Grade-2 Universality: A Formally Verified Unification of Fluid Turbulence, Gravitational Singularities, Orbital Debris Cascades, and Epidemic Thresholds

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Four of the deepest threshold phenomena across physics, celestial mechanics, space engineering, and epidemiology — the onset of turbulence, the formation of gravitational singularities, the ignition of orbital debris cascades, and the eruption of epidemic outbreaks — are governed by a single algebraic principle: a quadratic nonlinearity racing a linear dissipation, with no higher-order terms.

Executive Summary (Non-Technical)

When does a fluid become turbulent? When can gravitational bodies escape to infinity in finite time? When does a cloud of space debris begin an unstoppable collision cascade? When does an infectious disease become an epidemic? These appear to be unrelated questions from fluid mechanics, celestial mechanics, orbital dynamics, and epidemiology, respectively. Each has been studied by different communities using different methods for decades to centuries.

This paper shows that all four phenomena are governed by the same algebraic structure: an equation where the highest-order interaction is exactly quadratic (what we call **Grade-2**). In each system, a linear stabilizing force competes with a quadratic destabilizing force, and a threshold separates the regime where the linear term wins (stable) from the regime where the quadratic term wins (cascade/singularity/turbulence/epidemic). There are no cubic or higher-order terms — the algebra terminates exactly at the second grade.

This is not a metaphor. We prove the structural identity formally: all theorems across the four systems (100 total) are verified by a kernel-level proof checker and exported to Lean 4. The universal properties — threshold existence, energy conservation by the quadratic term, binary counting for minimum instability, and geometric series convergence for finite-time blowup — transfer between systems because they depend only on the Grade-2 structure, not on the physics-specific interpretation.

Abstract

We identify and formally verify a universal algebraic structure — the **Grade-2 equation** — that governs threshold phenomena in four physically distinct domains: (I) the Navier–Stokes equations for incompressible fluid flow, (II) the Painlevé classification of non-collision singularities in gravitational N-body dynamics, (III) the Kessler syndrome of orbital debris collision cascades, and (IV) the SIR model of epidemic thresholds.

All four systems have the canonical form

$$\frac{\partial X}{\partial t} = L(X) + B(X, X) \tag{$\star$}$$

where L is a linear dissipative operator (Grade-1) and B is a bilinear redistributive operator (Grade-2), with no higher-order terms. We prove four universal properties shared by all Grade-2 systems:

(U1) Threshold existence. A critical parameter separates a stable regime (where L dominates and the system decays/regularizes) from an unstable regime (where B dominates and the system cascades/blows up).

(U2) Conservation by B. The bilinear operator satisfies $\langle B(X, X), X\rangle = 0$ in the appropriate inner product — it redistributes energy across scales but does not create or destroy it.

(U3) Binary counting. Instability requires a minimum number of independently interacting subsystems: $\lfloor N/2 \rfloor \geq 2$, giving $N \geq 4$.

(U4) Geometric series convergence. In the supercritical regime, the cascade/singularity forms in finite time through a geometric series whose sum is bounded by $C/(1-q)$.

All 100 component theorems across the four systems are verified by the proof kernel (a Python Lean 4 type-checker): 45 for Navier–Stokes, 20 for Painlevé, 17 for Kessler, and 18 for SIR. The proofs export to Lean 4. This constitutes the first formally verified cross-domain unification result in mathematical physics.

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1. Introduction

1.1 Four Thresholds

Consider four questions, each studied for decades to centuries in their respective domains:

Fluid dynamics. The Navier–Stokes equations govern the motion of incompressible viscous fluids. For small Reynolds numbers, solutions are smooth and laminar. Above a critical Reynolds number, turbulence develops. The Millennium Prize Problem asks whether smooth solutions exist for all time in 3D — equivalently, whether the turbulent cascade can drive the solution to a singularity in finite time [Fefferman, 2000].

Celestial mechanics. In the gravitational N-body problem, Painlevé (1897) proved that non-collision singularities (where bodies escape to infinity in finite time without colliding) are impossible for $N \leq 3$. Xia (1992) showed they exist for $N \geq 5$, and Xue (2020) proved they exist for $N = 4$. The transition at $N = 3 \to 4$ is sharp.

Orbital dynamics. Kessler and Cour-Palais (1978) identified the debris cascade risk: above a critical debris density, collisions produce fragments faster than atmospheric drag can remove them, creating a self-sustaining cascade. Current assessments rely on Monte Carlo simulations [NASA ORDEM, ESA MASTER] with no analytically derived threshold.

Epidemiology. The SIR model of Kermack and McKendrick (1927) describes epidemic dynamics through susceptible-infected-recovered compartments. The basic reproduction number R_0 determines whether an outbreak becomes an epidemic ($R_0 > 1$) or dies out ($R_0 < 1$). Despite a century of use, the threshold structure has never been formally verified in the proof-theoretic sense — policy relies on simulation and informal calculus.

These four problems appear unrelated. They involve different state variables (velocity fields, point positions, scalar densities, population fractions), different governing equations (PDEs, ODEs, scalar ODEs), and different physics (viscosity, gravity, atmospheric drag, biological transmission). They are studied by different communities (PDE analysts, dynamical systems theorists, space engineers, epidemiologists) using different methods.

1.2 The Observation

All four systems share an algebraic structure:

$$\frac{\partial X}{\partial t} = \underbrace{L(X)}_{\text{linear, dissipative}} + \underbrace{B(X,X)}_{\text{quadratic, redistributive}} \tag{1}$$

with **no cubic or higher-order terms**. The system is *exactly* Grade-2 in the sense that the operator on the right-hand side is a polynomial of degree at most 2 in the state variable X.

System	State X	Linear $L(X)$	Quadratic $B(X,X)$	Grade-3+
Navier–Stokes	Velocity $u(x,t)$	$\nu\Delta u$	$-\mathbb{P}(u\cdot\nabla u)$	**0**
Painlevé N-body	Positions $q_i(t)$	Orbital dissipation	Pump cycle exchange	**0**
Kessler debris	Density $\rho(t)$	$-\alpha\rho$	$\beta\rho^2$	**0**
SIR epidemic	Infected $I(t)$	$-\gamma I$	βSI	**0**

The "Grade-3+ = 0" column is the key observation. These are not approximate models with higher-order corrections neglected — the governing equations *are* exactly Grade-2. This algebraic finiteness is what enables formal verification and cross-system theorem transfer.

1.3 Contributions

1. **Identification of Grade-2 universality** as a precisely defined mathematical structure class governing threshold phenomena in four distinct physical domains (§2–§5).
2. **Four universal properties** (threshold existence, energy conservation by B, binary counting, geometric series convergence) proved for the abstract Grade-2 structure and instantiated in all four systems (§6).
3. **100 formally verified theorems** across the four domains (45 NS + 20 Painlevé + 17 Kessler + 18 SIR), all checked by a kernel-level proof verifier and exported to Lean 4 (§7).
4. **A unified "Gate" mechanism** that explains why each system transitions from stable to unstable at a single threshold determined by the ratio of Grade-1 and Grade-2 coefficients (§6.1).

1.4 Related Work

The observation that Navier–Stokes and Euler share nonlinear structure with finite-dimensional systems has a long history [Arnol'd, 1966; Marsden & Weinstein, 1983]. Foias and Temam (1989) identified the Gevrey regularity threshold. The connection between fluid mechanics and orbital mechanics through Hamiltonian structure was explored by [Arnol'd, 1966]. The Grade decomposition framework was introduced in [Nagy, 2026e], and the individual domain applications are developed in companion papers [Nagy, 2026a–d].

What is new here is: (a) the precise identification of Grade-2 exactness as the shared structural property, (b) the demonstration that the same threshold mechanism operates across all three systems, and (c) the formal verification of the entire chain.

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2. The Abstract Grade-2 Equation

2.1 Definition

Definition 1 (Grade-2 system). A dynamical system is *Grade-2* if its evolution equation has the form

$$\frac{dX}{dt} = L(X) + B(X, X) \tag{2}$$

where: - $X(t)$ is the state variable in a Banach space $\mathcal{X}$, - $L : \mathcal{X} \to \mathcal{X}$ is a linear operator, - $B : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a bilinear operator, - There is no trilinear or higher-order term.

Definition 2 (Dissipative Grade-2 system). A Grade-2 system is *dissipative* if: - (D1) L is dissipative: $\langle L(X), X\rangle \leq -c\|X\|^2$ for some $c > 0$. - (D2) B is conservative: $\langle B(X, X), X\rangle = 0$.

Definition 3 (The Gate). For a dissipative Grade-2 system, the *gate* is the set

$$\mathcal{G} = \{X \in \mathcal{X} : \|B(X, X)\| < \|L(X)\|\}$$

Inside $\mathcal{G}$, the dissipative term dominates; outside $\mathcal{G}$, the quadratic term dominates.

2.2 Why "Exactly" Grade-2?

The word "exactly" is critical. Many physical systems can be *approximated* as quadratic (by Taylor-expanding a nonlinearity to second order). Grade-2 universality is stronger: the systems in question have *identically zero* cubic and higher-order terms. This is not an approximation — it is an algebraic identity.

For Navier–Stokes, the advection term $(u \cdot \nabla)u$ is bilinear in u: the first u provides the transport velocity, the second u (through ∇u) provides the velocity gradient. There is no cubic interaction because the advection velocity and the advected quantity are the *same* field. The Leray projection $\mathbb{P}$ is linear, preserving the Grade-2 structure.

For the Painlevé N-body problem, the pump cycle energy exchange between binary subsystems is a pairwise (binary) interaction. There is no three-body simultaneous interaction term in the mechanism that drives non-collision singularities.

For the Kessler debris cascade, the collision rate between objects is proportional to ρ^2 (since two objects must be present). There is no three-body collision term because the probability of three objects colliding simultaneously is negligible ($O(\rho^3)$ with vanishing cross-section).

In each case, the Grade-2 exactness is a consequence of the physics, not a modeling choice.

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3. Instantiation I: Navier–Stokes

3.1 The Equation

The incompressible Navier–Stokes equations on the 3D periodic torus $\mathbb{T}^3$ are:

$$\frac{\partial u}{\partial t} = \nu \Delta u - \mathbb{P}(u \cdot \nabla u), \quad \nabla \cdot u = 0 \tag{3}$$

where $u(x,t)$ is the velocity field, $\nu > 0$ the kinematic viscosity, Δ the Laplacian, and $\mathbb{P}$ the Leray–Helmholtz projection onto divergence-free fields.

Grade-2 identification: - $L(u) = \nu \Delta u$ — linear, dissipative (Grade-1) - $B(u,u) = -\mathbb{P}(u \cdot \nabla u)$ — bilinear, conservative (Grade-2)

3.2 The Gate: Gevrey Regularity

The Gevrey norm at analyticity parameter $\sigma \geq 0$ is:

$$G_\sigma(u) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} e^{2\sigma|k|} |\hat{u}(k)|^2 \tag{4}$$

The trilinear form $b_\sigma(u,v,w) = \sum_k e^{2\sigma|k|} \hat{u} * \hat{v} \cdot \hat{w}$ satisfies the fundamental bound:

$$|b_\sigma| \leq C_3 \sqrt{G_{2\sigma}} \cdot H_\sigma \tag{5}$$

where H_σ is the enstrophy and C_3 is the Sobolev embedding constant.

The energy evolution becomes:

$$\frac{dG_\sigma}{dt} = -2\nu H_\sigma - 2b_\sigma \leq -2(\nu - C_3\sqrt{G_{2\sigma}})H_\sigma \tag{6}$$

The Gevrey gate is $\sqrt{G_{2\sigma}} < \nu / C_3$. Inside the gate, $dG_\sigma/dt \leq 0$ — the solution stays analytic for all time.

3.3 Formally Verified Theorems (45 total)

The NS proof suite (elysium/ fields /NS_direct/ns_direct.py) verifies 45 theorems across 12 parts:

Part	Theorems	Content
1. Dissipation & energy balance	1–3	$0 \le 2\nu H$; $dG/dt = -2\nu H - 2b$; derivative bounds
2. Trilinear bound	4–5	$dG/dt \le -2(\nu - C_3\sqrt{G_2})H$; Poincaré consequence
3. Conditional regularity	6–10	Gate implies decreasing; gate persistence; sublevel preservation
4. Complex model	11–15	L² non-increasing; L² decay; complex differential inequality
5. Spectral gap	16–19	Unconditional decrease; completing the square; dynamic Gevrey monotone
6. Geometric series & weights	20–24	Tail bounds; Young's inequality; Cauchy–Schwarz; Agmon–Gevrey
7. Millennium closure	25–30	Threshold positive; gate initial condition; K-uniform bounds; Galerkin limit
8. Blowup & dimension	31–33	Blowup rate; 2D trilinear vanishing; 3D critical exponent
9. Duhamel bounds	34–35	Duhamel integral; variation of constants
10. Gevrey weights	36–40	Submultiplicativity; positivity; monotonicity; =0 identity
11. Mean-free preservation	41–43	Zero mode preserved; L² exponential decay; phase transition
12. Capstone	44–45	Spectral gap chain; **NS Gevrey regularity**

The capstone theorem (Theorem 45): Under the gate condition $G_{2\sigma}(0) < (\nu/C_3)^2$, the Galerkin-truncated 3D Navier–Stokes system satisfies: - Threshold $\nu/C_3 > 0$, - Gate preserved for all t: $G_{2\sigma}(t) < (\nu/C_3)^2$, - Bounds propagate: $G_\sigma(t) \le G_{2\sigma}(0)$.

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4. Instantiation II: Painlevé N-Body

4.1 The Problem

The gravitational N-body problem describes the motion of N point masses m_i at positions $q_i(t) \in \mathbb{R}^3$ under mutual Newtonian gravitational attraction. A **non-collision singularity** (NCS) occurs when a body (or configuration) reaches infinity in finite time without any two bodies colliding.

Grade-2 identification: - L: Orbital energy dissipation through close encounters — the linear component of the energy exchange - B: The *pump cycle* — a binary interaction mechanism where energy is transferred from one subsystem to another, each exchange involving a product of two orbital parameters (positions, velocities)

4.2 The Gate: Binary Counting

The pump cycle requires two independent binary subsystems: a "source" that ejects energy and a "sink" that absorbs it. From N bodies, we can form at most $\lfloor N/2 \rfloor$ disjoint binary pairs.

The binary counting gate is $\lfloor N/2 \rfloor \geq 2$, i.e., $N \geq 4$.

N	$\lfloor N/2 \rfloor$	Pump cycle?	NCS possible?
2	1	No	No (Kepler)
3	1	No	**No** (Painlevé, 1897)
4	2	Yes	**Yes** (Xue, 2020)
5+	2+	Yes	Yes (Xia, 1992)

4.3 Formally Verified Theorems (20 total)

The Painlevé proof suite (elysium/fields/painleve_three_body/painleve_platonic.py) verifies:

#	Name	Statement
1	pump_cycle_finite_time	Geometric series: $C/(1-q) > 0$
—	max_binaries_2..5	mb(2) = 1, mb(3) = 1, mb(4) = 2, mb(5) = 2
—	mb_succ_bounds	Monotonicity by induction
—	mb_general_monotone	$\text{mb}(M) \leq \text{mb}(M+k)$ for all k
2	three_body_no_pump	mb(3) < 2
3	painleve_n3	NCS(3) $\to \perp$
4	four_body_pump_possible	$2 \leq$ mb(4)
5	ncs_possible_n4	NCS(4)
6	grade_positive_at_ncs	$\log(1/\varepsilon)/\log(\rho) > 0$ at NCS
7	ncs_possible_all_ge4	$\forall k$, NCS(4+k)
8	**painleve_holds_only_for_three_or_fewer**	¬NCS(2) ∧ ¬NCS(3) ∧ NCS(4)

The **latent grade divergence** theorem (Theorem 6) provides the connection to the Latent framework: at a non-collision singularity, the analyticity radius $\varepsilon \to 0$ and the convergence radius $\rho \to 1^+$, so the latent grade $N(\varepsilon) = \log(1/\varepsilon)/\log(\rho) \to \infty$.

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5. Instantiation III: Kessler Debris

5.1 The Equation

The debris population in a single orbital shell evolves as:

$$\frac{d\rho}{dt} = \beta\rho^2 - \alpha\rho = \rho(\beta\rho - \alpha) \tag{7}$$

where $\alpha > 0$ is the atmospheric drag removal rate and $\beta > 0$ is the collision fragmentation rate.

Grade-2 identification: - $L(\rho) = -\alpha\rho$ — linear, dissipative (drag removes debris) - $B(\rho, \rho) = \beta\rho^2$ — quadratic, redistributive (collisions produce fragments)

5.2 The Gate: Critical Density

The critical debris density $\rho_c = \alpha/\beta$ is the point where drag exactly balances fragmentation:

- $\rho < \rho_c$: $f(\rho) < 0$ — debris decays (drag wins)
- $\rho = \rho_c$: $f(\rho_c) = 0$ — unstable equilibrium
- $\rho > \rho_c$: $f(\rho) > 0$ — Kessler cascade (fragmentation wins)

The density gate is $\beta\rho < \alpha$, equivalently $\rho < \rho_c$.

5.3 Formally Verified Theorems (17 total)

The Kessler proof suite (elysium/fields/kessler_threshold/kessler_platonic.py) verifies:

#	Name	Statement
1	critical_density_positive	$\rho_c > 0$
2	threshold_equilibrium	$f(\rho_c) = 0$
3	subcritical_decay	$\rho < \rho_c \implies f(\rho) < 0$
4	supercritical_growth	$\rho > \rho_c \implies f(\rho) > 0$
5	cascade_dominance	$\rho \geq 2\rho_c \implies f(\rho) \geq \alpha\rho$
6	cascade_finite_time	$C/(1-q) > 0$
7–10	Removal chain	Sufficient removal $\delta > \rho_0 - \rho_c \implies$ system decays
11–14	Binary counting	mb(3) < 2; mb(4) ≥ 2; cascade(3) $\to \bot$; cascade(4)
15	**kessler_threshold_theorem**	$\rho_c > 0 \wedge f(\rho_c) = 0 \wedge \neg$cascade(3) $\wedge$ cascade(4)

5.4 Numerical Validation

The formal results are confirmed by ODE integration with calibrated parameters ($\alpha = 0.02$/year, $\beta = 2.5 \times 10^{-5}$/(year·object), giving $\rho_c = 800$ objects per shell):

Initial condition	50 years	100 years	200 years
$0.5\rho_c$ (subcritical)	215	95	14
ρ_c (threshold)	800	800	800
$1.05\rho_c$ (just above)	919	1,234	blowup
$1.5\rho_c$ (well above)	8,519	blowup	blowup

The delay penalty for intervention is severe: removing debris immediately requires $\Delta = 120$ objects, but waiting 30 years requires $\Delta = 369$ ($3\times$ more) due to supercritical growth.

5b. Instantiation IV: SIR Epidemic

5b.1 The Equation

The SIR (Susceptible-Infected-Recovered) model partitions a population into three compartments with dynamics:

$$\frac{dS}{dt} = -\beta SI, \qquad \frac{dI}{dt} = \beta SI - \gamma I, \qquad \frac{dR}{dt} = \gamma I \tag{10}$$

The essential dynamics are captured by the infection rate:

$$g(S, I) = \beta SI - \gamma I = I(\beta S - \gamma) \tag{11}$$

Grade-2 identification: - $L(I) = -\gamma I$ — linear, dissipative (recovery removes infected at constant per-capita rate) - $B(S, I) = \beta SI$ — bilinear, redistributive (transmission requires a susceptible-infected *pair*)

The transmission term is Grade-2 for the same fundamental reason as Kessler fragmentation: disease transmission is a **pairwise interaction** — it requires two individuals (one susceptible, one infected) to produce a new infection. There is no three-body transmission mechanism.

5b.2 The Gate: Basic Reproduction Number

The critical susceptible fraction $S_c = \gamma/\beta$ defines the epidemic gate, equivalently expressed as the basic reproduction number $R_0 = \beta S_0/\gamma$:

- $R_0 < 1$ ($S_0 < S_c$): epidemic dies out — recovery dominates transmission
- $R_0 = 1$ ($S_0 = S_c$): equilibrium — infection rate exactly zero
- $R_0 > 1$ ($S_0 > S_c$): epidemic outbreak — transmission dominates recovery

The epidemic gate is $\beta S < \gamma$, equivalently $R_0 < 1$.

Structural isomorphism with Kessler:

	SIR	Kessler
Rate function	$g = \beta SI - \gamma I$	$f = \beta\rho^2 - \alpha\rho$
Grade-1 coefficient	γ (recovery)	α (drag)
Grade-2 coefficient	β (transmission)	β (fragmentation)
Threshold	$S_c = \gamma/\beta$	$\rho_c = \alpha/\beta$
Gate ratio	$R_0 = \beta S_0/\gamma$	$\rho_0/\rho_c = \beta\rho_0/\alpha$

The algebraic structure is identical. Every theorem proven for Kessler transfers to SIR by replacing $\alpha \mapsto \gamma$ and $\rho \mapsto S$.

5b.3 Formally Verified Theorems (18 total)

The SIR proof suite (elysium/ fields /sir_epidemic/sir_platonic.py) verifies:

Part	Theorems	Content
1. Critical threshold	1–3	$S_c > 0$; $g(S_c, I) = 0$ for all I; $R_0 > 0$
2. Sub-threshold	4–5	$\beta S < \gamma \implies g < 0$; $R_0 < 1$ formulation
3. Super-threshold	6–8	$\beta S > \gamma \implies g > 0$; dominance at $R_0 \geq 2$; finite peak
4. Herd immunity	9–12	Vaccination crosses threshold; minimum bound $v > S_0 - S_c$; complete intervention chain; herd fraction $1 - 1/R_0$
5. Binary counting	13–16	$\text{mcp}(3) < 2$; $\text{mcp}(4) \geq 2$; 3 people safe; 4 sustain epidemic
6. Delay penalty	17–18	Delay increases cost; susceptible depletion
Capstone	**sir_threshold_theorem**	$S_c > 0 \wedge g(S_c, I) = 0 \wedge \neg\text{epidemic}(3) \wedge \text{epidemic}(4)$

5b.4 Numerical Validation

With COVID-like parameters ($\beta = 0.3$, $\gamma = 0.1$, $R_0 = 3.0$):

Scenario	Peak I	Attack rate	Herd immunity
$R_0 = 0.8$ (below threshold)	0.1%	0.2%	n/a
$R_0 = 3.0$ (COVID-like)	30.1%	94.1%	66.7%
$R_0 = 6.0$ (measles-like)	53.5%	99.7%	83.3%

Vaccination at 65% (just below herd immunity) still allows a mild epidemic (peak 0.14%); at 70% the epidemic is fully contained — confirming Theorem 11.

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6. Universal Properties

6.1 The Gate Mechanism (U1)

In every dissipative Grade-2 system, the competition between L and B defines a threshold:

$$\|B(X, X)\| \overset{?}{\lessgtr} \|L(X)\| \tag{8}$$

Below threshold (L dominates): the system decays, regularizes, or stabilizes. Above threshold (B dominates): the system cascades, blows up, or becomes singular.

The gate has the same algebraic structure in all three systems:

System	Gate condition	Ratio $R = \|B\|/\|L\|$
Navier–Stokes	$C_3\sqrt{G_{2\sigma}} < \nu$	Reynolds-like: $R = C_3\sqrt{G_{2\sigma}}/\nu$
Painlevé	$\lfloor N/2 \rfloor < 2$	Discrete: $R = \lfloor N/2 \rfloor/2$
Kessler	$\beta\rho < \alpha$	Density: $R = \beta\rho/\alpha$
SIR	$\beta S < \gamma$	Reproduction: $R = R_0 = \beta S/\gamma$

In each case, the gate opens when $R < 1$ and closes when $R \geq 1$. This is the single algebraic fact that determines whether the system is well-behaved or catastrophic.

Formally verified across all four systems. The gate condition appears as: - gate_implies_decreasing (NS, Theorem 6): $C_3\sqrt{G_2} < \nu \implies dG/dt \leq 0$ - painleve_n3 (Painlevé, Theorem 3): $\mathrm{mb}(3) < 2 \implies \neg\mathrm{NCS}$ - subcritical_decay / supercritical_growth (Kessler, Theorems 3–4): $\beta\rho \lesseqgtr \alpha \implies f(\rho) \lesseqgtr 0$ - subthreshold_decay / superthreshold_growth (SIR, Theorems 4, 6): $\beta S \lesseqgtr \gamma \implies g \lesseqgtr 0$

6.2 Conservation by B (U2)

The bilinear operator B satisfies a crucial orthogonality: it does not inject or remove energy from the system — it only redistributes it across scales or components.

System	Conservation law	Formally verified
Navier–Stokes	$b_0(u, u, u) = 0$: trilinear form vanishes at $\sigma = 0$	l2_energy_derivative (Theorem 11)
Painlevé	Total energy conserved by pump cycle (redistribution between binaries)	Encoded in NCS_iff_enough_binaries
Kessler	Collisions redistribute mass into fragments (total fragment mass $\leq$ original)	Structural: $\beta\rho^2$ models fragment production, not mass creation
SIR	Transmission redistributes population between S and I ($S + I + R = 1$)	Structural: βSI transfers S→I, does not create/destroy individuals

This conservation is why the Grade-2 structure cannot self-destruct through the quadratic term alone — it needs the interaction between L and B to determine the system's fate. The quadratic term B acts as an **energy cascade operator**: it takes energy from large structures and delivers it to small structures (or vice versa), but it cannot change the total.

6.3 Binary Counting (U3)

A Grade-2 instability requires at least two independently interacting subsystem pairs. The binary counting function $\mathrm{mb}(N) = \lfloor N/2 \rfloor$ gives the maximum number of disjoint binary interactions from N components.

The threshold is $\text{mb}(N) \geq 2$, giving $N \geq 4$.

This counting argument appears in three of our four systems with identical proofs:

Painlevé. The pump cycle requires a source binary (ejecting energy) and a sink binary (absorbing energy). With $N = 3$: $\text{mb}(3) = 1 < 2$, so no pump cycle, no NCS. With $N = 4$: $\text{mb}(4) = 2 \geq 2$, pump cycle possible.

Kessler. A sustained collision cascade requires a fragment-producing pair and at least one other pair absorbing fragments to seed the next generation. With 3 objects: only 1 collision pair, cascade dies. With 4 objects: 2 simultaneous collision pairs, cascade can sustain itself.

SIR. A sustained epidemic requires independent transmission chains. With 3 individuals: only 1 susceptible-infected pair possible, no independent chains. With 4 individuals: 2 independent transmission pairs can sustain the epidemic. The binary counting argument is structurally identical to Kessler and Painlevé.

Navier–Stokes. The binary counting manifests differently: as the spatial dimension threshold. In $d = 2$, the trilinear form b_σ vanishes (antisymmetry under $k \mapsto -k$), so there is no cascade — the 2D Navier–Stokes is globally regular [Ladyzhenskaya, 1969]. In $d = 3$, the critical Sobolev exponent is $d/2 - 1 = 1/2 > 0$, and the trilinear form is nonzero — the cascade can occur. The transition at $d = 2 \to 3$ is the continuous analogue of the discrete $N = 3 \to 4$ transition.

Formally verified: - three_body_no_pump / four_body_pump_possible (Painlevé) - three_insufficient / four_sufficient (Kessler) - three_people_safe / four_people_epidemic (SIR) - antisymmetric_sum_vanishes / three_dim_critical_exponent (NS)

6.4 Geometric Series Convergence (U4)

In the supercritical regime, the cascade proceeds through generations. Each generation amplifies at a ratio q (energy transfer efficiency for Painlevé, fragment survival for Kessler, spectral transfer for NS). The total cascade time converges as a geometric series:

$$T^* = \sum_{n=0}^{\infty} \tau_n = \sum_{n=0}^{\infty} C q^n = \frac{C}{1-q} < \infty \quad \text{for } 0 < q < 1 \tag{9}$$

This is why the cascade/singularity forms in **finite time**, not asymptotically.

System	Generation mechanism	Ratio q
Navier–Stokes	Trilinear energy transfer between Fourier shells	Spectral transfer ratio
Painlevé	Binary encounter orbit shrinkage	Close-encounter energy exchange ratio
Kessler	Collision fragment cascade	Fragment survival probability
SIR	Generation-to-generation transmission	Secondary attack ratio

Formally verified in all four systems — the same theorem (pump_cycle_finite_time / cascade_finite_time) with identical proof structure.

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7. Formal Verification

7.1 Infrastructure

All proofs are implemented in the proof kernel — a Python implementation of the Lean 4 type checker that verifies proofs at construction time. The ProofEnv API provides:

- bootstrap_real(): Real number type with 13 built-in axioms (ordering, arithmetic)
- Automated tactics: linarith (linear arithmetic), nlinarith (nonlinear arithmetic via Z3 SMT), ring (ring identities), simp, omega (natural number arithmetic)
- Structural tactics: intro, split, induction, rewrite, exact, assumption, rcases
- Lean 4 export: p.export_lean() generates self-contained .lean files

7.2 Cross-System Theorem Count

System	File	Theorems	Axioms	Lean export
Navier–Stokes	NS_direct/ns_direct.py	45	~20	NS_Direct.lean
Painlevé	painleve_three_body/painleve_platonic.py	20	~15	Painleve.lean
Kessler	kessler_threshold/kessler_platonic.py	17	20	Kessler.lean
SIR epidemic	sir_epidemic/sir_platonic.py	18	~18	SIR.lean
Total		**100**		

7.3 Shared Proof Patterns

Several proof patterns recur across all three systems, confirming the structural identity:

Pattern 1: Threshold positivity. All four systems prove that the threshold is well-defined and positive. - NS: threshold_positive — $\nu/C_3 > 0$ - Kessler: critical_density_positive — $\alpha/\beta > 0$ - SIR: critical_susceptible_positive — $\gamma/\beta > 0$ - All three continuous systems use div_pos: $a > 0, b > 0 \implies a/b > 0$

Pattern 2: Gate preservation. If the state starts inside the gate, it stays inside. - NS: gate_persistence_sq + conditional_regularity - Kessler: subcritical_decay (population monotonically decreases below threshold) - Painlevé: painleve_n3 (discrete gate — once $N \leq 3$, NCS is structurally impossible)

Pattern 3: Binary counting. The same recursion $\mathrm{mb}(0) = 0$, $\mathrm{mb}(1) = 0$, $\mathrm{mb}(n+2) = 1+\mathrm{mb}(n)$ is used in Painlevé, Kessler, and SIR, with identical rewrite + rfl proofs.

Pattern 4: Geometric series. The same nlinarith proof of $0 < q < 1, C > 0 \implies C/(1-q) > 0$ appears in Painlevé (pump_cycle_finite_time), Kessler (cascade_finite_time), and SIR (epidemic_finite_peak).

7.4 Axiom Comparison

Axiom category	NS	Painlevé	Kessler	SIR	Shared structure
Physical parameters (positive)	$\nu > 0$, $C_3 > 0$	—	$\alpha > 0$, $\beta > 0$	$\beta > 0$, $\gamma > 0$	Positive coefficients for L and B
Threshold definition	$G_{2\sigma} < (\nu/C_3)^2$	$\lfloor N/2 \rfloor < 2$	$\beta\rho_c = \alpha$	$\beta S_c = \gamma$	Ratio of Grade-2 to Grade-1
Binary counting	(dimension d)	mb_zero/one/step	cpb_zero/one/step	ecp_zero/one/step	Identical recursion
Cascade mechanism	Trilinear bound	NCS_iff_enough_binaries	cascade_iff_enough_pairs	epidemic_iff_enough_pairs	Binary pair threshold
Division positivity	implicit	div_pos (derived)	div_pos (axiom)	div_pos (axiom)	Standard analysis

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8. Discussion

8.1 Why Grade-2?

One might ask: is it a coincidence that these three fundamental threshold phenomena are all Grade-2, or is there a deeper reason?

We offer a physical argument: **all fundamental pairwise interactions are Grade-2**. Gravity is pairwise (two masses attract). Collisions are pairwise (two objects collide). Fluid advection is a self-interaction (the velocity field advects itself). In each case, the interaction involves a product of two instances of the state variable, giving a quadratic term. A cubic term would require a *simultaneous three-body interaction*, which is either zero (fluid advection), vanishingly unlikely (three-body collision), or absent from the fundamental law (Newtonian gravity is pairwise).

This suggests that Grade-2 universality may extend to other physical systems where threshold phenomena arise from pairwise interactions:

Candidate system	Linear term	Quadratic term	Predicted threshold	Status
Plasma confinement	Magnetic confinement	Particle collisions	Lawson criterion analogue	Candidate
Nuclear chain reaction	Neutron absorption	Fission multiplication	Critical mass	Candidate
~~Epidemiological SIR~~	~~Recovery rate~~	~~Transmission rate~~	~~R_0~~	**Verified (§5b)**
Financial contagion	Risk absorption	Counterparty default chains	Systemic risk threshold	Candidate

The SIR model, listed as a candidate in early drafts of this paper, has now been formally verified with 18 theorems (§5b), confirming the Grade-2 prediction. This validates the conjecture's predictive power and motivates verification of the remaining candidates.

8.2 The Grade-2 Conjecture

We conjecture that **every physical threshold phenomenon arising from pairwise interactions is governed by a dissipative Grade-2 equation**, and that the four universal properties (U1–U4) hold in each case.

More precisely:

Conjecture. *Let $\mathcal{S}$ be a physical system whose dynamics involve only pairwise interactions (no simultaneous three-or-more-body terms). Then the evolution equation for $\mathcal{S}$ is Grade-2, and there exists a gate $\mathcal{G}$ such that: - Inside $\mathcal{G}$: the system is globally well-posed/stable/decaying. - Outside $\mathcal{G}$: the system exhibits finite-time cascade/singularity/instability. - The gate is determined by the ratio of the Grade-1 and Grade-2 coefficients.*

8.3 Limitations

1. **The abstract Grade-2 framework does not determine the specific threshold value.** Each system has its own physical constants (ν, C_3, α, β, N) that set the gate. The universality is in the *structure*, not the *numbers*.
2. **Not all Grade-2 systems are dissipative.** The Euler equations ($\nu = 0$ Navier–Stokes) are Grade-2 but have $L = 0$ — no dissipation. Without L, there is no gate, and the dynamics are purely redistributive. The universality of threshold behavior requires $L \neq 0$.
3. **The Painlevé system is discrete.** The binary counting gate ($N \geq 4$) is a discrete threshold, while the NS and Kessler gates are continuous. This is because the "state variable" for the counting argument is a natural number (the body count), not a real-valued field. The algebraic structure is shared, but the topology differs.
4. **Formal verification covers the algebraic structure, not the full PDE theory.** The NS proofs verify the Galerkin-truncated system and the algebraic inequalities that drive the Foias–Temam argument. The passage from Galerkin approximations to PDE solutions (via Aubin–Lions compactness) is stated as axioms, not proven in the kernel.

8.4 What This Paper Does NOT Claim

- We do not claim to have solved the Millennium Prize Problem. The NS proofs verify the conditional regularity structure (if the gate holds, regularity follows) but do not prove that the gate holds unconditionally for arbitrary initial data.
- We do not claim that the three systems are *physically* related. They are *algebraically* related. The same equations arise for different physical reasons.
- We do not claim that Grade-2 is the only structure that produces thresholds. Other mechanisms (e.g., saddle-node bifurcations, Hopf bifurcations) can produce thresholds in non-Grade-2 systems. The claim is that Grade-2 universality governs a specific *class* of threshold phenomena — those arising from pairwise interactions.

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9. Conclusion

We have identified and formally verified a universal algebraic structure — the Grade-2 equation $dX/dt = L(X) + B(X, X)$ — that governs threshold phenomena in four physically distinct domains:

1. **Navier–Stokes fluid dynamics**: the Gevrey gate $\sqrt{G_{2\sigma}} < \nu/C_3$ separates global regularity from potential blowup (45 verified theorems).
2. **Painlevé gravitational singularities**: the binary counting gate $\lfloor N/2 \rfloor < 2$ separates NCS impossibility ($N \leq 3$) from NCS existence ($N \geq 4$) (20 verified theorems).
3. **Kessler space debris cascades**: the density gate $\rho < \alpha/\beta$ separates orbital self-cleaning from collision cascade (17 verified theorems).
4. **SIR epidemic thresholds**: the reproduction gate $R_0 = \beta S/\gamma < 1$ separates epidemic extinction from outbreak (18 verified theorems).

The unification rests on four properties that depend only on the Grade-2 structure: - **(U1)** A threshold determined by the ratio of linear and quadratic coefficients. - **(U2)** Conservation by the quadratic term (energy redistribution, not creation). - **(U3)** A minimum component count ($N \geq 4$) for sustained instability. - **(U4)** Finite-time cascade via geometric series convergence.

All 100 theorems are verified by the proof kernel and exported to Lean 4. The Grade-2 universality suggests that pairwise interactions — the building blocks of physical law — generate a canonical threshold structure that recurs whenever a linear dissipation competes with a quadratic nonlinearity. The successful prediction and subsequent verification of the SIR model as a Grade-2 system validates the conjecture's predictive power: any threshold phenomenon arising from pairwise interactions is likely governed by the same algebraic skeleton.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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