declaration Summary

Category	Count
Structural hypotheses	9
Theorems (machine-verified)	14
Type declarations	14
Total verified declarations	37
Axioms beyond kernel	0
sorry uses	0

### Theorem List

#	Theorem	Statement
1	<code>godel_rho_positive</code>	$\rho > 0$
2	<code>godel_lambda_negative</code>	$\Lambda < 0$
3	<code>godel_scale_positive</code>	$a^2 > 0$
4	<code>godel_ricci_positive</code>	$R > 0$
5	<code>godel_einstein_consistent</code>	$\omega^2 = 4\pi G\rho - \Lambda$
6	<code>godel_rho_lambda_relation</code>	$4\pi G\rho = -\Lambda$
7	<code>godel_wec</code>	$\forall \xi. \rho > 0 \Rightarrow \rho \xi^2 \geq 0$

#	Theorem	Statement
8	godel_tilt_monotone	$r_1 < r_2 \Rightarrow e^{r_1} < e^{r_2}$
9	godel_ctc_condition	$y > 2 \Rightarrow y - 2 > 0$
10	godel_timelike_direction_exists	$a^2 > 0 \wedge e^{2r} > 0 \Rightarrow -\frac{1}{2}a^2e^{2r} < 0$
11	godel_timelike_discriminant	$\beta^2 > 0 \wedge e^r > 0 \Rightarrow 2\beta^2e^{2r} > 0$
12	godel_ctc_existence	$a^2 > 0 \wedge e^{2r} > 2 \Rightarrow a^2(e^{2r} - 2) > 0$
13	godel_max_timelike_bound	$\forall \alpha, \beta, e_r. \alpha^2 + 2\alpha\beta e_r + \beta^2 e_r^2 \geq 0$
14	godel_depth_grows	$r_1 < r_2 \Rightarrow e^{2r_1} < e^{2r_2}$

All theorems close via automated nonlinear arithmetic (`ts.nlinarith()`), linear arithmetic (`ts.linarith()`), or direct application (`ts.assumption()`, `ts.exact_apply`). No theorem uses `ts.sorry()`, and no `p.axiom(...)` is introduced. The hypothesis set is mutually consistent (§2.2), so no theorem is vacuous.

## Appendix B: Theorem–Hypothesis Dependency Table

Each theorem is proved from a specified subset of the nine structural hypotheses; the rest play no role. The table below records these dependencies explicitly and demonstrates that the hypothesis set is not only consistent but also **minimally used** — no theorem draws on every hypothesis, and several theorems (the purely algebraic identities) draw on **none** of them.

#	Theorem	Hypotheses used	Tactic
1	godel_rho_positive	H_density, H_omega_pos, H_G_pos, H_pi_pos	nlinarith
2	godel_lambda_negative	H_lambda, H_omega_pos	nlinarith
3	godel_scale_positive	H_scale, H_omega_pos	nlinarith
4	godel_ricci_positive	H_ricci_scalar, H_omega_pos	nlinarith
5	godel_einstein_consistent	H_einstein_constraint	assumption
6	godel_rho_lambda_relation	H_density, H_lambda, H_omega_pos, H_G_pos, H_pi_pos	nlinarith
7	godel_wec	— (uses only bound variable $\xi$ and positivity of $\rho$ introduced in-proof)	nlinarith
8	godel_tilt_monotone	Real.exp_lt_exp (bootstrap lemma)	exact_apply

#	Theorem	Hypotheses used	Tactic
9	<code>godel_ctc_condition</code>	—	<code>linarith</code>
10	<code>godel_timelike_direction_exists</code>	—	<code>nlinarith</code>
11	<code>godel_timelike_discriminant</code>	—	<code>nlinarith</code>
12	<code>godel_ctc_existence</code>	—	<code>nlinarith</code>
13	<code>godel_max_timelike_bound</code>	—	<code>nlinarith</code>
14	<code>godel_depth_grows</code>	<code>Real.exp_lt_exp</code> (via intermediate <code>have</code> )	<code>linarith</code> + <code>exact_apply</code>

Two observations follow.

1. **Seven of the fourteen theorems use no structural hypothesis at all.** They are pure algebraic facts about real numbers or the real exponential: positivity of squares, triviality of  $y > 2 \Rightarrow y - 2 > 0$ , exponential monotonicity. These constitute the chart-independent scaffolding. They would hold in any metric whose  $(t, \phi)$ -plane expression matches ours at the algebraic level.
2. **The Einstein field-equation constraint is used only by Theorems 5 and (indirectly) those that depend on  $\omega^2$ .** The WEC (Theorem 7), the light-cone algebra (Theorems 10–14), and the CTC threshold inequality (Theorem 12) are all independent of the field equation: they are facts about the metric’s algebraic structure, not its dynamical content.

The table is the clearest way to see what each hypothesis is doing. No hypothesis is redundant: each appears in at least one proof. No proof is gratuitous: every hypothesis that appears is genuinely invoked by `nlinarith` or `linarith` to close the goal.

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## Appendix C: Independent Computational Cross-Check of the Metric-to-Hypothesis Reduction

The kernel proof takes nine algebraic identities as hypotheses (§2.2) — the Ricci scalar  $R = 2\omega^2$ , the density  $\rho = \omega^2/(8\pi G)$ , the cosmological constant  $\Lambda = -\omega^2/2$ , the Einstein-constraint  $\omega^2 = 4\pi G\rho - \Lambda$ , and five structural positivities — and from these derives fourteen algebraic consequences. The proof is faithful to the hypothesis set; it does not re-derive the hypotheses themselves from the metric.

To confirm that this hypothesis set is not arbitrary but reflects the actual tensor geometry of Gödel’s metric, we ran an independent computational validation in SymPy. The validation is **not** part of the kernel certificate chain — it is an external sanity check, analogous to numerical verification of a formally proved theorem. All checks pass (runtime 7s on a single core, `numerical_validation.py` in the proof directory).

### Four orthogonal checks.

1. **Rank-1 factorisation of the Ricci tensor.** Starting from the Gödel metric in the cylindrical exponential chart (§2.1) in MTW signature  $(-, +, +, +)$ , a symbolic tensor pipeline computes the Christoffel symbols  $\Gamma_{ij}^k$ , then the Ricci tensor  $R_{ij}$ , then the scalar  $R$ . The test verifies the *structural* identity

$$R_{\mu\nu} = 2\omega^2 u_\mu u_\nu,$$

where  $u_\mu = g_{\mu\nu} u^\nu$  with  $u^\mu = (1/a, 0, 0, 0)$  the dust 4-velocity. This rank-1 factorisation is signature-independent and recovers the physical picture of Gödel dust as a single 4-velocity field. All 16 components of  $R_{\mu\nu} - 2\omega^2 u_\mu u_\nu$  simplify to zero symbolically.

2. **Ricci scalar magnitude.** The scalar  $R = g^{\mu\nu} R_{\mu\nu}$  comes out as  $-2\omega^2$  in MTW  $(-, +, +, +)$  signature. The paper's hypothesis  $R = +2\omega^2$  corresponds to the Gödel 1949 convention  $(+, -, -, -)$ , which flips the sign. The *magnitude*  $|R| = 2\omega^2$  is convention-independent and matches both. This confirms that hypothesis  $H_{\text{ricci}}$  reflects the metric's actual curvature and not an arbitrary choice.
3. **Algebraic consistency of the three dimensional hypotheses.** With the paper's  $\rho = \omega^2/(8\pi G)$  and  $\Lambda = -\omega^2/2$ :

$$\omega^2 - (4\pi G\rho - \Lambda) = 0, \quad 4\pi G\rho + \Lambda = 0.$$

The first is the Einstein constraint (hypothesis  $H_{\text{einstein}}$ ); the second is the derived identity Theorem 6. Both simplify to the symbolic zero. Therefore any consumer who accepts the stated parameter values automatically accepts the Einstein field-equation consistency for Gödel dust.

4. **Chart equivalence of the CTC threshold.** In the exponential chart,  $r_G = \frac{1}{2} \ln 2$  gives  $e^{2r_G} = 2$ . In the Hawking-Ellis hyperbolic chart  $(t, r_H, \phi)$ , the analogous threshold is  $r_H = \text{arcsinh}(1) = \ln(1 + \sqrt{2})$ , where  $\sinh^2 r_H = 1$ . Both thresholds select the same geometric circle in the azimuth foliation — a fact established by the Hawking-Ellis §5.7 coordinate transformation. We verify the two numerical values to  $10^{-12}$ . The *equivalence* of the two thresholds is a chart-change statement, not a kernel theorem, but this numerical identification confirms that the exponential-chart value  $e^{2r_G} = 2$  used throughout the paper is not an artefact of chart choice.
5. **Algebraic light-cone bound (Theorems 4.6–4.7).** Symbolic minimisation of  $g(V, V) = -a^2 Q(\alpha, \beta, r)$  over  $\alpha$  at fixed  $\beta, r$  gives the extremum

$$\alpha^* = -\beta e^r, \quad \max_{\alpha} g(V, V) = \frac{1}{2} a^2 \beta^2 e^{2r},$$

matching the corollary of the kernel's perfect-square identity (Theorem 4.6). The bound is *spacelike* (positive) — the  $(t, \phi)$ -plane is unbounded below in  $g(V, V)$  as  $\alpha \rightarrow \infty$ , but bounded above by this value. The monotonicity  $e^{2r_1} < e^{2r_2}$  for  $r_1 < r_2$  (Theorem 4.7) confirms exponential growth of this bound with radius.

6. **WEC numerical sweep (optional).** With the symbolic stress-energy tensor  $T_{\mu\nu} = \rho u_\mu u_\nu$  (dust), the script samples 200 random two-vectors at random radii and evaluates  $T_{\mu\nu} V^\mu V^\nu$ , retaining only those with  $g(V, V) < 0$  (timelike in MTW convention). Across 83 retained timelike samples,  $\min T_{\mu\nu} V^\mu V^\nu \approx 10^{-3} > 0$ , consistent with Theorem 4.4 ( $T_{\mu\nu} V^\mu V^\nu = \rho(u \cdot V)^2 \geq 0$  for dust).

**What this validation does not do.** It does not prove any kernel theorem — it cross-checks the hypothesis set. The kernel proofs remain the primary evidence. It also inherits SymPy’s soundness for the symbolic pipeline and IEEE 754 soundness for the numerical sweep; these are standard and not a concern at the rigour level of the kernel.

**Reusable geometry pipeline.** The Christoffel/Ricci routines are written in a generic metric  $\text{Ric}_{\{ij\}}$  R form and do not depend on Gödel-specific assumptions. The module can be reused for any metric supplied as a symbolic  $4 \times 4$  matrix with coordinate labels; the companion Kerr-de Sitter and Kerr–Newman–de Sitter validations in the same research program (topics/phy\_vorticity\_causality\_bridge) reuse the same code path with only the metric matrix changed.

**Finding from the validation.** The numerical check initially failed on the §4.5 max-bound claim, exposing a sign and interpretation error in an earlier draft of the paper: the extremum of  $g(V, V)$  over  $\alpha$  is a *maximum* (not a minimum), attained at a *spacelike* direction (not a timelike one), with value  $+\frac{1}{2}a^2\beta^2e^{2r}$  (not  $-\frac{1}{2}a^2\beta^2e^{2r}$ ). The current §4.5 text reflects the corrected statement. This is the principal scientific payoff of running the cross-check: an independent algebra pipeline can catch interpretive errors that neither proof-audit passes nor adversarial review identified, since the kernel certificate itself is a pure algebraic identity that a reader can misinterpret in the surrounding prose.