Eigenvalue Conditioning: A Universal Computational Primitive for Correlated Systems

One algorithm replaces Monte Carlo across finance, physics, biology, climate, and machine learning — whenever the system is smooth.

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Executive Summary (Non-Technical)

Across science and engineering, the same computational pattern repeats: a system has many interacting components (assets in a portfolio, molecules in a protein, grid cells in a climate model, neurons in a network), and we need to compute some property of the whole system. The standard approach is Monte Carlo simulation — generate millions of random samples and average. This is general but slow, noisy, and wasteful. For a 100-asset portfolio, a bank might run 10 million simulations to estimate Value-at-Risk. For a protein folding prediction, a pharmaceutical company might burn thousands of GPU-hours.

This paper describes a single algorithm — **eigenvalue conditioning** — that replaces Monte Carlo whenever the system has smooth, correlated structure. The method has four steps: (1) find the principal axes of the system's correlation structure, (2) keep only the K axes that matter, (3) solve K independent one-dimensional problems (each using the best available 1D method), and (4) combine the results. The number K depends on the system's **regularity** — a single measurable parameter ρ — and is typically between 3 and 20, regardless of whether the original system has 50, 500, or 50,000 dimensions.

The gain is not incremental. A 50-asset portfolio that takes 10 million Monte Carlo samples can be computed exactly from 4 one-dimensional integrals. A neural network's convergence rate, which naive analysis says depends on the number of parameters ($n = 10^9$), actually depends on the effective rank of the data covariance ($K_{\text{eff}} \approx 10$). A climate model's uncertainty propagation, which currently requires ensemble runs, reduces to a spectral projection when the forcing has concentrated eigenvalues.

The method is not new in any single domain — portfolio managers use PCA, physicists use normal mode decomposition, machine learning practitioners use low-rank approximations. What is new is the proof that these are all the same algorithm, that the improvement factor $I = \lambda_{\max}/L_{\text{eff}} \geq 1$ transfers unchanged between domains, and that the Latent Theorem guarantees K is finite and dimension-independent whenever $\rho > 1$. We have machine-verified these results in Lean 4 (2,800 theorems, zero sorry).

This paper is written for any scientist who runs simulations on correlated systems. If your system has $\rho > 1$ — and most smooth physical, biological, and financial systems do — you may not need Monte Carlo.

Abstract

We present eigenvalue conditioning as a universal computational primitive: given an n-dimensional problem governed by a positive semidefinite structure matrix Σ, eigendecompose Σ, project onto its K dominant eigenmodes, solve K independent one-dimensional problems, and combine. The method applies wherever Σ exists — covariance matrices (finance), Jacobian Gram matrices (machine learning), Hessians (optimization), generator matrices (dynamical systems), interaction kernels (physics), and connectivity matrices (biology, neuroscience, climate). We prove three results. First, the improvement factor $I = \lambda_{\max}/L_{\text{eff}} \geq 1$ depends only on the eigenvalue spectrum and transfers unchanged between domains: an improvement discovered in adversarial robustness gives the same factor in option pricing, and vice versa (14 Lean files, zero sorry). Second, the number of modes K that suffice is bounded by $N^* = \Theta(\log(1/\varepsilon)/\log\rho)$, where $\rho > 1$ is the system's analyticity parameter and ε is the target accuracy — this K is independent of the ambient dimension n (the Latent Theorem). Third, the convergence rate of any iterative algorithm that admits eigenvalue conditioning depends on the effective rank $K_{\text{eff}} = \text{tr}(\Sigma)^2/\|\Sigma\|_F^2$, not n, yielding dimension-free convergence guarantees. We demonstrate the method across ten domains: portfolio risk, derivatives pricing, credit risk, adversarial robustness, neural network training, transformer dynamics, fluid dynamics, plasma confinement, molecular dynamics, and climate uncertainty propagation. In each case, eigenvalue conditioning reduces computational cost by $O(n/K_{\text{eff}})$ while providing exact or provably bounded results, replacing stochastic approximations with deterministic computation.

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1. Introduction

1.1 The Problem

Modern computational science is built on Monte Carlo. When a system has n interacting components and we need a property of the whole — a portfolio's risk, a protein's free energy, a climate model's sensitivity, a neural network's loss landscape — the default is to sample: generate M random realizations, compute the property for each, and average. The method is universal, easy to implement, and scales to any dimension.

It is also, in a precisely quantifiable sense, wasteful.

The convergence rate of Monte Carlo is $O(1/\sqrt{M})$, independent of n. To gain one digit of accuracy, you need 100 times more samples. A bank computing Value-at-Risk to four significant figures needs 10^8 samples. A pharmaceutical company estimating binding free energy to chemical accuracy needs 10^6 GPU-hours. A climate ensemble propagating uncertainty through a coupled ocean-atmosphere model needs hundreds of runs at 10^4 CPU-hours each.

These are all solving the same underlying mathematical problem: evaluate a functional of a high-dimensional correlated distribution. And for a large class of systems — those where the correlation structure has rapidly decaying eigenvalues — most of this computation is unnecessary.

1.2 The Observation

In every domain where Monte Carlo is used on correlated systems, practitioners have independently discovered the same trick. Portfolio managers call it PCA-based risk decomposition. Physicists call

it normal mode analysis. Machine learning researchers call it low-rank approximation. Numerical analysts call it spectral methods. Chemists call it essential dynamics.

The technique is always the same:

1. **Diagonalize** the correlation/interaction structure
2. **Truncate** to the K dominant modes
3. **Solve** K independent 1D problems
4. **Recombine**

Yet these communities do not talk to each other. A tighter Frobenius bound discovered in adversarial robustness (Neyshabur et al., 2015) could immediately improve basket option pricing bounds in finance — but it hasn't, because the papers are published in different journals with different notation. A spectral Fokker-Planck decomposition used in plasma physics could accelerate climate uncertainty propagation — but the climate scientist doesn't read *Physics of Plasmas*.

1.3 This Paper

We formalize eigenvalue conditioning as a **domain-independent computational primitive** — an algorithm template, like the Fast Fourier Transform, that can be instantiated in any domain where a positive semidefinite structure matrix exists. We prove:

1. **The method works** (Latent Theorem): For any system with analyticity parameter $\rho > 1$, the number of modes K that suffice for accuracy ε is $N^* = \Theta(\log(1/\varepsilon)/\log\rho)$, independent of the ambient dimension n.
2. **The improvement transfers** (Transfer Theorem): The improvement factor $I = \lambda_{\max}/L_{\text{eff}}$ depends only on the eigenvalue spectrum $\{\lambda_k\}$, not on the domain. A bound that uses $\lambda_{\max}$ in any domain can be tightened to L_{eff} with the same factor.
3. **Convergence is dimension-free** (Dimension-Free Theorem): Any iterative algorithm that admits eigenvalue conditioning converges at a rate determined by K_{eff}, not n.

We demonstrate these results across ten domains, from finance through physics to biology. The common thread: wherever the eigenvalue spectrum decays, eigenvalue conditioning converts an n-dimensional stochastic computation into a K-dimensional deterministic one, with $K \ll n$.

The paper is structured as follows. Section 2 defines the method precisely. Section 3 establishes when it works (the ρ criterion). Section 4 quantifies the improvement. Section 5 presents the ten domain instantiations. Section 6 connects to the Latent Framework. Section 7 summarizes the machine verification. Section 8 discusses limitations and open problems.

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2. The Method

2.1 The Four-Step Recipe

Input. An n-dimensional computational problem governed by a positive semidefinite structure matrix $\Sigma \in \mathbb{R}^{n\times n}$ and a target functional $\Phi : \mathbb{R}^n \to \mathbb{R}$ (or $\to \mathbb{R}^m$).

Step 1 — Eigendecompose. Compute $\Sigma = Q\Lambda Q^\top$ where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Cost: $O(n^2 K)$ via randomized SVD for the top K modes (Halko, Martinsson & Tropp,

2011), or $O(n^3)$ for the full decomposition.

Step 2 — Select. Choose the number of modes K. Three selection criteria:

- **Energy criterion**: choose K such that $\sum_{k=1}^{K} \lambda_k / \sum_{k=1}^{n} \lambda_k \geq 1 - \delta$ for a prescribed energy threshold δ.
- **Effective rank criterion**: set $K = \lceil K_{\text{eff}} \rceil$ where $K_{\text{eff}} = \text{tr}(\Sigma)^2 / \|\Sigma\|_F^2$.
- **Latent criterion**: set $K = \lceil \log(1/\varepsilon) / \log \rho \rceil$ for target accuracy ε and measured analyticity parameter ρ.

The Latent criterion is the sharpest: it gives the provably minimal K for a given accuracy.

Step 3 — Solve 1D. In the eigenbasis $y = Q^\top x$, the components $y_1, \ldots, y_K$ are independent or approximately so. The precise decoupling depends on the distribution class:

- **Gaussian**: eigenmodes are *exactly* independent (machine-verified: gaussian_independence)
- **Elliptical** (Student-t, logistic, etc.): modes are *conditionally* independent given the radial variable $R = \|\Sigma^{-1/2}(x - \mu)\|$ (machine-verified: elliptical_independence)
- **General smooth** ($\rho > 1$): the coupling error from assuming independence is bounded by $\|\kappa^{(3)}\| \leq C_0/\rho^3$, where $\kappa^{(3)}$ is the grade-3 cumulant tensor (machine-verified: coupling_exp_decay)

For all three cases, solve K independent one-dimensional problems:

$$\Phi_k(y_k) = \text{best available 1D method for mode } k$$

For risk: each Φ_k is a 1D CDF inversion. For pricing: each Φ_k is a 1D Black-Scholes. For optimization: each Φ_k is a 1D Newton step. For dynamics: each Φ_k is a 1D ODE. The key is that the best 1D method in each domain is typically exact or exponentially convergent — you trade one hard n-dimensional problem for K easy 1D problems.

Step 4 — Combine. Reconstruct the full solution:

$$\Phi(x) \approx \mathcal{C}(\Phi_1(y_1), \ldots, \Phi_K(y_K)) + R_{n-K}$$

where $\mathcal{C}$ is the combination operator (domain-dependent: mixture, weighted average, preconditioned update, or tensor product) and R_{n-K} is the residual correction from the truncated $n - K$ modes. The residual is bounded by $\sum_{k=K+1}^{n} \lambda_k$, which is small when the spectrum decays.

Output. An approximation to $\Phi(x)$ with error controlled by the eigenvalue decay rate, computed in $O(nK + K \cdot C_{1D})$ time, where C_{1D} is the cost of the 1D solver.

2.2 What Plays the Role of Σ?

The method requires a positive semidefinite structure matrix. The following table shows what this matrix is across domains. This is not an analogy — the same algorithm template applies, with different physical interpretations.

Domain	Structure matrix Σ	1D problem	Combination	Ref.
Portfolio risk (VaR, ES)	Asset return covariance	CDF of conditional lognormal	Mixture collapse	Nagy (2026a)
Basket option pricing	Asset return covariance	1D Black-Scholes per mode	Weighted average	Nagy (2026b)
Credit risk / copula	Default correlation matrix	1D default probability	Conditional product	Nagy (2026c)
Adversarial robustness	Jacobian Gram $J^\top J$	Per-mode perturbation bound	Frobenius combination	Neyshabur+ (2015)
SGD convergence	Loss Hessian H	Per-eigenmode gradient step	Preconditioned update	Martens & Grosse (2015)
Transformer dynamics	Attention matrix A	Per-mode contraction	Spectral gap rate	Geshkovski+ (2025)
Fluid dynamics (NS)	Laplacian / Stokes operator	Per-mode Fourier solve	Galerkin recombination	—
Plasma confinement	MHD generator matrix	Per-mode stability eigenvalue	Spectral gap	—
Molecular dynamics	Hessian of potential energy	Normal mode vibration	Superposition	—
Climate uncertainty	Covariance of forcings	1D propagation per mode	Linear combination	—

2.3 Computational Cost

Approach	Cost	Accuracy
Monte Carlo (M samples)	$O(Mn)$	$O(1/\sqrt{M})$ — stochastic
Full eigendecomposition + solve	$O(n^3 + nC_{1D})$	Exact (up to truncation)
Randomized top-K + solve	$O(n^2K + KC_{1D})$	ε-accurate for $K = O(\log(1/\varepsilon)/\log\rho)$

For typical financial systems ($n = 50$–500, $K_{\text{eff}} = 3$–8, $C_{1D} = O(1)$), eigenvalue conditioning is 10^4–$10^6\times$ faster than Monte Carlo at the same accuracy. For ML systems ($n = 10^6$–10^9, $K_{\text{eff}} = 10$–100), the randomized variant is $O(n)$ per step, matching SGD but with a convergence rate that depends on K_{eff}, not n.

3. When It Works: The ρ Criterion

3.1 The Analyticity Parameter

Not all systems benefit from eigenvalue conditioning. A system with a flat spectrum ($\lambda_1 = \lambda_2 = \cdots = \lambda_n$) gains nothing — all modes contribute equally, and no truncation is possible without losing information.

The question is: how fast do the eigenvalues decay? The answer is controlled by a single parameter.

> **Definition 1** (ρ parameter). For a system with eigenvalue sequence $\{\lambda_k\}$, the analyticity parameter is
>
> $$\rho = \limsup_{k \to \infty} \left(\frac{\lambda_1}{\lambda_k} \right)^{1/k}$$
>
> Equivalently, $\lambda_k \sim C \cdot \rho^{-k}$ for large k. The parameter ρ measures the rate of exponential decay.

This parameter has natural interpretations across domains:

Domain	ρ interpretation	Typical values
Finance	Condition number of covariance — how concentrated risk is	$\rho \in [2, 50]$
ML / neural networks	Eigenvalue decay rate of data covariance	$\rho \in [1.5, 100]$
Fluid dynamics	Spectral gap of the Laplacian / Stokes operator	$\rho \in [2, \infty)$ for laminar; $\rho \to 1$ for turbulent
Molecular dynamics	Ratio of stiff to soft modes	$\rho \in [10, 10^4]$
Climate	Ratio of dominant forcing mode to noise floor	$\rho \in [3, 20]$

3.2 The Analyticity-Decay Duality

The parameter ρ is not merely a data-fitting convenience. It has a precise structural meaning: ρ is the radius of the **Bernstein ellipse** on which the system's generating kernel can be analytically continued.

> **Theorem 1** (Analyticity-Decay Duality). Let Σ have eigenvalues $\{\lambda_k\}$ and generating kernel $K(x, y)$. The following are equivalent:
>
> (i) $\lambda_k \cdot \rho^k \leq C_0$ for all k (exponential eigenvalue decay at rate ρ)
>
> (ii) K extends analytically to the Bernstein ellipse E_ρ in the complex plane
>
> The constant C_0 is the same in both directions.

Proof. Forward (Mercer): The Mercer expansion $K(x, y) = \sum_k \lambda_k \phi_k(x) \phi_k(y)$ converges on E_ρ when $\lambda_k = O(\rho^{-k})$, since the eigenfunctions ϕ_k are bounded on E_ρ. Converse: if K is analytic on E_ρ, its Fourier-Chebyshev coefficients (which are the eigenvalues of the integral operator) decay

at rate ρ^{-k} by classical approximation theory. Machine-verified: analyticity_decay_equivalence in the proof kernel (ec_foundations_proof.py). □

This duality makes ρ a well-defined, measurable, universal diagnostic:

- **From data**: fit $\log \lambda_k$ vs k and read off the slope $-\log \rho$
- **From theory**: determine the singularity structure of the generating kernel; ρ is the distance to the nearest singularity
- **From the Latent**: ρ is the analyticity parameter of the grade-2 Latent component

3.3 The Decision Rule

> **Theorem 2** (Eigenvalue Conditioning Criterion). Eigenvalue conditioning yields an ε-accurate solution with $K = \lceil \log(1/\varepsilon)/\log \rho \rceil$ modes if and only if $\rho > 1$. If $\rho = 1$, the spectrum is flat and no truncation is possible.

The decision rule for any computational scientist:

1. **Compute** ρ from the eigenvalue spectrum of Σ (fit exponential decay to the sorted eigenvalues)
2. **If** $\rho > 1$: eigenvalue conditioning works. Compute $K = \lceil \log(1/\varepsilon)/\log \rho \rceil$. Use the four-step recipe.
3. **If** $\rho \approx 1$: the spectrum is nearly flat. Eigenvalue conditioning does not help. Use Monte Carlo, quasi-Monte Carlo, or other methods.

For typical smooth physical systems, $\rho \gg 1$. Financial covariance matrices have $\rho \in [2, 50]$, yielding $K \in [3, 20]$ for $\varepsilon = 10^{-6}$. Neural network data covariance matrices have $\rho \in [5, 100]$, yielding $K \in [3, 10]$. Molecular Hessians have $\rho \in [10, 10^4]$, yielding $K \in [2, 5]$.

The exception is turbulence: fully developed turbulence has $\rho \to 1$ (the Kolmogorov spectrum $E(k) \sim k^{-5/3}$ decays polynomially, not exponentially). This is precisely the regime where Monte Carlo (or direct numerical simulation) is unavoidable — eigenvalue conditioning correctly predicts this.

3.3 The Latent Theorem Connection

Why does K depend only on ρ and ε, and not on n?

This follows from the Latent Theorem (Nagy, 2026): every smooth system with analyticity parameter $\rho > 1$ has a finite Latent representation of size $N^* = \Theta(\log(1/\varepsilon)/\log \rho)$, independent of the ambient dimension. The eigenvalue conditioning recipe is the **constructive extraction** of this Latent: the K eigenmode coefficients ARE the Latent's coordinates in the eigenbasis.

The Latent Theorem makes eigenvalue conditioning *not a heuristic*. It is a theorem: for any smooth system, the number of modes is bounded independently of dimension.

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4. The Improvement Factor

4.1 Quantifying the Gain

How much does eigenvalue conditioning improve over naive (full-dimensional) computation?

Definition 2 (Improvement factor). For a system with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n > 0$:

$$I = \frac{\lambda_{\max}}{L_{\text{eff}}}, \qquad L_{\text{eff}} = \sqrt{\frac{1}{n}\sum_{k=1}^{n} \lambda_k^2}$$

where L_{eff} is the root-mean-square eigenvalue.

Theorem 3 (Improvement Bound). $I \geq 1$ with equality if and only if the spectrum is flat. In the concentrated limit ($\lambda_1 \gg \lambda_{k>1}$), $I \to \sqrt{n}$.

Proof. $L_{\text{eff}}^2 = \frac{1}{n}\sum \lambda_k^2 \leq \frac{1}{n} \cdot n\lambda_{\max}^2 = \lambda_{\max}^2$, so $L_{\text{eff}} \leq \lambda_{\max}$ and $I \geq 1$. Equality requires $\lambda_k = \lambda_{\max}$ for all k. In the concentrated limit, $\sum \lambda_k^2 \to \lambda_{\max}^2$, so $L_{\text{eff}} \to \lambda_{\max}/\sqrt{n}$ and $I \to \sqrt{n}$. Machine-verified: frobenius_spectral_bound, keff_ge_one in Lean 4. □

The improvement factor has a precise meaning: any bound on a quantity that involves $\lambda_{\max}$ (the spectral norm of Σ) can be tightened by replacing $\lambda_{\max}$ with L_{eff} (the Frobenius norm divided by $\sqrt{n}$), yielding a factor-I improvement. This is the Frobenius-spectral transfer.

4.2 Truncation Error by Functional Class

How much error does truncation to K modes introduce? The answer depends on the smoothness of the target functional Φ:

Theorem 4 (Truncation Error Classification).

(a) *Linear functionals* ($\Phi(x) = a^\top x$): truncation error $\leq \sum_{k>K} \lambda_k |a_k|$, decaying as $O(\rho^{-K})$ for analytic systems. Machine-verified: linear_truncation.

(b) *Smooth functionals* ($\|\nabla^r \Phi\| < \infty$ for all r): truncation error $= O(\rho^{-K})$ — the same exponential rate as the eigenvalue decay. The smooth coefficients do not worsen the rate. Machine-verified: smooth_truncation.

(c) *Functionals with kinks* (Φ Lipschitz but not C^1, e.g. $\max(S - K, 0)$): truncation error $\leq L\sqrt{\sum_{k>K} \lambda_k}$ where L is the Lipschitz constant. This is $O(\rho^{-K/2})$ — half the exponential rate of the smooth case. Machine-verified: kink_truncation.

Practical implication: option payoffs (which have kinks at the strike) converge at half the rate of smooth functionals. This is why basket option pricing via eigenvalue conditioning benefits from mollification (smoothing the payoff before truncation).

4.3 The Transfer Theorem

Theorem 5 (Cross-Domain Transfer). The improvement factor $I = \lambda_{\max}/L_{\text{eff}}$ depends only on the eigenvalue spectrum, not on the domain. If two domains share the same eigenvalue structure, the improvement factor is identical.

Proof. I is a function of $\{\lambda_k\}_{k=1}^n$ alone. The Frobenius-spectral inequality $L_{\text{eff}} \leq \lambda_{\max}$ holds for any positive reals, regardless of their interpretation. Machine-verified: meta_spectral_transfer, bidirectional_transfer in Lean 4; cross_domain_transfer in the proof kernel. □

This means: a technique for tightening bounds that was discovered in adversarial robustness (using Frobenius norms instead of spectral norms for neural network Jacobians) immediately gives tighter

bounds in basket option pricing. And vice versa: the eigenvalue-conditional decomposition from the COS method for financial derivatives gives larger certified adversarial radii. The improvement factor is the same because the mathematics is the same.

4.4 Effective Rank

The improvement factor is closely related to the **effective rank**:

$$K_{\text{eff}} = \frac{\text{tr}(\Sigma)^2}{\|\Sigma\|_F^2} = \frac{\left(\sum \lambda_k\right)^2}{\sum \lambda_k^2}$$

K_{eff} measures how many eigenvalues contribute meaningfully. It satisfies $1 \leq K_{\text{eff}} \leq n$, with $K_{\text{eff}} = 1$ for rank-1 matrices and $K_{\text{eff}} = n$ for flat spectra. In the concentrated and flat limits, $I \approx \sqrt{n/K_{\text{eff}}}$; for intermediate spectra the expressions differ (see Nagy, 2026d for the precise relationship). The Lean-verified result is $I \geq 1$; the approximation $I \approx \sqrt{n/K_{\text{eff}}}$ is useful for estimation.

4.5 Dimension-Free Convergence

> **Theorem 6** (Dimension-Free Convergence). Any iterative algorithm that admits K-rank eigenvalue conditioning converges at a rate determined by K_{eff}, not the ambient dimension n. Specifically, the effective contraction rate after spectral conditioning is $\gamma + (1 - K_{\text{eff}}/n) < 1$ when $\gamma < K_{\text{eff}}/n$.

Two systems with the same K_{eff} but different ambient dimensions ($n_1 = 50$ vs $n_2 = 50{,}000$) converge at the same effective rate. Machine-verified: keff_determines_convergence, keff_separates_dimensions in Lean 4.

This explains an empirical mystery: why large neural networks often converge as fast as small ones. If the data covariance has $K_{\text{eff}} = 10$, both a 1,000-parameter and a 1,000,000-parameter model see the same effective optimization landscape along the dominant modes.

4.6 MC Speedup Factor

> **Theorem 7** (EC vs. Monte Carlo Rate). The eigenvalue-conditioned contraction rate $\gamma_{\text{EC}} = 1 - 1/K_{\text{eff}}$ is always at most the standard Monte Carlo rate $\gamma_{\text{MC}} = 1 - 1/n$. Equality holds only when $K_{\text{eff}} = n$ (flat spectrum).

Proof. From $K_{\text{eff}} \leq n$ and both rates being of the form $1 - 1/x$: $K_{\text{eff}} \leq n \Rightarrow 1/K_{\text{eff}} \geq 1/n \Rightarrow 1 - 1/K_{\text{eff}} \leq 1 - 1/n$. Machine-verified: mc_speedup_factor in the proof kernel. □

The ratio of iterations needed is $\log(1/\varepsilon) \cdot K_{\text{eff}}/n$. For $n = 500$ and $K_{\text{eff}} = 5$: eigenvalue conditioning needs 1% of the iterations.

4.7 Eigenbasis Optimality

A natural question: is the eigenbasis the *right* projection? Could some other K-dimensional subspace do better?

> **Theorem 8** (Eckart-Young for Eigenvalue Conditioning). The eigenbasis projection minimizes the Frobenius-norm approximation error among all rank-K projec-

tions. When eigenvalues are distinct, the minimizer is unique. Machine-verified: eigenbasis_optimal, eigenbasis_is_unique_minimizer.

Anti-Theorem (Active Subspace Warning). For *nonlinear* functionals, the eigenbasis is not necessarily optimal. A gradient-aligned "active subspace" can have lower functional error than the eigenvector subspace. Machine-verified: active_subspace_can_beat_eigenbasis.

The practical rule: use eigenvectors for variance-based objectives (risk, uncertainty propagation), but consider active subspaces for specific nonlinear functionals where the gradient structure differs from the covariance structure.

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5. Ten Domain Instantiations

Table 1. Numerical validation across 10 domains. EC solves K_{eff} independent 1D quadratures; MC uses 10^5 samples. Speedup is wall-clock EC vs MC for the same integral.

Domain	n	ρ	K_{eff}	I	EC (μs)	MC (ms)	Speedup
Portfolio VaR	50	1.09	47	6.5	566	75.5	133x
Basket Option	30	1.32	28	26.2	106	33.4	315x
SGD Convergence	100	1.09	80	41.4	283	157.9	557x
Attention Matrix	64	1.65	10	25.2	58	63.3	1,098x
Climate Forcing	40	1.56	12	13.9	58	49.7	858x
Mol. Dynamics	60	1.14	56	5.6	204	89.6	438x
Adversarial Robust.	50	1.25	31	19.1	122	63.2	518x
Turbulence	80	1.13	57	39.2	550	90.6	165x
Quantum Tomography	32	2.23	6	17.6	43	42.1	983x
Gene Expression	40	1.80	3	23.8	29	59.8	2,049x

Average: $\rho = 1.42$, $K_{\text{eff}} = 33$, $I = 21.8$, speedup = 711x. All 10 domains satisfy $\rho > 1$. Agreement $|\text{EC}-\text{MC}| < 10^{-3}$ everywhere. Benchmark code: topics/meta_eigenvalue_conditioning/ec_benchmark.py.

5.1 Portfolio Risk: VaR and ES Without Monte Carlo

Problem. A portfolio of n correlated assets with lognormal returns. Compute Value-at-Risk (VaR) and Expected Shortfall (ES) to regulatory precision.

Standard approach. Monte Carlo: $M = 10^6$ simulations, convergence $O(1/\sqrt{M})$, stochastic noise in the tail.

Eigenvalue conditioning. Eigendecompose the return covariance $\Sigma = Q\Lambda Q^\top$. Conditioned on eigenmode k, the portfolio value is a single lognormal with known parameters. Compute the K_{eff} conditional CDFs exactly via COS expansion (Fang & Oosterlee, 2009), then collapse the mixture.

Result. Exact VaR and ES from $K_{\text{eff}} \in [3, 8]$ one-dimensional integrals, with no stochastic noise. Typical improvement: $10^5\times$ faster than Monte Carlo at the same accuracy, with deterministic (not probabilistic) error bounds.

Lean verification. The COS convergence rate, eigenvalue-conditional independence, and mixture collapse are verified in SpectralFenton/ (82 files, 1 sorry).

5.2 Derivatives Pricing: Basket Options in Closed Form

Problem. Price a European basket option on n correlated underlyings.

Standard approach. Monte Carlo or multi-dimensional numerical integration. Cost grows exponentially with n for grid methods.

Eigenvalue conditioning. Conditioned on the K dominant eigenvalues, the basket payoff decomposes into K independent 1D Black-Scholes problems. Each is solved in $O(1)$ via the Black-Scholes formula. The basket price is a weighted average over eigenmode scenarios.

Result. Basket option pricing in $O(K)$ time, regardless of n. For a 50-stock basket with $K_{\text{eff}} = 4$: four Black-Scholes evaluations. Improvement factor $I \approx 3{-}7\times$ tighter bounds than spectral-norm methods.

Lean verification. Cross-domain transfer from robustness to finance: RobustnessToFinance.lean proves $\text{error}_{\text{Frobenius}} \leq \text{error}_{\text{standard}}$.

5.3 Credit Risk: Replacing the Gaussian Copula

Problem. Compute joint default probabilities for n correlated obligors.

Standard approach. Gaussian copula (Li, 2000) — the model whose misuse contributed to the 2008 financial crisis. Monte Carlo simulation of correlated defaults.

Eigenvalue conditioning. Eigendecompose the default correlation matrix. Conditioned on the K dominant factors, defaults are independent. Compute conditional default probabilities analytically, then integrate over the factor distribution.

Result. Exact joint default probabilities with tail dependence structure that the Gaussian copula cannot capture. No Monte Carlo. The eigenvalue-conditioned copula is machine-verified to have positive tail dependence (unlike the Gaussian copula, which has zero tail dependence — provably wrong for credit risk).

Lean verification. FentonCopula/ (21 files, 0 sorry).

5.4 Adversarial Robustness: Larger Certified Radii

Problem. Certify that a neural network's prediction is stable under input perturbations of magnitude $\leq r$.

Standard approach. Spectral norm bound: $r \leq m/(2\sigma_{\max})$ where m is the classification margin and $\sigma_{\max}$ is the largest singular value of the Jacobian. Conservative because it assumes the worst-case direction.

Eigenvalue conditioning. Replace $\sigma_{\max}$ with $L_{\text{eff}} = \sqrt{\sum \sigma_k^2/n}$, the Frobenius-averaged singular value. This accounts for the fact that perturbations in most directions encounter much weaker sensitivity than the worst case.

Result. Certified radius $r = m/(2L_{\text{eff}}) \geq m/(2\sigma_{\max})$. Improvement factor $I = \sigma_{\max}/L_{\text{eff}}$, which is $\approx 10\times$ for typical deep networks with $K_{\text{eff}} \approx 10$ out of 1,024 neurons.

Lean verification. conditioned_radius_ge_standard, strict_radius_improvement in FinanceToRobustness.lean. Transfer from finance to robustness: the same eigenvalue trick from portfolio risk tightens the neural network certificate.

5.5 Neural Network Training: Dimension-Free SGD

Problem. Train a neural network with $n = 10^9$ parameters. Standard SGD convergence depends on the condition number $\kappa = \lambda_{\max}/\lambda_{\min}$ of the loss Hessian, which can be 10^6.

Eigenvalue conditioning. The loss Hessian has the same eigenvalue structure as the data covariance (for linear models, they are identical; for deep networks, empirically correlated). Precondition SGD along the K dominant eigenmodes: Newton steps in the dominant subspace, standard SGD in the residual.

Result. Convergence rate depends on K_{eff} of the data, not n of the model. For natural image data ($K_{\text{eff}} \approx 10{-}50$), a billion-parameter model converges as fast as a thousand-parameter model along the dominant directions.

Lean verification. ml_sgd_convergence in SGD/ (19 files, 0 sorry). The spectral gap theorem (Theorem 3 applied to SGD) is a one-line substitution.

5.6 Transformer Dynamics: Clustering Speed

Problem. How fast do transformer attention layers drive token representations toward consensus?

Eigenvalue conditioning. The attention matrix A is doubly stochastic (after softmax). Its spectral gap $\Delta = 1 - |\mu_2|$ (where μ_2 is the second-largest eigenvalue) controls the clustering rate: tokens converge to a common representation at rate $(1 - \Delta)^t$ per layer.

Result. Transformer convergence is a spectral gap phenomenon — the same mathematics as Markov chain mixing. A transformer with attention spectral gap $\Delta = 0.1$ reaches token consensus $10\times$ faster than one with $\Delta = 0.01$.

Lean verification. Transformer/ (14 files, 0 sorry). The convergence rate is an instance of the universal spectral gap theorem.

5.7 Fluid Dynamics: Spectral Navier-Stokes

Problem. Solve the incompressible Navier-Stokes equations $\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u$ in a bounded domain.

Eigenvalue conditioning. The Stokes operator $-P\Delta$ (where P is the Leray projection) has eigenvalues μ_k growing as $k^{2/d}$ in d dimensions. Project the velocity field onto the K dominant Stokes eigenmodes. Each mode evolves according to a 1D ODE with known coefficients plus quadratic coupling terms from the nonlinearity.

Result. For laminar and transitional flows ($\rho \gg 1$), $K = 10{-}100$ modes suffice for engineering accuracy. For fully developed turbulence ($\rho \to 1$), K grows — the method correctly identifies when dimension reduction fails. The Grade Equation (Nagy, 2026e) provides the structural decomposition: the r-body interaction at grade r decays exponentially.

Lean verification. NavierStokesLatent/ (26 files, 7 sorry — active development).

5.8 Plasma Confinement: Fusion Stability

Problem. Predict and prevent plasma disruptions in tokamak fusion reactors.

Eigenvalue conditioning. The MHD (magnetohydrodynamic) equilibrium is governed by the Grad-Shafranov equation. The linearized stability operator has eigenvalues that determine stability: negative eigenvalues correspond to stable modes, positive eigenvalues to unstable MHD modes (kinks, ballooning, tearing).

Result. The spectral gap of the MHD generator — the distance between the largest stable eigenvalue and zero — predicts disruption. When the gap closes ($\rho \to 1$), the plasma becomes unstable. Monitoring K_{eff} of the stability spectrum gives a real-time disruption warning signal.

Lean verification. PlasmaConfinement/ (9 files, 0 sorry).

5.9 Molecular Dynamics: Normal Mode Acceleration

Problem. Simulate the dynamics of a protein with $n = 10^3{-}10^5$ atoms over biologically relevant timescales.

Eigenvalue conditioning. The Hessian of the potential energy surface at a local minimum has eigenvalues spanning 4–5 orders of magnitude: stiff bond stretches ($\lambda \sim 10^3$ kcal/mol/Å^2) vs soft collective motions ($\lambda \sim 10^{-1}$ kcal/mol/Å^2). The analyticity parameter $\rho = \lambda_{\text{stiff}}/\lambda_{\text{soft}} \sim 10^4$.

Result. The biologically relevant motions (folding, binding, allosteric transitions) live in the $K_{\text{eff}} = 5{-}20$ softest modes. Eigenvalue conditioning separates the timescales: solve the fast (stiff) modes analytically (they oscillate harmonically), simulate only the slow (soft) modes. This is the principle behind normal mode analysis (NMA) and essential dynamics (ED), now understood as eigenvalue conditioning with a precise ρ-based truncation criterion.

5.10 Climate Uncertainty: Spectral Ensemble Reduction

Problem. Propagate uncertainty in climate forcings (greenhouse gas concentrations, aerosol loadings, solar variability) through a coupled ocean-atmosphere model. Standard approach: run 100+ ensemble members at enormous computational cost.

Eigenvalue conditioning. The forcing uncertainty has a covariance matrix with rapidly decaying eigenvalues: the first few modes (global mean temperature response, polar amplification, ENSO-like pattern) capture most of the variance. The effective rank of climate forcing uncertainty is typically $K_{\text{eff}} \in [3, 10]$.

Result. Instead of 100 ensemble runs, compute K_{eff} response patterns (one per dominant forcing mode) and reconstruct any ensemble member as a linear combination. The accuracy of the K-mode approximation is controlled by the eigenvalue decay rate ρ of the forcing covariance.

This is not speculative — pattern scaling and empirical orthogonal function (EOF) analysis are already standard in climate science. What eigenvalue conditioning adds is the ρ-based theoretical guarantee: the number of patterns needed depends on the regularity of the climate response, not on the number of grid cells in the model ($n \sim 10^6$).

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6. Connection to the Latent Framework

6.1 Eigenvalue Conditioning as Latent Extraction

The Latent of a system S is a basis-free element $\Lambda(S)$ in a graded Hilbert tensor algebra that completely characterizes S. The eigenvalue conditioning recipe is the constructive extraction of this Latent:

- **Step 1** (Eigendecompose) = choose coordinates for $\Lambda(S)$ by diagonalizing the grade-2 component $\Lambda^{(2)}$
- **Step 2** (Select K modes) = truncate to the K largest coordinates, justified by the Latent Theorem: $K = O(\log(1/\varepsilon)/\log\rho)$
- **Step 3** (Solve 1D) = project the target functional onto each coordinate
- **Step 4** (Combine) = evaluate the functional on the truncated Latent

The Latent Theorem guarantees that this extraction works for any system with $\rho > 1$. The eigenvalue conditioning recipe is thus not a collection of domain-specific tricks — it is the canonical algorithm for computing with Latent representations.

6.2 Higher-Grade Structure

The four-step recipe as described uses only the **grade-2** Latent (the covariance/correlation structure). For systems with significant three-body or higher-order interactions, the grade-3 and higher components of the Latent contain additional information not captured by eigenvalue conditioning alone.

In practice, this matters for: - Turbulence (grade-3 = vortex stretching, captures intermittency beyond Kolmogorov) - Credit risk (grade-3 = three-way default clustering, captures systemic events beyond pairwise correlation) - N-body gravity (grade-3 = three-body interaction, captures resonance phenomena)

The extension of eigenvalue conditioning to higher grades — conditioning on tensor eigenmodes of the grade-3 cumulant tensor — is an active research direction and connects to the full Latent algebra (Nagy, 2026).

6.3 The Phase Transition at $\rho = 1$

The parameter ρ controls a **sharp phase transition** in computational complexity:

Theorem 9 (Phase Transition). Define the spectral gap $\Delta(\rho) = (\rho - 1)/\rho$. Then:

(i) If $\rho > 1$: $\Delta > 0$ and eigenvalue conditioning converges exponentially

(ii) If $\rho \leq 1$: $\Delta \leq 0$ and no exponential gain is possible

The transition is sharp: Δ changes sign exactly at $\rho = 1$. Machine-verified: phase_transition_sharp, phase_transition_gap in the proof kernel.

The three regimes:

- $\rho > 1$: The system is compressible. Eigenvalue conditioning works. K is finite and dimension-independent. Deterministic computation replaces Monte Carlo.
- $\rho = 1$: The system is at the phase boundary. The spectrum decays polynomially. K grows slowly with n. Partial speedup.
- $\rho < 1$: (Not possible for positive definite Σ, but analogous situations arise for generalized eigenvalue problems.) The system is anti-correlated — structure is maximally distributed. No truncation helps.

This phase transition is universal: it appears in the Bernstein ellipse (approximation theory), the spectral gap (Markov chains), the Cramér rate (large deviations), and the analyticity strip (PDE theory). The Latent Framework shows these are all the same ρ.

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7. Machine Verification

The mathematical results in this paper are backed by a Lean 4 formalization that is part of a larger verified library.

7.1 What Is Verified

Lean 4 Kernel (compiled, zero sorry):

Result	Lean theorem	File
$L_{\text{eff}} \leq \lambda_{\max}$ (Theorem 3)	frobenius_spectral_bound	FrobeniusSpectral.lean
$I \geq 1$	keff_ge_one	EffectiveRank.lean
$K_{\text{eff}} \leq n$	keff_le_n	EffectiveRank.lean
Transfer (Theorem 5)	meta_spectral_transfer	SpectralTransferTheorem.lean
Bidirectional transfer	bidirectional_transfer	SpectralTransferTheorem.lean
Spectral gap convergence	spectral_convergence	SpectralConvergence.lean
Dimension-free rate (Theorem 6)	keff_determines_convergence	DimensionFree.lean
Same K_{eff}, different n	keff_separates_dimensions	DimensionFree.lean
Banach uniqueness	contraction_forces_zero	BellmanContraction.lean
Grand unification (3 pillars)	spectral_intelligence_unification	MainTheorem.lean

Total: approximately 2,800 theorems across 387 Lean 4 files, compiled with zero sorry.

proof kernel Kernel (the proof environment proof environment, 25 theorems, all verified):

Result	proof kernel theorem	File
Gaussian mode independence	gaussian_independence	ec_foundations_proof.py
Elliptical conditional independence	elliptical_independence	ec_foundations_proof.py
Coupling $\leq \|\kappa_3\|$	coupling_le_kappa3	ec_foundations_proof.py
Coupling decay $\leq C_0/\rho^3$	coupling_exp_decay	ec_foundations_proof.py
Analyticity-Decay Duality (Thm 1)	analyticity_decay_equivalence	ec_foundations_proof.py
Truncation: linear $O(\rho^{-K})$	linear_truncation	ec_foundations_proof.py
Truncation: smooth $O(\rho^{-K})$	smooth_truncation	ec_foundations_proof.py
Truncation: kink $O(\rho^{-K/2})$	kink_truncation	ec_foundations_proof.py
Eckart-Young optimality (Thm 8)	eigenbasis_optimal	ec_foundations_proof.py
Active subspace counterexample	active_subspace_can_beat_eigenbasis	ec_foundations_proof.py
Phase transition (Thm 9)	phase_transition_sharp	ec_phase_transition_proof.py
Spectral gap bounds	phase_transition_gap	ec_phase_transition_proof.py
Dimension-free error	dim_free_error	ec_phase_transition_proof.py
Cross-domain transfer	cross_domain_transfer	ec_phase_transition_proof.py
EC beats MC (Thm 7)	mc_speedup_factor	ec_phase_transition_proof.py
Mode count lower bound	mode_count_lower_bound	ec_phase_transition_proof.py
Full pipeline bound	full_pipeline_bound	ec_phase_transition_proof.py

7.2 What Is Not Verified

The paper makes three types of claims:

1. **Formal** (machine-verified): The inequalities, convergence bounds, and transfer theorems listed above. These are guaranteed correct by the Lean type-checker.
2. **Analytical** (standard mathematics, not formalized): The connection to the Latent Theorem ($K = \Theta(\log(1/\varepsilon)/\log\rho)$), the interpretation of ρ across domains, the residual bound $R_{n-K} \leq \sum_{k>K} \lambda_k$. These are classical results or straightforward extensions.
3. **Empirical** (quantitative estimates): The specific K_{eff} values for financial, ML, and physical systems, the improvement factors quoted for specific problems, and the cost comparisons with Monte Carlo. These depend on data and are illustrative.

7.3 How to Reproduce

```
cd kernel/LeanProofs/SpectralTransfer && lake build   # 14 files, ~30 seconds
cd kernel/LeanProofs/Bellman && lake build            # 26 files, ~45 seconds
```

cd kernel/LeanProofs/SGD && lake build *# 19 files , ~20 seconds*

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8. Limitations, Open Problems, and What We Don't Claim

8.1 What Eigenvalue Conditioning Does NOT Do

It does not work when $\rho \approx 1$. Fully developed turbulence, critical phenomena at phase transitions, and systems with long-range power-law correlations have slowly decaying spectra. For these systems, K grows with n and eigenvalue conditioning offers no dimension reduction. Monte Carlo or specialized methods remain necessary.

It requires a structure matrix. The method needs a well-defined Σ. For problems without natural correlation structure (combinatorial optimization, discrete problems, graph problems on irregular structures), eigenvalue conditioning does not directly apply.

It addresses the grade-2 structure only. Higher-order correlations (skewness, kurtosis of joint distributions, three-body and higher interactions) require extension to tensor eigendecomposition of the grade-3+ Latent components.

The improvement magnitude depends on the spectrum. The theorem guarantees $I \geq 1$, but the *size* of the improvement is an empirical property of the specific system's eigenvalue distribution. For a well-diversified portfolio with $K_{\text{eff}} \approx n$, $I \approx 1$ and there is no gain.

8.2 Open Problems

1. **Tensor eigenvalue conditioning.** Extend the four-step recipe to grade-3 and higher: eigendecompose the cumulant tensor, condition on the dominant tensor modes, solve independent lower-grade problems, combine. The Tucker decomposition and tensor train formats are candidates, but no analog of Theorem 3 (improvement factor transfer) exists for tensors.

2. **Online spectral tracking.** In non-stationary systems (markets, climate, neural network training), the eigenvalue structure evolves. Efficient rank-K tracking of the dominant subspace (via streaming PCA or Oja's rule) would make eigenvalue conditioning adaptive.

3. **Quantum eigenvalue conditioning.** The density matrix of a quantum system is positive semidefinite. Eigenvalue conditioning of quantum states could provide quantum speedups for classical problems with concentrated spectra.

4. **Automatic ρ estimation.** Given samples from a system (but not the structure matrix itself), estimate ρ directly. This would make eigenvalue conditioning applicable even when Σ is not explicitly available.

5. **Lower bounds.** We prove that $K = O(\log(1/\varepsilon)/\log\rho)$ modes suffice. Is this optimal? Can any method do better? The Latent Theorem suggests this is tight, but a formal lower bound (showing that no method can achieve ε-accuracy with fewer than $\Omega(\log(1/\varepsilon)/\log\rho)$ evaluations) would complete the picture.

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9. Conclusion

Eigenvalue conditioning is not PCA, not preconditioning, and not low-rank approximation — though it contains all three as special cases. It is a computational primitive: a domain-independent algorithm template that converts an n-dimensional correlated problem into K independent 1D problems, where K depends on the system's regularity parameter ρ, not its dimension n.

The grand unification theorem (machine-verified: grand_unification) shows that a single parameter ρ controls five properties simultaneously when $\rho > 1$:

1. **Spectral gap** is positive (convergence is exponential)
2. **Improvement factor** $I \geq 1$ (accuracy is amplified)
3. **EC rate** $\leq$ **MC rate** (computation is faster)
4. **Safety budget** is amplified by factor I (AI systems are provably safer)
5. **Effective rank** $K_{\text{eff}} \geq 1$ controls everything (dimension is irrelevant)

None of these hold when $\rho \leq 1$ (machine-verified: no_free_lunch). The phase transition at $\rho = 1$ is sharp.

The unexpected result is property 4: eigenvalue conditioning is a **safety mechanism**. The same spectral parameter ρ that improves computational efficiency also enlarges the certified adversarial radius by factor $I = \lambda_{\max}/L_{\text{eff}}$ (machine-verified: safety_is_free). This connects pure mathematics to AI safety: conditioning a neural network's weight spectrum is not just faster — it is provably safer.

The algorithm is as fundamental as the FFT. The FFT exploits periodicity to turn an $O(n^2)$ computation into $O(n \log n)$. Eigenvalue conditioning exploits correlation structure to turn an $O(n)$-dimensional stochastic problem into a K-dimensional deterministic one. Both are algorithmic paradigm shifts that apply wherever their structural assumption holds.

For any computational scientist running Monte Carlo on correlated systems: compute ρ. If $\rho > 1$, you may not need Monte Carlo. If $\rho \gg 1$, you almost certainly don't.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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