

# Nash-Boltzmann Duality: Statistical Mechanics as Game Theory in the Latent Algebra

Phase Transitions, Equilibria, and Renormalization — Unified by Interaction Grade

*The partition function IS the quantal response equilibrium. The cluster expansion IS the ANOVA decomposition. Renormalization IS grade truncation.*

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## Executive Summary (Non-Technical)

Two of the most powerful intellectual frameworks of the last century — **game theory** (the mathematics of strategic interaction) and **statistical mechanics** (the physics of many-particle systems) — have developed in near-complete isolation. Economists compute Nash equilibria; physicists compute partition functions. The mathematical structures are eerily similar but the communities rarely talk.

This paper shows they are **the same mathematics**. The partition function of a physical system IS the equilibrium of a game played by the system’s components. The cluster expansion that physicists use to approximate many-body interactions IS the ANOVA decomposition that statisticians use for multivariate functions. And the **renormalization group** — Kenneth Wilson’s Nobel Prize-winning technique for understanding phase transitions — IS the Latent’s grade truncation applied to the interaction Hamiltonian.

The key that unlocks this unification is **interaction grade** — the order of multi-body coupling. In physics, grade 2 means pairwise interactions (the Ising model). In game theory, grade 2 means bilateral strategic competition (polymatrix games). In both cases, the central question is the same: **do higher-order interactions matter, or is the system dominated by pairwise effects?** The Latent framework answers this with a single number  $\rho$ : the exponential decay rate of interaction grades.

The duality reveals that:

- **Phase transitions** in physics (water boiling, magnets demagnetizing) and **complexity transitions** in game theory (games becoming intractable) are the **same phenomenon** — the Latent Number  $\rho$  crossing 1.
- **Mean-field theory** in physics and **mean-field games** in economics are the **same approximation** — grade-1 truncation.
- The **Ising model** is literally a **congestion game** where spins compete for alignment.

This is not an analogy. It is an identity within the Latent algebra.

# Abstract

We establish an exact duality between strategic games and statistical mechanical systems through the Latent algebra’s grade decomposition. The **Nash-Boltzmann map** identifies the Hamiltonian  $H(\sigma)$  of a physical system with the negative payoff  $-u(s)$  of a game, the inverse temperature  $\beta$  with the rationality parameter, and the Boltzmann distribution  $\pi(\sigma) \propto e^{-\beta H(\sigma)}$  with the logit quantal response equilibrium. Under this identification, three fundamental correspondences emerge: (i) the **cluster expansion** of statistical mechanics is the **ANOVA decomposition** of the payoff function — both decompose a multivariate function by interaction order; (ii) the **Kadanoff-Wilson renormalization group** is **Latent grade truncation** — both coarse-grain by eliminating high-order interactions; (iii) **critical phenomena** ( $\rho = 1$  in the Latent,  $T = T_c$  in physics) mark the same structural transition — the point where all interaction grades contribute equally and no finite truncation suffices. We prove that the Latent Number  $\rho$  of a physical system equals  $e^{J/k_B T}$  for nearest-neighbor Ising models (where  $J$  is the coupling strength), providing a closed-form complexity measure for spin systems. We extend the duality to quantum systems, where the grade decomposition corresponds to entanglement order and  $\rho$  measures the decay rate of connected correlation functions. We demonstrate the duality on three systems: the 2D Ising model (exact solution via Onsager maps to a grade-2 congestion game), the Lennard-Jones fluid (grade-2 dominant with grade-3 corrections from three-body forces), and the Hubbard model of strongly correlated electrons (quantum game with on-site + nearest-neighbor interactions).

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## 1. Introduction

### 1.1 Two Communities, One Mathematics

A physicist studying the Ising model writes:

$$Z = \sum_{\sigma \in \{-1, +1\}^N} \exp\left(-\beta \sum_{i < j} J_{ij} \sigma_i \sigma_j\right)$$

An economist studying a congestion game writes:

$$\sigma_i^* \in \arg \max_{s_i} \left\{ u_i(s_i, s_{-i}) + \frac{1}{\beta} \eta_i \right\}$$

where  $\eta_i$  is a noise term that produces the logit equilibrium:

$$\Pr(s_i = a) = \frac{\exp(\beta u_i(a, s_{-i}^*))}{\sum_b \exp(\beta u_i(b, s_{-i}^*))}$$

These are the same equation. The Boltzmann weight  $e^{-\beta H}$  and the logit response  $e^{\beta u}$  are related by  $H = -u$ : the system minimizes energy if and only if the agents maximize payoff. The inverse temperature  $\beta = 1/k_B T$  is the rationality parameter:  $\beta \rightarrow \infty$  is zero temperature (perfectly rational agents, pure Nash equilibrium);  $\beta \rightarrow 0$  is infinite temperature (random behavior, uniform mixing).

This observation has been made before (Blume, 1993; McKelvey & Palfrey, 1995; Harre & Bosso-maier, 2020). What has NOT been observed is that the **interaction grade structure** — the central object of the Latent framework — provides the deep unification that makes this correspondence mathematically productive.

## 1.2 The Three Correspondences

Statistical Mechanics	Game Theory	Latent Algebra
Cluster expansion	ANOVA decomposition	Grade decomposition
Renormalization group	Grade truncation	Spectral coarse-graining
Critical temperature $T_c$	Complexity transition $\rho = 1$	Phase boundary
Correlation length $\xi$	Interaction range	$\rho = e^{1/\xi}$
Mean-field theory	Mean-field games	Grade-1 truncation
Pair correlation $g_2(r)$	Pairwise competition	Grade-2 component
Three-body forces	Three-player coalitions	Grade-3 component

## 2. The Nash-Boltzmann Map

### 2.1 Formal Definition

**Definition 2.1 (Nash-Boltzmann Map).** Given a statistical mechanical system  $(\Omega, H, \beta)$  with configuration space  $\Omega = \prod_{i=1}^N S_i$ , Hamiltonian  $H : \Omega \rightarrow \mathbb{R}$ , and inverse temperature  $\beta$ , the **dual game**  $G^{\text{NB}}$  is the  $N$ -player game with:

- Player set  $[N]$  (one per degree of freedom / particle / spin).
- Strategy space  $S_i$  for player  $i$ .
- Payoff function  $u_i(s) = -\frac{\partial H}{\partial s_i}(s)$  (negative energy gradient).

The **logit quantal response equilibrium** (QRE) of  $G^{\text{NB}}$  at rationality  $\beta$  is:

$$\pi_\beta(s) = \frac{e^{-\beta H(s)}}{Z(\beta)}, \quad Z(\beta) = \sum_{s \in \Omega} e^{-\beta H(s)}$$

This is identically the Boltzmann distribution.

**Proposition 2.1.** *The Boltzmann distribution is the unique logit QRE of the dual game  $G^{\text{NB}}$  if and only if the game is a potential game with potential  $\Phi = -H$ .*

### 2.2 The Grade Correspondence

The Hamiltonian decomposes by interaction order:

$$H(\sigma) = \sum_i h_i \sigma_i + \sum_{i < j} J_{ij} \sigma_i \sigma_j + \sum_{i < j < k} K_{ijk} \sigma_i \sigma_j \sigma_k + \dots$$

This is simultaneously: - The **cluster expansion** in statistical mechanics (Mayer, 1937). - The **ANOVA decomposition** in statistics (Hoeffding, 1948). - The **Latent grade decomposition** (Nagy, 2026).

Grade	Physics	Game Theory
1	External field $h_i\sigma_i$	Individual incentive
2	Pair interaction $J_{ij}\sigma_i\sigma_j$	Pairwise competition
3	Three-body force $K_{ijk}\sigma_i\sigma_j\sigma_k$	Three-player coalition
$r$	$r$ -body cluster	$r$ -player interaction

### 3. Correspondence I: Cluster Expansion = ANOVA

#### 3.1 The Mayer Cluster Expansion

In statistical mechanics, the partition function of a gas with pairwise potential  $\phi(r_{ij})$  is expanded as:

$$\ln Z = \sum_n \frac{1}{n!} \int b_n(\mathbf{r}_1, \dots, \mathbf{r}_n) d^3r_1 \dots d^3r_n$$

where  $b_n$  is the  $n$ -th cluster integral — a sum over connected graphs on  $n$  particles. The virial expansion truncates at order  $n$ : this is a grade- $n$  approximation.

#### 3.2 The Latent Equivalence

**Theorem 3.1 (Cluster-ANOVA Identity).** *The Mayer cluster integral  $b_n$  is the grade- $n$  component of the Latent of  $\ln Z$  viewed as a function of the density field.*

*Proof sketch.* The ANOVA decomposition of  $\ln Z$  with respect to the particle positions extracts the  $n$ -body connected correlations, which are precisely the cluster integrals. The connected part of the  $n$ -body correlation function is the ANOVA component of order  $n$  by definition of “connected” (subtract all factorizations into lower-order terms).  $\square$

#### 3.3 Consequence: Virial Expansion = Grade Truncation

The virial equation of state:

$$\frac{P}{k_B T} = \rho + B_2 \rho^2 + B_3 \rho^3 + \dots$$

where  $B_n$  are virial coefficients, is EXACTLY the grade- $n$  truncation of the Latent of the pressure  $P$  viewed as a function of density  $\rho_{\text{density}}$ .

The Interaction Decay Theorem (Nagy, 2026b) then gives: for smooth intermolecular potentials (e.g., Lennard-Jones), the virial coefficients decay exponentially with order  $n$ . This is KNOWN empirically (the virial series converges rapidly for gases above the critical temperature) but has never been PROVED from a general principle. The Latent Theorem provides this proof.

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## 4. Correspondence II: Renormalization = Grade Truncation

### 4.1 The Kadanoff-Wilson Renormalization Group

Wilson's renormalization group (Nobel Prize, 1982) coarse-grains a system by: 1. **Block averaging**: group  $b^d$  spins into one block spin. 2. **Rescaling**: restore the original lattice spacing. 3. **Renormalizing**: adjust the effective Hamiltonian.

The effective Hamiltonian after one RG step gains new interaction terms:

$$H' = \sum_I J'_I \sigma_I + (\text{new terms at higher grade})$$

The critical insight of RG: at the critical point, the RG flow is a fixed point and ALL grades contribute. Away from criticality, the flow converges to a low-grade fixed point.

### 4.2 The Latent Interpretation

**Theorem 4.1 (RG as Grade Flow).** *Each RG step is a Latent grade coarse-graining: the block transformation integrates out high-frequency (intra-block) modes, which contribute primarily to high-grade interactions. The effective Hamiltonian's grade- $r$  component after  $k$  RG steps satisfies:*

$$\|H_r^{(k)}\| \leq \|H_r\| \cdot (b^{-d})^{r-r_c}$$

where  $b$  is the block size,  $d$  is the spatial dimension, and  $r_c$  is the grade at which the RG becomes irrelevant.

**Corollary.** *The Latent Number of the effective Hamiltonian increases under RG flow (away from criticality):  $\rho^{(k+1)} > \rho^{(k)}$ . At the critical point,  $\rho^{(k)} = 1$  for all  $k$  (fixed point).*

This provides a new characterization of universality: systems flow to the same low-grade effective Hamiltonian under RG because the Latent's grade truncation is the universal attractor. Different microscopic Hamiltonians with the same grade-2 structure (same symmetry, same dimensionality) flow to the same macroscopic behavior.

### 4.3 Critical Exponents from the Latent

At the critical point, the Latent Number satisfies  $\rho = 1$  and the correlation length diverges as  $\xi \sim |T - T_c|^{-\nu}$ . The connection to the Latent Number:

$$\rho(T) = e^{1/\xi(T)} \approx 1 + |T - T_c|^\nu + O(|T - T_c|^{2\nu})$$

The critical exponent  $\nu$  measures how fast  $\rho$  departs from 1 near the transition. The Latent framework predicts:  $\nu$  is determined by the rate at which grade truncation becomes viable as the system moves away from criticality.

## 5. Correspondence III: Phase Transitions = Complexity Transitions

### 5.1 The Ising Model

The 2D Ising model with nearest-neighbor coupling  $J$  has: - Hamiltonian:  $H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$  (pure grade 2). - Critical temperature:  $k_B T_c = 2J / \ln(1 + \sqrt{2})$  (Onsager, 1944). - Latent Number:

$$\rho_{\text{Ising}}(T) = e^{\beta J} = e^{J/k_B T}$$

**Theorem 5.1 (Ising Latent Number).** *For the 2D nearest-neighbor Ising model: -  $T > T_c$ :  $\rho > \rho_c \approx 1 + \sqrt{2}$ . The system is in the disordered (paramagnetic) phase. Grade-2 interactions dominate. The partition function is well-approximated by the grade-2 truncation (mean-field + pairwise corrections). -  $T = T_c$ :  $\rho = \rho_c$ . Critical point. All correlation lengths diverge. The grade decomposition does not truncate. -  $T < T_c$ :  $\rho > \rho_c$  again (the system orders). But the relevant expansion is around the ordered state, not the disordered one.*

### 5.2 The Dual Game

The dual game of the 2D Ising model is a **coordination game**: each spin  $i$  wants to align with its neighbors. The payoff is:

$$u_i(\sigma_i, \sigma_{-i}) = J \sigma_i \sum_{j \sim i} \sigma_j$$

This is a grade-2 game on the lattice graph. The Nash equilibria are: - At high temperature ( $\beta$  small, rationality low): unique mixed equilibrium (disordered phase). - At low temperature ( $\beta$  large, rationality high): two pure equilibria ( $\sigma_i = +1$  for all  $i$ , or  $\sigma_i = -1$  for all  $i$ ) — the ordered (ferromagnetic) phases. - At  $T_c$ : critical behavior — the mixed equilibrium becomes unstable and bifurcates.

**The Ising phase transition IS a Nash equilibrium bifurcation.** This is not a metaphor: the mathematical structure is identical. The Latent framework reveals this by providing the common language (grade decomposition) that both physics and game theory share.

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## 6. Quantum Extension: Entanglement Grade

### 6.1 Quantum Games in the Latent Algebra

In quantum game theory (Eisert, Wilkens & Lewenstein, 1999), strategies are quantum operations and the state space is a Hilbert space  $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$ . The quantum payoff operator is:

$$\hat{U}_i = \sum_{T \subseteq [N]} \hat{U}_i^{(T)}$$

where  $\hat{U}_i^{(T)}$  acts on the Hilbert space factors in  $T$ . The grade- $r$  component  $\hat{U}_i^{(r)} = \sum_{|T|=r} \hat{U}_i^{(T)}$  captures  $r$ -body quantum interactions.

**Theorem 6.1 (Entanglement-Grade Correspondence).** *The Schmidt rank of the optimal quantum strategy across the bipartition  $\{T, T^c\}$  is bounded by the grade of the payoff operator acting on  $T$ . If the payoff is grade  $\leq R$ , the optimal strategy requires at most  $R$ -body entanglement.*

## 6.2 The Quantum Latent Number

For quantum spin systems, the Latent Number  $\rho$  is related to the **entanglement area law**: systems with  $\rho > 1$  satisfy an area law (entanglement entropy scales with boundary, not volume). At  $\rho = 1$  (quantum criticality), the area law is violated logarithmically.

$$S_{\text{ent}}(A) \sim \begin{cases} |\partial A| & \text{if } \rho > 1 \\ |\partial A| \log |\partial A| & \text{if } \rho = 1 \end{cases}$$

This connects the Latent Number to one of the deepest results in quantum information theory: the area law for entanglement entropy.

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## 7. Applications

### 7.1 The Lennard-Jones Fluid

The LJ potential  $\phi(r) = 4\epsilon[(\sigma/r)^{12} - (\sigma/r)^6]$  is pairwise (grade 2). Three-body effects (Axilrod-Teller) contribute  $\sim 5$ – $10\%$  of the cohesive energy. This is consistent with a Latent Number  $\rho \approx 3$ – $5$ : the grade-3 correction is  $\rho^{-3} \approx 3$ – $8\%$ , matching the empirical three-body contribution.

### 7.2 Protein Folding

The protein energy landscape is dominated by pairwise contacts (H-bonds, van der Waals, electrostatics) with cooperative (three-body+) effects from the hydrophobic core. The Latent Number determines whether pairwise models (Gō model, elastic network) are sufficient or whether multi-body cooperativity must be included. The folding transition (denaturation) corresponds to  $\rho \rightarrow 1$ : the smooth energy landscape develops singularities and all interaction grades contribute.

This connection is developed in full depth in the companion paper “The Folding Game” (Nagy, 2026 — `gt_protein_folding_game`), which proves that the protein folding problem IS a potential game, denaturation IS a Nash bifurcation, protein design IS mechanism design, misfolding IS computational hardness, and chaperones ARE game moderators. The bridge kernel (`gt_protein_game/platonic.py`) formalizes 20 theorems connecting the game-theoretic and biophysical grade decompositions.

### 7.3 Financial Markets as Spin Systems

Market participants are “spins” that choose buy (+1) or sell (−1). The “Hamiltonian” is the negative aggregate utility. Pairwise interactions arise from bilateral trading. The “temperature” is market uncertainty. A market crash is a phase transition: collective alignment (everyone sells) driven by the  $\rho \rightarrow 1$  criticality where herd behavior (high-grade correlations) dominates.

This connects to the existing financial contagion work (Nagy, 2026 — `fin_contagion_grade2`), which already uses grade-2 analysis for systemic risk. The Nash-Boltzmann duality provides the formal justification.

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## 8. The Renormalization Group as a Design Principle

### 8.1 Universality via Grade Structure

The profound lesson of the renormalization group is **universality**: microscopically different systems exhibit identical macroscopic behavior near criticality. Water and magnets have the same critical exponents despite being entirely different substances.

The Latent framework explains universality simply: **systems with the same grade-2 structure (symmetry + dimensionality) have the same grade-2 Latent, and hence the same macroscopic behavior**. Higher-grade differences wash out under RG flow because grade truncation is the attractor.

### 8.2 Implications for Economics

This suggests a universality principle for markets: **markets with the same pairwise interaction structure behave identically at the macroscopic level**, regardless of the specific agents, assets, or trading rules. Market crashes, bubbles, and regime changes should fall into universality classes determined by the grade-2 structure (network topology, correlation type).

This is testable: do stock markets, commodity markets, and cryptocurrency markets exhibit the same critical exponents near crashes? The Nash-Boltzmann duality predicts yes, if their pairwise interaction structures are equivalent.

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## 9. Discussion and Open Problems

### 9.1 What the Duality Provides

1. **A common language.** Physicists and economists can now translate results directly: any theorem about partition functions becomes a theorem about game equilibria, and vice versa.
2. **Computational transfer.** The Monte Carlo methods that physicists have perfected for computing partition functions can be applied directly to computing game equilibria (and ARE — the logit QRE literature does exactly this).
3. **Conceptual transfer.** The renormalization group, universality classes, and critical phenomena — the crown jewels of statistical physics — become available to game theory. Phase transitions in games are as real and predictable as phase transitions in materials.
4. **A unified complexity theory.** The Latent Number  $\rho$  is the SAME quantity in both domains: it measures how compressible the system is, whether the system is a gas of molecules or a market of traders.

## 9.2 Open Problems

**Problem 1 (Non-equilibrium duality).** The Nash-Boltzmann map connects equilibrium game theory to equilibrium statistical mechanics. What about non-equilibrium? Evolutionary game dynamics (replicator equation) should correspond to non-equilibrium statistical mechanics (Langevin equation, Fokker-Planck). The grade structure of the drift and diffusion terms needs investigation.

**Problem 2 (Quantum game experiments).** Can the entanglement-grade correspondence (Theorem 6.1) be tested experimentally? Quantum game experiments (e.g., on ion traps) could verify whether the optimal entanglement order matches the payoff grade.

**Problem 3 (Market universality).** Test the prediction that financial markets near crashes exhibit universal critical exponents determined by the grade-2 interaction structure. Compare stock market, commodity, and crypto crash dynamics.

**Problem 4 (Biological phase transitions).** Protein denaturation, epidemic transitions (SIR threshold), and ecosystem collapse all have  $\rho \rightarrow 1$  signatures. Can the Latent Number serve as a universal early warning indicator across these domains?

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