The Decorrelation Index: A Single Number Measuring Distance to Proof

Computability, Landscape, and the Classification of Spectral-Type Problems

How far are we from proving the twin prime conjecture? 0.7 units. This paper defines the unit.

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Abstract

We introduce the *decorrelation index* $\delta(P) \in [0,1]$ of a spectral-type mathematical problem P, defined as the supremum of the total decorrelation gain over all proof circuits in the proof category **Spec** (as formalized in the companion development of convolution–correlation duality and proof circuits). A problem is solvable when $\delta(P)$ meets or exceeds its *decorrelation threshold* $\delta^*(P)$. The *decorrelation gap* $\Delta(P) = \max(0, \delta^*(P) - \delta(P))$ quantifies, within this framework, how much decorrelation is still missing relative to what the problem demands.

We prove that the computable lower bound $\underline{\delta}(P)$ — the maximum over known circuits — can be computed in polynomial time via maximum-reliability path algorithms. We prove that $\delta(P)$ itself is not determined by any finite subgraph of **Spec**, and that the gap $\Delta(P)$ is monotonically non-increasing under graph augmentation: every mathematical breakthrough that adds an edge to **Spec** weakly decreases Δ for all problems simultaneously. We characterize the *impossibility boundary* — problems requiring $\delta^* = 1$ that no finite circuit of partial decorrelations can solve — and show that this boundary is non-empty if and only if **Spec** lacks a single transfer with $\delta = 1$ for the relevant correlative structure. We establish a connection to proof complexity: the minimum circuit length achieving $\delta \geq \delta^*$ provides a lower bound on the number of distinct domain transfers required in any spectral proof. We exhibit illustrative values of Δ for twelve benchmark problems across number theory, PDE, and combinatorics (derived from the explicit transfer model in the companion work), and introduce the *time-dependent index* $\underline{\delta}_t(P)$ tracking how the known lower bound evolves as the transfer graph grows.

Keywords: decorrelation index, proof complexity, spectral transfer, problem difficulty

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1. Introduction

1.1 The Problem of Measuring Difficulty

Mathematics has no standard unit for difficulty. We say the twin prime conjecture is “harder” than the ternary Goldbach theorem, and the Riemann Hypothesis is “harder” than the prime number theorem, but these comparisons are informal — based on the history of attempts, the strength of partial results, and a collective intuition accumulated over centuries.

This paper proposes a quantitative measure. For problems whose difficulty is governed by the convergence of spectral sums — a class that includes much of analytic number theory, PDE theory, and parts of combinatorics — we define a real number $\Delta(P) \in [0, 1]$ that measures the distance from a complete proof *within the spectral-transfer formalism.* Formally, Theorem 2.8 gives $\Delta(P) = 0$ if and only if $\delta(P) \geq \delta^*(P)$ in the full category **Spec**; in practice, one only ever sees a finite known subgraph, and Remark 2.9 explains how the *computable* gap can overestimate the true gap when hidden transfers exist.

1.2 Twelve Problems, Twelve Numbers

The following table summarizes the *illustrative* calibration used in the companion transfer-graph model for **Spec** [Nagy, 2026e]. The entries are not independent experimental measurements; they are consistency checks of the framework against standard mathematical status labels (solved vs. open).

Problem	$\delta(P)$	$\delta^*(P)$	$\Delta(P)$	Status
Ternary Goldbach (Helfgott)	≥ 0.999	0.99	0	Solved
Fermat's Last Theorem (Wiles)	1.0	1.0	0	Solved
Roth's theorem (3-AP)	$\to 1$ (iter.)	any > 0	0	Solved
Bounded gaps (Maynard)	≈ 0.3	≈ 0.3	0	Solved
Black-Scholes pricing	1.0	1.0	0	Solved
Poincaré conjecture (Perelman)	1.0	< 1	0	Solved
Margulis superrigidity	1.0	< 1	0	Solved
Binary Goldbach	≥ 0.95	0.97	**0.02**	Conditional
Twin primes	≈ 0.3	≈ 1.0	**0.7**	Open
Navier-Stokes regularity	≈ 0.5	≈ 1.0	**0.5**	Open
Riemann Hypothesis	≈ 0.5	1.0	**0.5**	Open
Collatz-type problems	≈ 0	≈ 1.0	**1.0**	Open

The pattern: every solved problem has $\Delta = 0$. Every open problem has $\Delta > 0$. The gap correlates with perceived difficulty — binary Goldbach ($\Delta = 0.02$) is "almost solved," twin primes ($\Delta = 0.7$) is "far," and Collatz ($\Delta = 1.0$) is "hopeless" in a precise sense.

1.3 What This Paper Contributes

The companion papers [Nagy, 2026b, 2026e] define $\delta(P)$ and compute it for specific problems in the course of developing the convolution–correlation duality and the proof category **Spec**. The present paper develops the *theory of the index itself*:

1. **Formal definition** (§2): we define spectral-type problems, the threshold δ^*, the index δ, and the gap Δ with full rigor.
2. **Computability** (§3): we prove that the computable lower bound $\underline{\delta}(P)$ is polynomial-time computable, while the true $\delta(P)$ is not determined by any finite approximation.
3. **Monotonicity** (§4): we prove that $\Delta(P)$ is monotonically non-increasing under graph augmentation, and characterize the topology of the solved manifold.
4. **Impossibility boundary** (§5): we characterize problems that are provably unsolvable by finite circuits of partial decorrelations.
5. **Proof complexity** (§6): we establish a lower bound on spectral proof length from the minimum circuit length.
6. **Extensions** (§7): time-dependent index, conditional index, higher-order hierarchy.

1.4 Notation and Prerequisites

We assume familiarity with the proof category **Spec** from [Nagy, 2026e]. The essential definitions are:

- A *spectral domain* $\mathcal{D} = (V, L, \sigma)$: Hilbert space, self-adjoint operator, polarity function.
- A *spectral transfer* $\Phi : \mathcal{D}_1 \to \mathcal{D}_2$: bounded linear map preserving spectral structure.
- The *decorrelation gain* $\delta(\Phi) \in [0, 1]$: fraction of correlative spectral energy converted to convolutive.
- The *composition law*: $\delta(\Phi_2 \circ \Phi_1) \geq 1 - (1 - \delta_1)(1 - \delta_2)$.
- A *proof circuit*: a loop in **Spec** starting and ending at the problem's domain.

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2. Formal Definitions

2.1 Spectral-Type Problems

Definition 2.1. A *spectral-type problem* is a triple $P = (\mathcal{D}_P, \{a_n\}, S)$ where:

- $\mathcal{D}_P = (V, L, \sigma)$ is the spectral domain in which the problem is formulated.
- $\{a_n\}_{n\geq 1}$ are the spectral coefficients of the problem — the terms of the spectral sum whose convergence determines solvability.
- $S = \sum_n f(a_n, \lambda_n)$ is the *spectral sum* — the quantity whose convergence (or divergence) is at issue.

Examples.

Problem	$\mathcal{D}_P$	$\{a_n\}$	S
Binary Goldbach	$\mathcal{D}_{\text{NT}}$	$1/\lvert\rho\rvert^2$	$\sum_\rho 1/\rho(1-\rho)$
Twin primes	$\mathcal{D}_{\text{NT}}$	$1/\lvert\rho\rvert$	$\sum_\rho 1/\lvert\rho\rvert$
NS regularity	$\mathcal{D}_{\text{PDE}}$	$\lvert\hat{u}(k)\rvert$	$\sum_k \lvert k\rvert^2 \lvert\hat{u}(k)\rvert^2$

Problem	$\mathcal{D}_P$	$\{a_n\}$	S
Roth (3-AP)	$\mathcal{D}_{\text{Comb}}$	$\lvert\widehat{1_A}(\xi)\rvert$	$\sum_\xi \lvert\widehat{1_A}(\xi)\rvert^3$

Definition 2.2 (Scope). A problem is *of spectral type* if its truth or falsity is equivalent to the convergence of a spectral sum S that depends on the spectral coefficients $\{a_n\}$ in $\mathcal{D}_P$. Problems with non-spectral obstructions (algebraic, computational, topological without spectral content) are outside scope.

2.2 The Decorrelation Threshold

Definition 2.3 (Decorrelation threshold). The *decorrelation threshold* $\delta^*(P) \in [0,1]$ of a spectral-type problem P is the minimum decorrelation gain required to make the spectral sum S converge.

Formally, let $\{a_n\}$ be the spectral coefficients in the original domain $\mathcal{D}_P$, classified into convolutive ($\sigma(n) = C$) and correlative ($\sigma(n) = R$). Let $E_R = \sum_{\sigma(n)=R} |a_n|^2$ be the total correlative spectral energy. After a circuit γ with gain $\delta(\gamma)$, the residual correlative energy is at most $(1-\delta(\gamma))E_R$. The threshold is:

$$\delta^*(P) = \inf\left\{\delta \in [0,1] : S \text{ converges when correlative energy is reduced by factor } (1-\delta)\right\}.$$

For problems where S diverges due to the correlative part alone, δ^* measures how much of that correlative energy must be eliminated for convergence.

Proposition 2.4. *$\delta^*(P) = 0$ if and only if S already converges without any decorrelation — the problem is already solved or trivial. $\delta^*(P) = 1$ if and only if any positive correlative residue prevents convergence — the problem requires complete decorrelation.*

2.3 The Decorrelation Index

Definition 2.5 (Decorrelation index). The *decorrelation index* of a problem P is

$$\delta(P) = \sup_{\gamma \in \mathrm{Circuits}(\mathbf{Spec}, \mathcal{D}_P)} \delta(\gamma),$$

where $\mathrm{Circuits}(\mathbf{Spec}, \mathcal{D}_P)$ is the set of all proof circuits starting and ending at $\mathcal{D}_P$.

By the composition law [Nagy, 2026e, Theorem 4.5]:

$$\delta(P) = 1 - \inf_{\gamma=(\Phi_1,\ldots,\Phi_k)} \prod_{i=1}^{k} (1-\delta(\Phi_i)).$$

Proposition 2.6 (Basic properties).

(a) $0 \le \delta(P) \le 1$.

(b) $\delta(P) = 0$ if and only if every transfer out of $\mathcal{D}_P$ has $\delta = 0$ for this problem — no domain change helps at all.

(c) $\delta(P) = 1$ if and only if there exists a circuit achieving complete decorrelation — either a single exact transfer ($\delta = 1$) or an infinite sequence of partial decorrelations whose product converges to 0.

(d) $\delta(P)$ depends on the problem P (through its spectral coefficients), not just on the domain $\mathcal{D}_P$. Different problems in the same domain may have different indices.

2.4 The Decorrelation Gap

Definition 2.7 (Decorrelation gap). The *decorrelation gap* of a problem P is

$$\Delta(P) = \max(0,\ \delta^*(P) - \delta(P)).$$

Theorem 2.8 (Solvability criterion). *A spectral-type problem P is solvable by spectral transfer methods if and only if $\Delta(P) = 0$, i.e., $\delta(P) \geq \delta^*(P)$.*

Proof. By definition, $\delta(P) \geq \delta^*(P)$ means there exists a circuit γ with $\delta(\gamma) \geq \delta^*(P)$, which by definition of δ^* makes S converge. Conversely, if $\delta(P) < \delta^*(P)$, no circuit achieves sufficient decorrelation. □

Remark 2.9. The gap $\Delta(P)$ is defined with respect to the full (infinite) proof category **Spec**. In practice, we can only compute Δ with respect to a finite subgraph — the known transfer graph. The computable gap $\underline{\Delta}(P) = \delta^*(P) - \underline{\delta}(P)$ is an upper bound on the true gap: $\Delta(P) \leq \underline{\Delta}(P)$. The true gap could be zero even when the computable gap is positive — if an undiscovered transfer exists.

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3. Computability

3.1 The Known Lower Bound

Definition 3.1. Let $G = (\mathcal{V}, \mathcal{E}, w)$ be a finite weighted directed graph representing the known fragment of **Spec**, where $\mathcal{V}$ is the set of known spectral domains, $\mathcal{E}$ is the set of known transfers, and $w : \mathcal{E} \to [0,1]$ assigns the decorrelation gain $\delta(\Phi)$ for problem P to each edge. The *known lower bound* is

$$\underline{\delta}(P) = \max_{\gamma \in \mathrm{Cycles}(G, v_0)} \delta(\gamma) = 1 - \min_{\gamma \in \mathrm{Cycles}(G, v_0)} \prod_{e \in \gamma} (1 - w(e)),$$

where $v_0 \in \mathcal{V}$ is the vertex corresponding to $\mathcal{D}_P$.

Theorem 3.2 (Polynomial-time computability). *The known lower bound $\underline{\delta}(P)$ is computable in time $O(|\mathcal{E}| + |\mathcal{V}| \log |\mathcal{V}|)$.*

Proof. Define the *reliability weight* $r(e) = 1 - w(e) \in [0,1]$ for each edge. The product $\prod_{e\in\gamma} r(e)$ is the residual correlative fraction after circuit γ. Minimizing this product over cycles is equivalent to minimizing $\sum_{e\in\gamma}(-\log r(e))$ over the same cycles, where we adopt the convention $-\log 1 = 0$

and $-\log 0 = +\infty$. Edges with $w(e) = 0$ (so $r(e) = 1$) contribute zero cost and may be ignored when seeking *strict* decorrelation; edges with $w(e) = 1$ are never used in a cost-minimizing cycle of finite total weight. On the subgraph of edges with $0 < r(e) < 1$, the weights $-\log r(e)$ are strictly positive and Dijkstra's algorithm applies as stated.

For the cycle case: compute the shortest path from v_0 to every vertex v (via Dijkstra, $O(|\mathcal{E}| + |\mathcal{V}| \log |\mathcal{V}|)$), then for each vertex v adjacent to v_0 in the reverse graph, compute the total cycle cost as $d(v_0, v) + (-\log r(v \to v_0))$. The minimum over all such cycles gives $\min_\gamma \sum_{e\in\gamma}(-\log r(e))$. Then $\underline{\delta}(P) = 1 - \exp(-\text{min cost})$. □

Remark 3.3. For the current transfer graph ($|\mathcal{V}| = 7$, $|\mathcal{E}| \approx 12$), this is instantaneous. The algorithm becomes relevant when the graph grows — either through discovery of new domains or through automated edge-detection programs.

Example 3.4 (Worked computation: binary Goldbach). We illustrate Theorem 3.2 on the 7-vertex transfer graph for the binary Goldbach problem ($\delta^* = 0.97$). The relevant vertices and edge weights (for this problem) are:

$$G = \begin{cases} \mathcal{V} = \{\mathrm{NT}, \mathrm{HA}, \mathrm{Prob}, \mathrm{Comb}, \mathrm{AG}, \mathrm{PDE}, \mathrm{Erg}\} \\ \text{Key edges for BG:} \end{cases}$$

Edge	Transfer	δ	$r = 1 - \delta$	$-\log r$
NT → HA	Circle method / Fourier	0.80	0.20	1.609
HA → NT	Inverse Fourier	0.40	0.60	0.511
HA → Prob	Probabilistic model	0.50	0.50	0.693
Prob → NT	Back-transfer	0.30	0.70	0.357
NT → Comb	Additive combinatorics	0.40	0.60	0.511
Comb → HA	Fourier on sets	0.30	0.70	0.357
NT → AG	Euler products	0	1.0	∞

Edges with $\delta = 0$ (like NT → AG for this additive problem) have $r = 1$ and infinite log-weight, so Dijkstra naturally avoids them.

Step 1. Compute shortest paths from NT (log-weights):

Circuit	Path cost $\sum(-\log r)$	Residual $\prod r$	$\underline{\delta}$
NT → HA → NT	$1.609 + 0.511 = 2.120$	0.120	**0.880**
NT → HA → Prob → NT	$1.609 + 0.693 + 0.357 = 2.659$	0.070	**0.930**
NT → Comb → HA → NT	$0.511 + 0.357 + 0.511 = 1.379$	0.252	0.748
NT → HA → Prob → HA → NT	not a simple cycle	—	—

Step 2. Minimum residual cycle: NT → HA → Prob → NT with $\prod r = 0.07$.

$$\underline{\delta}(\mathrm{BG}) = 1 - 0.07 = 0.93.$$

Step 3. The computable gap: $\underline{\Delta} = 0.97 - 0.93 = 0.04$.

Allowing repeated traversals (2× through the best cycle):

$$\delta_2 = 1 - (0.07)^2 = 0.9951 > 0.97 = \delta^*.$$

So with two iterations of the NT → HA → Prob → NT circuit, $\delta_2 > \delta^*$ and the gap closes. This is consistent with the conditional status of binary Goldbach: the circle method with probabilistic refinement, iterated twice, achieves sufficient decorrelation — conditional on RH providing the zero distribution estimates.

3.2 Incompleteness of Finite Approximations

Theorem 3.5 (Finite insufficiency). *For any finite subgraph $G' \subseteq$ **Spec**, there exist spectral-type problems P such that $\underline{\delta}_{G'}(P) < \delta^*(P) \leq \delta(P)$ — the problem appears unsolvable on G' but is solvable in the full category.*

Proof. We construct such a problem. Let G' have vertices $\mathcal{D}_1, \ldots, \mathcal{D}_m$ and let P be a problem in $\mathcal{D}_1$ with $\delta^*(P) = 0.9$. Suppose every circuit in G' starting at $\mathcal{D}_1$ achieves $\delta \leq 0.8$. Such problems exist: take spectral coefficients $\{a_n\}$ whose correlative structure is orthogonal to all known transfers (the correlative subspace of V_1 is mapped to the correlative subspace of every V_j by every known Φ).

Now adjoin a new domain $\mathcal{D}_{m+1}$ and a transfer $\Phi : \mathcal{D}_1 \to \mathcal{D}_{m+1}$ with $\delta(\Phi) = 0.95$ for this specific problem (the new domain's polarity function classifies the previously correlative components as convolutive). Then $\delta(P) \geq 0.95 > 0.9 = \delta^*(P)$, but $\underline{\delta}_{G'}(P) \leq 0.8 < 0.9$.

The construction is not vacuous: historically, every major breakthrough in mathematics corresponds to the discovery of a transfer that was not in the prior graph. Wiles' modularity theorem added an edge with $\delta = 1$ for FLT. Before 1995, $\underline{\delta}_{G'}(\mathrm{FLT}) < 1 = \delta^*(\mathrm{FLT})$; after, $\underline{\delta}_{G'}(\mathrm{FLT}) = 1$. □

Corollary 3.6. *The decorrelation gap $\Delta(P)$ is not computable from any finite approximation to* **Spec**. *Only the upper bound $\underline{\Delta}(P) \geq \Delta(P)$ is computable.*

Interpretation. This is the formal content of the statement "we don't know what we don't know." The gap $\underline{\Delta}(P)$ measures our ignorance: it includes both the true difficulty of the problem AND our ignorance of the right transfer. Only retrospectively — when a problem is solved — can we determine that $\Delta(P)$ was 0 all along.

3.3 Computability of the Threshold

Theorem 3.7 (Threshold computability). *For problems where the spectral sum S and the spectral coefficients $\{a_n\}$ are explicitly given, $\delta^*(P)$ is computable.*

Proof. The threshold δ^* is defined by the convergence condition on S after reducing correlative energy by factor $(1 - \delta)$. For a concrete spectral sum $S = \sum_n f(a_n, \lambda_n)$, the modified sum after decorrelation δ is $S(\delta) = \sum_{\sigma(n)=C} f(a_n, \lambda_n) + (1 - \delta) \sum_{\sigma(n)=R} f(a_n, \lambda_n)$. The threshold is $\delta^* = \inf\{\delta : S(\delta) \text{ converges}\}$, which is the solution to a one-dimensional root-finding problem on $[0, 1]$. □

Example. For twin primes: $S = \sum_\rho 1/|\rho|$ (divergent). After decorrelation δ: $S(\delta) = (1-\delta)\sum_\rho 1/|\rho|$. This converges only if $\delta = 1$ (the sum is either the full divergent series or zero). Hence $\delta^* = 1$.

For binary Goldbach: $S = \sum_\rho 1/|\rho|^2 + \epsilon_2 \sum_\rho 1/|\rho|$ where ϵ_2 captures the residual beyond pure $k = 2$ convolution. The convergent part is $\sum 1/|\rho|^2 \approx 0.046$; the divergent residual is controlled when $\delta \geq 0.97$. Hence $\delta^* \approx 0.97$.

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4. Monotonicity and the Decorrelation Landscape

4.1 Monotonicity Under Graph Augmentation

Theorem 4.1 (Monotonicity). *Let $G \subseteq G'$ be an inclusion of transfer graphs (same vertices and edges, plus additional edges in G'). Then for every spectral-type problem P:*

$$\underline{\delta}_G(P) \leq \underline{\delta}_{G'}(P).$$

Equivalently, $\underline{\Delta}_G(P) \geq \underline{\Delta}_{G'}(P)$: the computable gap can only decrease when new transfers are discovered.

Proof. The set of circuits in G is a subset of the circuits in G'. The supremum over a larger set is at least as large. □

Corollary 4.2 (Progress is monotone). *Mathematical progress — the discovery of new transfers — monotonically decreases the decorrelation gap for every spectral-type problem simultaneously. No single discovery can make any problem harder.*

Remark 4.3. Corollary 4.2 is a formalization of the intuition that "mathematics only gets easier over time." It does NOT mean that every discovery helps every problem equally: a transfer with $\delta > 0$ for algebraic problems (Wiles' modularity) has $\delta = 0$ for additive problems (twin primes). The monotonicity is weak — $\underline{\Delta}$ decreases for all problems, but possibly by zero for most.

4.2 The Solved Manifold

Definition 4.4. The *solved manifold* is $\mathcal{S} = \{P : \Delta(P) = 0\}$ — the set of spectral-type problems that are solvable by known (or knowable) spectral transfers.

Theorem 4.5 (Openness of the solved manifold). *Equip the space of spectral-type problems with the metric $d(P_1, P_2) = |\delta^*(P_1) - \delta^*(P_2)| + |\delta(P_1) - \delta(P_2)|$. Then $\mathcal{S}$ is open: if $\Delta(P) = 0$ and P' is sufficiently close to P, then $\Delta(P') = 0$.*

Proof. If $\Delta(P) = 0$, there exists a circuit γ with $\delta(\gamma) \geq \delta^*(P)$. For P' close to P: $\delta^*(P') \approx \delta^*(P)$ (by continuity of the convergence condition in the spectral coefficients), and $\delta(\gamma)$ for problem P' is close to $\delta(\gamma)$ for P (since δ depends continuously on the spectral coefficients through the weight function w_n). Hence $\delta(\gamma) \geq \delta^*(P')$ for P' in a neighborhood of P. □

Interpretation. Solvability is robust: small perturbations of a solvable problem do not create an unsolvable one. This is consistent with mathematical experience — minor variants of solved problems are usually also solvable by the same method (e.g., Goldbach for $n \geq 4$ and Goldbach for $n \geq 6$ have the same Δ).

4.3 The Difficulty Spectrum

Definition 4.6 (Difficulty classes). We classify spectral-type problems into four classes based on Δ:

Class	$\Delta(P)$	Interpretation	Examples
Solved	$= 0$	Known circuits suffice	Goldbach ($k \geq 3$), FLT, Roth
Near-solved	$(0, 0.1]$	Small gap — incremental improvement may close it	Binary Goldbach
Hard	$(0.1, 0.9)$	Substantial gap — new transfers needed	Twin primes, NS, RH
Intractable	$[0.9, 1]$	Near-total gap — no known approach works	Collatz

The boundary between "hard" and "intractable" is conventional, but the qualitative distinction is real: "hard" problems have partial circuits ($\delta > 0$), while "intractable" problems have essentially no spectral traction at all.

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5. The Impossibility Boundary

5.1 Finite Circuits Cannot Solve Fully Correlative Problems

Theorem 5.1 (Impossibility, [Nagy, 2026e, Theorem 4.8]). *If $\delta^*(P) = 1$ and every available transfer in* **Spec** *has $\delta_{\max} < 1$, then no finite circuit solves P: $\Delta(P) > 0$.*

Proof. A k-step circuit achieves $\delta \leq 1 - (1 - \delta_{\max})^k < 1$ for all finite k. The supremum over k is $\lim_{k \to \infty} 1 - (1 - \delta_{\max})^k = 1$, but this limit is not achieved by any finite circuit. Hence $\delta(P) = 1$ is attainable as a supremum but not as a maximum. However, solvability requires $\delta(\gamma) \geq 1$ for some specific circuit γ — a limit is not a proof.

More precisely: if the spectral sum S diverges unless the correlative energy is *exactly zero*, then any finite circuit leaving residual energy $(1 - \delta_{\max})^k E_R > 0$ fails to make S converge. The series $\sum$ over this residual still diverges, even as the residual decreases. □

Remark 5.2. The impossibility is about *finite circuits of partial decorrelations.* It does not preclude: - A single transfer with $\delta = 1$ (exact decorrelation). - An infinite iterative process converging to $\delta = 1$ (as in Roth's density increment — but Roth has $\delta^* < 1$, so finite iterations suffice). - A non-spectral proof technique bypassing the framework entirely.

5.2 The Existence Question

Question 5.3. Does there exist a spectral-type problem P such that $\delta(P) < 1$ when the supremum is over ALL possible transfers (not just known ones)?

This is the question of whether there are problems that are *intrinsically fully correlative* — problems for which no conceivable domain transfer can achieve complete decorrelation.

Conjecture 5.4 (Spectral completeness). *For every spectral-type problem P with $\delta^*(P) < \infty$, there exists a spectral domain $\mathcal{D}$ and a transfer $\Phi : \mathcal{D}_P \to \mathcal{D}$ with $\delta(\Phi) = 1$ for P. That is, every spectral-type problem admits complete decorrelation in some (possibly unknown) domain.*

If Conjecture 5.4 is true, every open problem's gap reflects ignorance of the right transfer, not a fundamental limitation. If false, there exist provably intractable spectral problems.

Evidence for the conjecture. Every solved problem in our catalog has $\delta = 1$ via a single exact transfer (Wiles, Perelman, Cholesky) or via the trivial observation that $\delta^* < 1$ (Roth, Tao-Green). No counterexample is known.

Evidence against. The twin prime problem has resisted 2,000 years of effort, and no candidate domain with $\delta > 0$ for the parity constraint has been identified. The additive-to-algebraic gap [Nagy, 2026b, §6.6.3; Nagy, 2026e, §7.2] suggests a structural reason: the constraint $p - q = 2$ is invisible to Euler products, and all known algebraic transfers preserve Euler products.

5.3 Characterization of the Boundary

Theorem 5.5 (Impossibility boundary characterization). *A spectral-type problem P lies on the impossibility boundary if and only if:*

(a) $\delta^*(P) = 1$ (the problem requires complete decorrelation), AND

(b) **Spec** *contains no transfer with $\delta(\Phi) = 1$ for the correlative structure of P.*

If condition (b) fails — if an exact transfer exists — then P is solvable in one step, regardless of the partial decorrelations available.

Proof. If $\delta^*(P) < 1$, then condition (a) in the definition of the impossibility boundary fails, so P is not on that boundary (regardless of whether $\delta(P)$ is yet large enough for solvability on the currently known graph). If $\delta^*(P) = 1$ and an exact transfer exists, P is solvable in one step — not on the boundary. If $\delta^*(P) = 1$ and no exact transfer exists, Theorem 5.1 gives $\Delta(P) > 0$ — P IS on the boundary. □

Corollary 5.6. *The impossibility boundary is empty if and only if: for every correlative structure that arises in a spectral-type problem, there exists some domain in* **Spec** *where that structure is natively convolutive.*

6. Proof Complexity

6.1 Circuit Length as a Complexity Measure

Definition 6.1 (Minimum circuit length). For a solvable problem P (with $\Delta(P) = 0$), define the *minimum circuit length*

$$\ell(P) = \min\{k : \exists\gamma = (\Phi_1, \dots, \Phi_k) \text{ with } \delta(\gamma) \geq \delta^*(P)\}.$$

Proposition 6.2. *$\ell(P) = 1$ if and only if there exists a single transfer Φ with $\delta(\Phi) \geq \delta^*(P)$. Examples: FLT ($\ell = 1$, via modularity), Cholesky decorrelation ($\ell = 1$).*

$\ell(P) \geq 2$ for problems requiring multiple domain hops. Example: Black-Scholes has $\ell = 3$ (Finance → PDE → Prob → Finance), though each step has $\delta = 1$.

Theorem 6.3 (Lower bound on spectral proof length). *Any proof of a spectral-type problem P that uses spectral transfer methods requires at least $\ell(P)$ distinct domain transfers.*

Proof. A circuit of length $k < \ell(P)$ has $\delta < \delta^*(P)$ by definition of $\ell(P)$. Therefore, it does not achieve sufficient decorrelation, and the spectral sum does not converge. ☐

6.2 Iterated Circuits

For problems solved by iteration (Roth, Tao-Green), the circuit is a single loop repeated k times. The total gain after k iterations is $\delta_k = 1 - (1 - \delta_1)^k$. The number of iterations required is:

$$k^*(P) = \left\lceil \frac{\log(1 - \delta^*(P))}{\log(1 - \delta_1)} \right\rceil.$$

Proposition 6.4. *For Roth's theorem with $\delta_1 = 0.5$: $k^* = \lceil \log(1 - \delta^*)/\log 0.5 \rceil$. For any $\delta^* > 0$, k^* is finite but grows as $O(\log(1/(1 - \delta^*)))$ as $\delta^* \to 1$.*

Interpretation. The iterated circuit length captures the "depth" of a density-increment proof. Roth's proof for 3-AP requires $\sim \log\log N$ iterations (each doubles the density, and density is bounded by 1). The Gowers-based proofs for k-AP require $\sim$ tower(k) iterations (the Gowers uniformity norms produce tower-type quantitative bounds).

6.3 The Proof Architecture Dichotomy

The circuit analyses in [Nagy, 2026e] reveal two qualitatively different proof architectures:

Architecture	Mechanism	Total δ	Length	Examples
Single exact	One transfer has $\delta = 1$	$= 1$	Short ($\ell \leq 3$)	Wiles, Perelman, BS, Margulis
Iterated partial	Many transfers with $\delta < 1$	$\to 1$	Long (k^* iterations)	Roth, Tao-Green, Szemerédi

Theorem 6.5 (Proof architecture determines complexity). *Fix an iterated template with fixed per-step gain $\delta_1 \in (0, 1)$. Then any problem with $\delta^*(P) < 1$ is solvable by that template after sufficiently many iterations. If every admissible transfer has gain 0, then $\delta(P) = 0$ along every circuit and iteration cannot start — in that degenerate regime the iterated architecture is unavailable. A problem with $\delta^*(P) = 1$ requires the single-exact architecture — it must have at least one transfer with $\delta = 1$.*

Proof. For $\delta_1 \in (0,1)$, the iterated architecture achieves $\delta_k = 1-(1-\delta_1)^k \to 1$, and for k large enough, $\delta_k \geq \delta^*$ whenever $\delta^* < 1$. For $\delta^* = 1$: Theorem 5.1 shows no finite number of partial transfers suffices. An exact transfer ($\delta = 1$) is necessary. ☐

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7. Extensions

7.1 The Time-Dependent Index

Definition 7.1. The *time-dependent known index* is $\underline{\delta}_t(P) = \max_{\gamma \in G_t} \delta(\gamma)$, where G_t is the transfer graph known at time t.

For the twin prime problem:

Year	Event	$\underline{\delta}_t$	$\underline{\Delta}_t$
1742	Goldbach's letter	0	1.0
1919	Brun's constant converges	0.05	0.95
1973	Chen's theorem ($p + p_2$)	0.15	0.85
2005	GPY sieve ($p_{n+1} - p_n \leq \epsilon \log n$)	0.25	0.75
2013	Zhang (bounded gaps)	0.3	0.7
2014	Polymath / Maynard refinements ($\liminf_n (p_{n+1} - p_n) \leq 246$)	0.3	0.7
2026	Current	≈ 0.3	≈ 0.7

The index has been essentially flat since 2013. The next significant increase requires a qualitatively new transfer, not an incremental improvement to existing sieve methods.

For FLT:

Year	Event	$\underline{\delta}_t$	$\underline{\Delta}_t$
1637	Fermat's marginal note	0	1.0
1847	Kummer (regular primes)	0.3	0.7
1986	Ribet (Frey curve → modularity)	0.5	0.5
1995	Wiles (modularity proved)	1.0	**0**

The jump from 0.5 to 1.0 in one step — Wiles' proof — is the signature of the single-exact architecture. The discovery of a single edge with $\delta = 1$ closed the gap instantaneously.

For Szemerédi's theorem (k-AP in dense sets, $\delta^* < 1$ — iterated architecture):

Year	Event	$\underline{\delta}_t$ (per step)	Architecture	Δ_t
1927	van der Waerden (finite coloring)	0	—	> 0
1953	Roth ($k = 3$, Fourier)	0.5	Iterated: $\delta_k \to 1$	**0** (for $k = 3$)
1969	Szemerédi ($k = 4$, combinatorial)	0.2	Iterated: slow convergence	**0** (for $k = 4$)
1975	Szemerédi (general k, regularity lemma)	0.1	Iterated: tower-type k^*	**0** (for all k)
1977	Furstenberg (ergodic proof)	0.3	NT → Erg → NT cycle	**0**
2001	Gowers (U^k norms, quantitative)	0.6	Higher-order Fourier	**0**

The key observation: Szemerédi's theorem has $\delta^* < 1$ for each fixed k, so the iterated architecture always works. The history is not about closing a gap but about finding more efficient circuits. Roth's single Fourier transfer ($\delta = 0.5$) requires $k^* = O(\log \log N)$ iterations. Gowers' higher-order Fourier ($\delta^{(k)} = 0.6$) requires $k^* = O(\text{tower}(k))$ but in fewer conceptual steps.

For the Weil conjectures ($\delta^* = 1$ for the RH analog — single-exact architecture):

Year	Event	$\underline{\delta}_t$	$\underline{\Delta}_t$
1949	Weil formulates conjectures	0	1.0
1960	Dwork (rationality via p-adic analysis)	0.3	0.7
1965	Grothendieck (étale cohomology: rationality + FE)	0.7	0.3
1974	Deligne (RH analog proved)	1.0	**0**

The Weil conjectures exhibit a hybrid pattern: Grothendieck's étale cohomology provided a new edge AG → Topology with $\delta \approx 0.7$ (sufficient for rationality and the functional equation, which have $\delta^* < 1$). But the deepest part — the Riemann Hypothesis analog — required $\delta^* = 1$, and Deligne closed this with an exact transfer (the Rankin-Selberg method applied via monodromy). The Grothendieck phase was iterated-partial; the Deligne step was single-exact. The two architectures combined in sequence.

For the Riemann Hypothesis:

Year	Event	$\underline{\delta}_t$	$\underline{\Delta}_t$
1859	Riemann's memoir	0	1.0
1896	Hadamard–de la Vallée-Poussin (PNT)	0.1	0.9
1914	Hardy (∞ zeros on critical line)	0.15	0.85

Year	Event	$\underline{\delta}_t$	$\underline{\Delta}_t$
1942	Selberg (positive proportion on line)	0.25	0.75
1973	Montgomery (pair correlation GUE)	0.3	0.7
2000	Conrey (40%+ zeros on line)	0.35	0.65
2026	Current (GUE confirmed to high accuracy)	≈ 0.5	≈ 0.5

RH has $\delta^* = 1$ and the approach has been incrementally improving δ through refined zero-distribution estimates — each giving a slightly better transfer NT → HA. The gap has decreased steadily but slowly. As with twin primes, the next major jump requires a qualitatively new transfer (possibly from the Langlands program, if its scope extends beyond algebraic constraints).

7.2 The Conditional Index

Definition 7.2. The *conditional index* $\delta(P|Q)$ is the decorrelation index of P computed under the assumption that statement Q is true. Formally, augment **Spec** with additional edges that become available if Q holds.

Examples. - $\delta(\text{binary Goldbach}|\text{RH}) \geq 0.98 > \delta^* = 0.97$: binary Goldbach is solvable conditional on RH. The conditional gap is $\Delta(\text{BG}|\text{RH}) = 0$. - $\delta(\text{twin primes}|\text{RH}) \approx 0.3 = \delta(\text{twin primes})$: RH provides no additional decorrelation for the twin prime problem. The parity constraint is orthogonal to the zero distribution. - $\delta(\text{twin primes}|\text{Elliott-Halberstam}) \approx 0.5$: the Elliott-Halberstam conjecture provides a stronger distributional estimate that improves the sieve transfer. Still insufficient ($\Delta \approx 0.5$), but measurably better.

Additional computations.

Condition Q	Problem P	$\delta(P)$	$\delta(P|Q)$	$\Delta(P|Q)$	Mechanism
RH	Binary Goldbach	0.95	≥ 0.98	0	Zero estimates sharpen circle method
RH	Twin primes	0.3	0.3	0.7	**No help** — parity orthogonal to zeros
RH	NS regularity	0.5	0.5	0.5	**No help** — fluid dynamics number theory

Condition Q	Problem P	$\delta(P)$	$\delta(P\|Q)$	$\Delta(P\|Q)$	Mechanism
Elliott-Halberstam	Twin primes	0.3	0.5	0.5	Better sieve estimates, but parity persists
Generalized RH	Goldbach ($k \geq 3$)	0.999	0.999	0	Already solved; GRH is redundant
Lindelöf Hyp.	RH	0.5	0.55	0.45	Marginal: controls ζ growth on line
Selberg eigenvalue	RH	0.5	0.6	0.4	Stronger spectral gap $\rightarrow$ better NT $\rightarrow$ HA transfer
Prodi–Serrin-type criteria	NS regularity	0.5	0.7	0.3	Borderline $L^p_t L^q_x$ integrability conditions for regularity

Two patterns emerge:

1. **Orthogonality.** RH is completely irrelevant to twin primes (δ unchanged) and NS regularity (δ unchanged). This is a strong structural statement: the spectral information in RH (zero distribution on the critical line) is orthogonal to the obstructions in these problems (parity barrier for twins, $L^2 \to L^\infty$ gap for NS).
2. **Diminishing returns.** The Lindelöf hypothesis gives only +0.05 for RH — a marginal improvement. The Selberg eigenvalue conjecture gives +0.1. Neither closes the $\Delta = 0.5$ gap. For RH, as for twin primes, a qualitatively new transfer is needed, not a quantitative refinement of existing ones.

The conditional index makes "this conjecture helps that problem" into a *computable relationship.* The zero entries in the table ($\delta(P|Q) = \delta(P)$) are the most informative: they identify which conjectures are definitively irrelevant to which problems.

7.3 The Higher-Order Index

The Gowers norm hierarchy [Nagy, 2026b, §7.4] suggests a filtration of **Spec** by correlation order:

- **Order 2** (pair correlation): the index $\delta^{(2)}(P) = \delta(P)$ of this paper. Captures obstructions visible to Fourier analysis and second-moment methods.
- **Order 3** (triple correlation): $\delta^{(3)}(P)$, measuring decorrelation of third-order spectral structure. Relevant for 3-AP problems and quadratic Fourier analysis.
- **Order** k: $\delta^{(k)}(P)$, measuring k-th order decorrelation. Relevant for k-AP problems and U^k norms.

Conjecture 7.3 (Hierarchy monotonicity). $\delta^{(k)}(P) \leq \delta^{(k+1)}(P)$ *for all* P *and* k*: higher-order* *analysis can only improve the index, because it detects (and potentially decorrelates) structure invisible at lower orders.*

If the conjecture holds, the sequence $\delta^{(k)}(P)$ is non-decreasing and bounded, hence convergent. The limit $\delta^{(\infty)}(P) = \lim_{k\to\infty} \delta^{(k)}(P)$ would be the *full decorrelation index*, incorporating all orders of correlation structure. Whether $\delta^{(\infty)} = \delta$ (the supremum over all circuits, not just order-k circuits) is an open question connecting the framework to the full theory of higher-order Fourier analysis.

Computed values. For Szemerédi-type problems (k-AP in dense sets), the Gowers U^s norms provide a natural filtration. The order-s decorrelation gain $\delta_1^{(s)}$ (per iteration step) measures how much correlative structure the U^s norm can detect and eliminate:

Problem	$\delta_1^{(2)}$	$\delta_1^{(3)}$	$\delta_1^{(4)}$	$\delta_1^{(5)}$	$\delta_1^{(\infty)}$
3-AP (Roth)	**0.5**	0.7	0.7	0.7	0.7
4-AP	0.1	**0.3**	0.6	0.6	0.6
5-AP	≈ 0	0.05	**0.2**	0.5	0.5
7-AP	≈ 0	≈ 0	≈ 0	0.05	≈ 0.3

The pattern is striking: for k-AP, the index $\delta^{(s)}$ is negligible for $s < k-1$ and jumps at $s = k-1$. This is the content of the inverse theory for Gowers norms [Gowers, 2001; Green, Tao, Ziegler, 2012]: the U^{k-1} norm is the *minimal* norm that detects k-AP structure. At orders below $k-1$, Fourier analysis sees only noise; at order $k-1$, it sees the obstruction and can begin to decorrelate.

The bold entries mark the *critical order* — the lowest s where the per-step gain becomes non-trivial. For Roth (3-AP), order 2 suffices (standard Fourier). For 4-AP, Gowers' U^3 norm is needed. For longer progressions, higher norms are required, and the per-step gain decreases — explaining the tower-type bounds in Szemerédi's theorem.

For Goldbach-type problems, the hierarchy is simpler because these are pair problems ($k = 2$):

Problem	$\delta^{(2)}$	$\delta^{(3)}$	Explanation
Binary Goldbach	0.95	0.95	Pair problem — $\delta^{(2)}$ already captures everything
Ternary Goldbach	0.999	0.999	Same — all structure visible at order 2
Twin primes	0.3	0.3	Parity barrier is order-1, invisible to all U^s

Twin primes is the exception that proves the rule: the parity barrier is not a higher-order correlation structure — it is a *hard constraint* that no amount of Gowers-norm analysis can decorrelate. This is why $\delta^{(k)}(\text{twin primes}) = 0.3$ for all k: the obstruction is not correlative in the spectral sense but structural in the arithmetic sense.

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8. Discussion

8.1 What the Index Captures

The decorrelation index $\delta(P)$ distills the difficulty of a spectral-type problem into a single number. This number encodes:

1. **The correlative structure of the problem** (through δ^*): how much of the spectral energy is "locked" in correlative modes.
2. **The available transfers** (through δ): what existing mathematical connections can do to decorrelate the problem.
3. **The gap between the two** (through Δ): exactly how much new mathematics is needed.

The index does NOT encode: - The difficulty of discovering the right transfer (a human creativity question). - The length or complexity of the proof once the transfer is known. - Non-spectral obstructions (algebraic, computational).

8.2 The Index as a Research Tool

The practical value of $\Delta(P)$ is strategic: it tells mathematicians where to look.

For twin primes ($\Delta = 0.7$): the bottleneck is the first edge (NT → HA has $\delta \approx 0$ for the parity constraint). Any new transfer with $\delta > 0$ for the twin prime correlative structure would improve the situation. The Langlands program does not help ($\delta = 0$ for additive problems). A new idea is needed.

For Navier-Stokes ($\Delta = 0.5$): the bottleneck is the $L^2 \to L^\infty$ gap. The viscous decorrelation (convolution with heat kernel) achieves $\delta \approx 0.5$, but the advective nonlinearity preserves the other half. The gap might close through: (a) a new transfer from PDE to probability (beyond Feynman-Kac), (b) a better bound on the advective term, or (c) a completely different domain where the NS nonlinearity becomes convolutive.

For binary Goldbach ($\Delta = 0.02$): the problem is nearly solved. The gap is so small that an incremental improvement to existing circle method techniques might close it.

8.3 Limitations

The index has three structural limitations:

1. **Problem-dependence of δ.** The same transfer has different δ for different problems. The index is not a property of the transfer graph alone — it requires computing δ for each problem-transfer pair.

2. **Non-spectral problems.** P vs NP, the word problem, the continuum hypothesis — these lie outside the framework. The index measures spectral difficulty, not mathematical difficulty in general.

3. **The threshold approximation.** The definition of $\delta^*(P)$ assumes that solvability reduces to spectral sum convergence. For some problems, this is exact (Goldbach: $r(n) > 0$ iff $\sum_\rho$ converges). For others, it is an approximation (NS: regularity involves more than spectral convergence of enstrophy).

8.4 Open Problems

Problem 8.1 (Spectral completeness). Is Conjecture 5.4 true? Does every spectral-type problem admit complete decorrelation in some domain?

Problem 8.2 (Topological structure). What is the topology of the difficulty landscape $P \mapsto \Delta(P)$? Is the boundary between $\Delta = 0$ and $\Delta > 0$ a manifold? A fractal? Are there problems arbitrarily close to the boundary?

Problem 8.3 (Information-theoretic bound). Can one prove a lower bound on $\Delta(P)$ for specific problems using information-theoretic arguments? A proof that $\delta(\text{twin primes}) < 1$ for all spectral transfers would be a new kind of impossibility result — not a proof that twins are infinite or finite, but a proof that spectral methods cannot decide the question.

Problem 8.4 (Higher-order convergence). Does $\delta^{(\infty)}(P) = \delta(P)$? If not, what is the relationship between the full index and the order-k indices?

Problem 8.5 (Historical reconstruction). Compute $\underline{\delta}_t(P)$ for additional problems (Szemerédi's theorem, the Gross-Zagier formula, the Weil conjectures) and plot the history of mathematical progress as a function on the transfer graph.

Acknowledgments

The decorrelation index was first defined in the companion paper on the convolution–correlation duality [Nagy, 2026b, §7.4, Problem 2]. The present paper develops it as a standalone mathematical object. The proof category **Spec** that underlies the definition was introduced in [Nagy, 2026e].

AI Disclosure

This paper was drafted with AI assistance under human direction. The mathematical framework — the definition of the decorrelation index, the computability and monotonicity theorems, the impossibility boundary characterization, and the proof complexity connection — represents the author's original contribution.

References

- Dijkstra, E. W. (1959). A note on two problems in connexion with graphs. *Numerische Mathematik*, 1, 269–271.
- Gowers, T. (2001). A new proof of Szemerédi's theorem. *Geometric and Functional Analysis*, 11, 465–588.
- Green, B., & Tao, T. (2008). The primes contain arbitrarily long arithmetic progressions. *Annals of Mathematics*, 167, 481–547.
- Green, B., Tao, T., & Ziegler, T. (2012). An inverse theorem for the Gowers $U^{s+1}[N]$-norm. *Annals of Mathematics*, 176(2), 1231–1372.
- Helfgott, H. A. (2013). Major arcs for Goldbach's theorem. *arXiv:1305.2897*.

- Maynard, J. (2015). Small gaps between primes. *Annals of Mathematics*, 181(1), 383–413.
- Nagy, T. (2026b). The convolution–correlation duality: A universal principle of spectral damping. Working paper.
- Nagy, T. (2026e). Proof circuits: Cross-domain transfer as mathematical method. Working paper.
- Perelman, G. (2002). The entropy formula for the Ricci flow and its geometric applications. *arXiv:math/0211159*.
- Ribet, K. (1990). On modular representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ arising from modular forms. *Inventiones Mathematicae*, 100, 431–476.
- Roth, K. F. (1953). On certain sets of integers. *Journal of the London Mathematical Society*, 28, 104–109.
- Wiles, A. (1995). Modular elliptic curves and Fermat's Last Theorem. *Annals of Mathematics*, 141, 443–551.
- Zhang, Y. (2014). Bounded gaps between primes. *Annals of Mathematics*, 179(3), 1121–1174.