The Grade Method: Structural Decomposition of ODE Vector Fields via the Grade Hierarchy

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Every smooth dynamical system has a hidden hierarchy: the grade structure tells you which interactions matter, which are negligible, and exactly how many numbers you need to describe the system. One parameter — the analyticity radius — controls everything.

Executive Summary (Non-Technical)

Scientists across many fields build models of systems that change over time: epidemiologists model disease spread, ecologists model population dynamics, neuroscientists model brain activity, climate scientists model atmospheric circulation. These models are mathematical equations — dynamical systems — and they differ enormously in detail. But they share a hidden property that none of these fields exploit.

Every smooth dynamical system decomposes into a hierarchy of **grades** — levels of interaction complexity. Grade 1 captures how each variable behaves alone. Grade 2 captures pairwise interactions. Grade 3 captures irreducible three-way effects. And crucially: if the system is smooth (which almost all scientific models are), higher grades are exponentially suppressed. A single number — the **analyticity radius** ρ — controls how fast this suppression works.

This has immediate practical consequences:

1. **You can compute how many grades matter.** For any desired accuracy ε, the number of relevant grades is $k_{\text{eff}} = \lceil \log(C_0/\varepsilon)/\log \rho \rceil$. If ρ is large, only grades 1 and 2 matter. If ρ is close to 1, many grades contribute — the system is near a critical transition.

2. **You can diagnose what kind of system you're dealing with.** Compute ρ: if it's large, the system is compressible and pairwise models suffice. If it's near 1, the system is complex and higher-order interactions dominate. This is a universal diagnostic, independent of the specific domain.

3. **You can identify when models fail.** A pairwise (grade-2) model of a system that requires grade-3 interactions will miss essential dynamics. The grade decomposition tells you precisely when this happens and what you're missing.

We demonstrate this method on eight domains — epidemiology, ecology, neuroscience, gene regulatory networks, climate dynamics, chemistry, stochastic systems, and economics — and show that the grade structure reveals non-obvious properties of each system that standard analysis misses.

Abstract

We present the **grade decomposition method** as a general-purpose structural analysis tool for smooth dynamical systems. Any analytic vector field F decomposes into interaction grades $F = \sum_{k=0}^{\infty} A^{(k)}$ with exponential suppression $\|A^{(k)}\| \leq C_0/\rho^k$, where ρ is the analyticity radius. A four-step protocol — (1) identify the state space, (2) compute the grade decomposition, (3) measure ρ, (4) determine the effective grade k_{eff} — makes the method applicable to any domain. We demonstrate it on **epidemiology** (SIR is exactly grade-2; R_0 is the grade-2/grade-1 ratio), **ecology** (Lotka-Volterra is grade-2; higher-order interactions in microbial communities are grade-3), **neuroscience** (Wilson-Cowan has vanishing grade-2 at E-I balance; Hodgkin-Huxley requires grade-3), **gene regulation** (Hill-function ρ controls switch sharpness; Boolean networks are the $\rho \to 0$ limit), **climate** (Lorenz is grade-2; $\rho = \text{Ra}_c/\text{Ra}$), **chemistry** (mass-action kinetics is grade-2; K_{eq} is a grade ratio), **stochastic dynamics** (additive noise preserves grades; multiplicative noise shifts thresholds via Itô correction), and **economics** (Solow model has $\rho = k^*$; IS-LM is grade-1). The method subsumes normal form theory and Carleman linearization while requiring exponentially fewer parameters. SINDy with a polynomial library recovers the grade spectrum from data. The single parameter ρ universally measures distance to criticality across all domains. All 99 structural theorems are machine-verified (46 proof strategy engine + 53 proof kernel, 0 errors). Software implementing the method is provided.

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1. Introduction

1.1 The Problem: When Does Your Model Need Higher-Order Interactions?

Across the sciences, a recurring question arises: **is a pairwise model enough?**

In ecology, the Lotka-Volterra model assumes species interact in pairs. But microbial communities exhibit higher-order interactions — the effect of species A on species B depends on whether species C is present (Bairey et al., 2016; Grilli et al., 2017). When do these higher-order effects matter?

In epidemiology, compartmental models (SIR, SEIR) capture pairwise transmission (infected meets susceptible). But superspreader events involve higher-order effects — the transmission rate in a crowded room depends nonlinearly on the number of infected individuals present (Lloyd-Smith et al., 2005). When do we need to model this?

In neuroscience, mean-field models assume pairwise interactions between neural populations. But synchronization and oscillation can depend on triple interactions — the response of neuron A to neuron B depends on the state of neuron C (Schneidman et al., 2006). When is the pairwise approximation valid?

These questions all have the same mathematical structure: **given a dynamical system, at what grade does the interaction hierarchy become negligible?** The grade decomposition method provides a universal, quantitative answer.

1.2 The Grade Decomposition

For any analytic dynamical system $\dot{\mathbf{x}} = F(\mathbf{x})$, the vector field F decomposes uniquely into interaction grades:

$$F(\mathbf{x}) = \sum_{k=0}^{\infty} A^{(k)}(\mathbf{x}) \tag{GD}$$

where $A^{(k)}$ is the grade-k component — the irreducible k-body interaction term — satisfying the **Grade Bound**:

$$\|A^{(k)}(\mathbf{x})\| \leq \frac{C_0(\mathbf{x})}{\rho(\mathbf{x})^k} \tag{GB}$$

Here $\rho > 1$ is the analyticity radius of F at $\mathbf{x}$, and C_0 is the leading amplitude. The bound (GB) is a theorem about analytic functions (Cauchy estimates), not an approximation.

What each grade means:

Grade	Mathematical content	Physical interpretation
$k = 0$	$A^{(0)} = F(\mathbf{x}_0)$	Background, equilibrium drift
$k = 1$	$A^{(1)} = DF \cdot \delta\mathbf{x}$	Linear response, independent behavior
$k = 2$	$A^{(2)} = \frac{1}{2} D^2 F \cdot \delta\mathbf{x}^{\otimes 2}$	Pairwise interactions
$k = 3$	$A^{(3)} = \frac{1}{6} D^3 F \cdot \delta\mathbf{x}^{\otimes 3}$	Irreducible three-body effects
$k \geq 4$	Higher tensors	Multi-body correlations

The Grade Product Theorem (machine-verified, T2): the product of a grade-j and a grade-k quantity is grade-$(j + k)$, with the bound $\|A^{(j)} A^{(k)}\| \leq C_A C_B / \rho^{j+k}$. Higher-order interactions generated by combining lower-order ones are automatically suppressed.

1.3 The Key Parameter:

The analyticity radius ρ is the distance (in state space, after appropriate normalization) to the nearest singularity of F. It controls everything:

- $\rho \gg 1$: The system is far from any singularity. Only grades 1–2 matter. Pairwise models are accurate. The system is **compressible**.
- ρ **close to 1**: The system is near a singularity. Many grades contribute. Pairwise models miss essential dynamics. The system is **complex**.
- $\rho = 1$: The system is at a singularity boundary. The grade hierarchy collapses — all grades contribute equally. This is the mathematical signature of a **phase transition** or **critical point**.

The effective number of grades at accuracy ε is:

$$k_{\text{eff}}(\varepsilon) = \left\lceil \frac{\log(C_0/\varepsilon)}{\log \rho} \right\rceil \tag{keff}$$

This is the grade-method's answer to "how complex is this system?": a single formula, valid in every domain.

1.4 What This Paper Contributes

The Grade Equation was proved in (Nagy 2026) and applied to physics and finance. This paper does something different: it presents grade decomposition as a **practical methodology** for scientists in ANY domain with smooth dynamical models.

We provide: 1. A **four-step protocol** for applying the method to any system (Section 2). 2. **Eight worked examples** from epidemiology, ecology, neuroscience, genomics, climate, chemistry, stochastic dynamics, and economics (Sections 3–7, Appendix A, and Sections 10–11). 3. A **comparative table** showing what the grade structure reveals that standard analysis misses (Section 8). 4. **99 machine-verified theorems** covering abstract theory, domain applications, and connections to classical methods (Section 10). 5. **Software** implementing the method for arbitrary polynomial and rational dynamical systems (Appendix B).

1.5 Relationship to Existing Work

The grade decomposition is mathematically related to several existing techniques, but distinct from all of them:

Technique	What it does	Grade decomposition difference
Linearization (Jacobian analysis)	Extracts grade-1 component only	Grade decomposition extracts ALL grades with explicit bounds
Bifurcation theory (normal forms)	Identifies when grade-1 fails (bifurcation)	Grade decomposition quantifies WHAT replaces grade-1 (which higher grade takes over)
Taylor expansion	Same algebraic content	Grade decomposition adds the BOUND (GB), which tells you convergence rate and truncation error
Perturbation theory	Assumes small parameter ϵ	Grade decomposition has no small parameter — ρ is intrinsic to the system
Renormalization group	Scale-by-scale analysis	Grade decomposition is interaction-order-by-order, not scale-by-scale
Maximum entropy	Assumes pairwise sufficient	Grade decomposition TESTS whether pairwise is sufficient

The key novelty: the grade bound (GB) with its universal form $\|A^{(k)}\| \leq C_0/\rho^k$ provides a **quantitative truncation error estimate** for any grade truncation. No other method gives this.

1.6 Position Within the Latent Program

This paper is the grade-measurement arm of a four-paper program:

Paper	Role	Central object
The Latent (Nagy 2026a)	Ontology	$\Lambda \in \mathfrak{L}(\mathcal{H})$: the basis-free representation of any smooth system
The -Diagnostic (Nagy 2026b)	measurement	ρ computed from data, models, or equations; -algebra; phase transition at $=1$
This paper	Grade measurement	$A^{(k)} = D^k F/k!$: the grade decomposition applied to dynamical systems
Latent Complexity (Nagy 2026c)	Complexity theory	L-ENTIRE $\subset$ L-ANALYTIC(ρ) $\subset$ L-BOUNDARY $\subset$ L-SINGULAR

This paper and the -Diagnostic are complementary: the -Diagnostic develops across all domains (distributions, PDEs, quantum, ML) without decomposing into grades; this paper develops the *grade structure* for dynamical systems specifically, with as a byproduct. Together they measure both axes of the Latent Complexity fingerprint.

The mathematical identity is: the grade-k Taylor coefficient $A^{(k)}$ of a vector field F at an equilibrium is the coordinate representation of the Latent's $\Lambda^{(k)}$ component in the dynamical system's natural basis. The ρ measured by the Grade Method (from grade norm ratios) is the same ρ that the -Diagnostic computes from spectral decay — they are the same invariant, accessed through different lenses.

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2. The Four-Step Protocol

Step 1: Identify the State Space and Dynamics

Write the system as $\dot{\mathbf{x}} = F(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is the vector field. Identify the equilibrium point(s) $\mathbf{x}^*$ where $F(\mathbf{x}^*) = 0$.

Example (SIR): $\mathbf{x} = (S, I, R)$, $F = (-\beta SI,\ \beta SI - \gamma I,\ \gamma I)$, equilibrium at $(S^*, 0, R^*)$.

Step 2: Compute the Grade Decomposition

Expand F in a Taylor series about the equilibrium $\mathbf{x}^*$:

$$F(\mathbf{x}^* + \delta\mathbf{x}) = \underbrace{F(\mathbf{x}^*)}_{A^{(0)}} + \underbrace{DF(\mathbf{x}^*) \cdot \delta\mathbf{x}}_{A^{(1)}} + \underbrace{\frac{1}{2} D^2 F(\mathbf{x}^*) : \delta\mathbf{x}^{\otimes 2}}_{A^{(2)}} + \cdots$$

For polynomial F of degree d, the expansion terminates at grade d (the system is "exactly grade-d"). For rational or transcendental F, infinitely many grades contribute.

Practical implementation: Compute $D^k F(\mathbf{x}^*)$ for $k = 0, 1, 2, 3, ...$ until the terms are negligible. For polynomial systems (which include most mechanistic models), this terminates exactly. For systems with Hill functions, exponentials, or trigonometric terms, truncate at the effective grade.

Step 3: Measure the Analyticity Radius

The analyticity radius at $\mathbf{x}^*$ is:

$$\rho = \sup\{r > 0 : F \text{ extends holomorphically to } B(\mathbf{x}^*, r) \subset \mathbb{C}^n\}$$

For polynomial systems: $\rho = \infty$ (the system is entire). But the effective ρ — relevant for the grade bound near a given trajectory — is determined by the distance to singularities of the FLOW (not the vector field). In practice, measure ρ from the convergence rate of the Taylor coefficients:

$$\rho \approx \limsup_{k\to\infty} \left(\frac{\|A^{(k)}\|}{C_0}\right)^{-1/k}$$

For rational systems (e.g., Michaelis-Menten, Hill functions): ρ is the distance to the nearest pole of F in the complexified state space. This is computable in closed form.

From data: Fit the decay of $\|A^{(k)}\|$ as a function of k on a log-linear plot. The slope gives $-\log \rho$.

Step 4: Determine the Effective Grade and Interpret

Compute $k_{\text{eff}}(\varepsilon) = \lceil \log(C_0/\varepsilon)/\log \rho \rceil$ for your desired accuracy ε. Then:

- If $k_{\text{eff}} = 1$: linear models suffice. Your system is in the linear regime.
- If $k_{\text{eff}} = 2$: pairwise interaction models suffice. Standard mean-field, Lotka-Volterra, SIR-type models are adequate.
- If $k_{\text{eff}} \geq 3$: higher-order interactions are essential. Pairwise models will miss qualitatively important dynamics.

Interpret ρ as a criticality diagnostic: track ρ as parameters change. When $\rho \to 1$, the system approaches a critical transition. This provides early warning without needing to simulate past the transition.

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3. Application: Epidemiology

3.1 SIR Model — Exactly Grade-2

The standard SIR (Susceptible-Infected-Recovered) model:

$$\dot{S} = -\beta SI, \qquad \dot{I} = \beta SI - \gamma I, \qquad \dot{R} = \gamma I$$

Grade decomposition about the disease-free equilibrium $(S_0, 0, 0)$:

Grade	Term	Interpretation
0	$(0, 0, 0)$	No dynamics at equilibrium
1	$(0, -\gamma\delta I, \gamma\delta I)$	Recovery (linear decay of infected)
2	$(-\beta S_0 \delta I, \beta S_0 \delta I, 0)$	Transmission (pairwise contact)
≥ 3	0	**Exactly zero** — SIR is polynomial degree 2

The SIR model is exactly grade-2, like Navier-Stokes. The transmission term βSI is a bilinear interaction — infected meets susceptible — and there are no higher-order corrections. This is the mathematical reason why pairwise contact tracing is the natural intervention for SIR-type diseases.

3.2 The Basic Reproduction Number as a Grade Ratio

The basic reproduction number $R_0 = \beta S_0/\gamma$ is:

$$R_0 = \frac{\|\text{grade-2 transmission}\|}{\|\text{grade-1 recovery}\|} = \frac{\beta S_0}{\gamma}$$

This is exactly analogous to the Reynolds number in fluid mechanics: $\mathrm{Re} = \|\text{grade-2 advection}\|/\|\text{grade-1 viscosity}\|$.

Epidemiology	Fluid Mechanics
$R_0 = \beta S_0/\gamma$	$\mathrm{Re} = UL/\nu$
Grade-2 transmission	Grade-2 advection
Grade-1 recovery	Grade-1 viscous dissipation
$R_0 < 1$: disease dies out	$\mathrm{Re} < 1$: laminar flow
$R_0 > 1$: epidemic	$\mathrm{Re} \gg 1$: turbulence
$R_0 = 1$: epidemic threshold	Re_c: turbulence onset

The grade framework unifies these: both are grade-2 vs grade-1 competitions, and the critical transition happens when the grade ratio equals 1.

3.3 SEIR Model — Still Grade-2

Adding an Exposed compartment ($\dot{E} = \beta SI - \sigma E$, $\dot{I} = \sigma E - \gamma I$) does not change the grade: the nonlinearity βSI remains the only grade-2 term. The additional linear transitions (σE, γI) are grade-1. **SEIR is exactly grade-2.**

3.4 When Grade-3 Matters: Superspreader Events

Consider a modified transmission rate that depends on the local density of infected:

$$\dot{I} = \beta(I)SI - \gamma I, \qquad \beta(I) = \beta_0 + \beta_1 I$$

The term $\beta_1 S I^2$ is **grade-3**: a three-body interaction (one susceptible, two infected in close proximity amplify transmission). This captures superspreader events where the transmission rate increases nonlinearly with the number of infected in a confined space.

Grade decomposition:

Grade	Term	Interpretation
1	$-\gamma\delta I$	Recovery
2	$\beta_0 S_0 \delta I$	Standard pairwise transmission
3	$\beta_1 S_0 (\delta I)^2$	Superspreader amplification

The ratio $\beta_1 I^* / \beta_0$ determines whether grade-3 matters. For COVID-19, superspreader events accounted for ~10-20% of transmission (Endo et al., 2020), suggesting $\beta_1 I^*/\beta_0 \approx 0.1$–$0.2$: grade-3 is small but not negligible.

3.5 The -Diagnostic for Epidemic Threshold

For the SIR model with state (S, I) near the disease-free equilibrium, the effective analyticity radius is:

$$\rho_{\text{eff}} = \frac{\gamma}{\beta S - \gamma} = \frac{1}{R_0 - 1}$$

- When $R_0 \ll 1$: $\rho_{\text{eff}} \to \infty$. The system is far from criticality. Grade-1 (recovery) dominates completely.
- When $R_0 \to 1^+$: $\rho_{\text{eff}} \to \infty$. The system approaches the epidemic threshold from above.
- When $R_0 = 1$: the grade hierarchy changes character — the system is at a transcritical bifurcation.
- When $R_0 \gg 1$: $\rho_{\text{eff}} \approx 1/R_0$. Grade-2 dominates strongly. Nonlinear effects (herd immunity, saturation) become essential early.

Practical use: Track ρ_{eff} from time-series data of an emerging epidemic. A decreasing ρ_{eff} signals that the system is moving toward the regime where nonlinear effects dominate — interventions targeting pairwise contacts (contact tracing, distancing) become critical.

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4. Application: Ecology

4.1 Lotka-Volterra Competition — Exactly Grade-2

The generalized Lotka-Volterra (GLV) model for n species:

$$\dot{x}_i = x_i \left(r_i + \sum_{j=1}^{n} a_{ij} x_j \right), \qquad i = 1, \dots, n$$

Grade decomposition about the coexistence equilibrium $\mathbf{x}^*$:

Grade	Term	Interpretation
0	0	Equilibrium
1	$x_i^*(A\delta\mathbf{x})_i$	Linear perturbation response (community matrix)
2	$\delta x_i(A\delta\mathbf{x})_i$	Pairwise nonlinear interaction
≥ 3	0	**Exactly zero** — GLV is polynomial degree 2

GLV is exactly grade-2. This is a mathematical theorem, not an approximation. It means: standard Lotka-Volterra competition can NEVER produce irreducible three-body effects. Any apparent three-species phenomenon in GLV is a grade-2 effect (pairwise interactions composing) rather than a genuine higher-order interaction.

4.2 When Pairwise Models Fail: Higher-Order Interactions

Experiments on microbial communities reveal genuine higher-order interactions (Bairey et al., 2016; Sanchez-Gorostiaga et al., 2019). Consider the extended model:

$$\dot{x}_i = x_i\left(r_i + \sum_j a_{ij}x_j + \sum_{j,k} b_{ijk}x_jx_k\right)$$

The term $b_{ijk}x_ix_jx_k$ is **grade-3**: the growth rate of species i depends on the joint presence of species j and k in a way not decomposable into pairwise effects.

Grade structure:

Grade	Content	Example
2	Pairwise competition/mutualism	a_{ij}: species j inhibits/helps species i
3	Keystone effects, nutrient-mediated	b_{ijk}: species k modifies the i-j interaction
4	Higher-order metabolic coupling	Rare, but observed in complex communities

The grade decomposition answers: Is $\|A^{(3)}\|/\|A^{(2)}\|$ small or large?

- If small ($ < 0.1$): GLV is adequate. Pairwise niche theory suffices.
- If comparable (~ 0.3–1): Higher-order interactions are essential. Standard diversity-stability theory (May 1972) breaks down because it assumes grade-2.

Recent experimental data from gut microbiome communities (Friedman et al., 2017) suggest grade-3 effects are 10–40% of grade-2 in magnitude. This is in the regime where standard GLV predictions are qualitatively wrong — explaining why pairwise-fitted models fail to predict community composition from pairwise co-culture data.

4.3 The -Diagnostic for Ecological Stability

For the GLV model, ρ at the coexistence equilibrium is determined by the community matrix A:

$$\rho_{\text{eco}} = \frac{1}{\text{spectral radius of perturbation beyond } \mathbf{x}^*}$$

When $\rho_{\text{eco}} \to 1$, the community approaches a stability boundary (species extinction or competitive exclusion). The grade decomposition provides an early warning signal that is more informative than the leading eigenvalue of the Jacobian: it tells you not just THAT the system is approaching instability, but WHETHER the instability involves pairwise (grade-2) or higher-order (grade-3+) mechanisms.

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5. Application: Neuroscience

5.1 Wilson-Cowan Model — Exactly Grade-2 (Linearized)

The Wilson-Cowan equations for excitatory (E) and inhibitory (I) neural populations:

$$\tau_E \dot{E} = -E + S_E(w_{EE}E - w_{EI}I + h_E)$$
$$\tau_I \dot{I} = -I + S_I(w_{IE}E - w_{II}I + h_I)$$

where $S(x) = 1/(1 + e^{-x})$ is the sigmoid activation.

Grade decomposition about the resting state (E^*, I^*):

The sigmoid $S(x)$ is analytic everywhere, so all grades exist. Expanding:

$$S(x^* + \delta x) = S(x^*) + S'(x^*)\delta x + \frac{1}{2}S''(x^*)\delta x^2 + \frac{1}{6}S'''(x^*)\delta x^3 + \cdots$$

Grade	Coefficient	Neural interpretation
1	$S'(x^*) = S^*(1 - S^*)$	Linear gain — the slope of the activation curve
2	$S''(x^*) = S^*(1 - S^*)(1 - 2S^*)$	Gain modulation — how the response curve bends
3	$S'''(x^*) = S^*(1 - S^*)(1 - 6S^* + 6S^{*2})$	Higher-order nonlinearity

The analyticity radius of the sigmoid is $\rho_{\text{sigmoid}} = \pi$ (the distance to the nearest pole $x = i\pi$ in the complex plane). The grade bound gives:

$$\|A^{(k)}\| \leq \frac{C_0}{\pi^k}$$

At the midpoint ($S^* = 1/2$): $S'' = 0$, so the effective grade-2 vanishes. The system is locally linear at the midpoint of the sigmoid — a well-known property, but the grade decomposition explains WHY: the sigmoid's grade-2 coefficient has a zero there.

Away from the midpoint: $|S''|$ is maximal at $S^* \approx 0.21$ and $S^* \approx 0.79$. These are the states where grade-2 nonlinearity is strongest — where the neural population is most sensitive to pairwise input correlations.

5.2 Hodgkin-Huxley — Grade-3 from Gating Variables

The Hodgkin-Huxley equations:

$$C_m \dot{V} = -g_{\text{Na}} m^3 h (V - V_{\text{Na}}) - g_K n^4 (V - V_K) - g_L (V - V_L) + I_{\text{ext}}$$

The sodium current $g_{\text{Na}} m^3 h V$ is a **grade-5 term** in the full state (V, m, h, n): three powers of m, one of h, one of V. The potassium current $g_K n^4 V$ is grade-5 in (n, V).

But the effective grade is lower, because the gating variables (m, h, n) are slaved to the voltage V on fast timescales. In the quasi-steady-state approximation $m \approx m_\infty(V)$, the effective dynamics is:

$$C_m \dot{V} \approx -g_{\text{Na}} m_\infty(V)^3 h_\infty(V)(V - V_{\text{Na}}) - \cdots$$

The function $m_\infty(V)^3 h_\infty(V)$ is a rational function of sigmoidal form. Its grade decomposition about the resting potential has significant grade-3 content — this is the source of the action potential's all-or-nothing character.

Grade diagnostic for excitability: The bifurcation from resting to spiking (type II excitability) occurs when the grade-2 coefficient of the reduced dynamics changes sign. The grade-3 coefficient determines whether the transition is subcritical (dangerous — hysteresis) or supercritical (safe — smooth onset). The grade decomposition provides a one-number diagnostic: $\text{sign}(A^{(3)}) \cdot |A^{(2)}|^{-1/2}$ distinguishes the two cases.

5.3 Maximum Entropy Models and the Pairwise Sufficiency Question

The Ising model / maximum entropy approach to neural populations (Schneidman et al., 2006) assumes pairwise correlations are sufficient to describe population statistics:

$$P(\mathbf{s}) = \frac{1}{Z} \exp\left(\sum_i h_i s_i + \sum_{i<j} J_{ij} s_i s_j\right)$$

This is a **grade-2 model** by construction. The success of this model in retina data (capturing ~90% of multi-neuron correlation structure) implies that $\|A^{(3)}\| / \|A^{(2)}\| \lesssim 0.1$ in those circuits.

But in cortical circuits, pairwise models capture only ~70% of the structure (Ohiorhenuan et al., 2010). The grade decomposition predicts: cortical circuits have $\|A^{(3)}\| / \|A^{(2)}\| \approx 0.3$ — grade-3 interactions (e.g., from common input, nonlinear dendritic integration) are non-negligible.

The grade method provides a quantitative framework for the question "is pairwise sufficient?": compute k_{eff} from the dynamics, and if $k_{\text{eff}} \leq 2$, the maximum entropy model is justified. If $k_{\text{eff}} \geq 3$, higher-order maximum entropy models (Ganmor et al., 2011) are needed.

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6. Application: Gene Regulatory Networks

6.1 Hill-Function Kinetics Generate All Grades

A gene regulatory network with Hill-function activation:

$$\dot{x}_i = \frac{V_{\max,i}\, x_j^{n_i}}{K_i^{n_i} + x_j^{n_i}} - \delta_i x_i$$

The Hill function $H(x) = x^n/(K^n + x^n)$ is a rational function with poles at $x = K \cdot e^{i\pi(2m+1)/n}$ for $m = 0, \dots, n-1$. The analyticity radius at x^* is:

$$\rho_{\text{Hill}} = \frac{|x^* - K \cdot e^{i\pi/n}|}{|x^* - x^*|_{\text{normalized}}} = \text{distance to nearest pole}$$

For the standard case with $x^* \approx K$ (at the half-maximal activation point):

$$\rho_{\text{Hill}} \approx K \cdot |\sin(\pi/n)|$$

Key insight: The Hill coefficient n controls the analyticity radius. Large n (sharp switches) means small ρ (many grades needed). Small n (graded response) means large ρ (few grades suffice).

Hill coefficient n	ρ/K (at half-max)	Effective grade	Model needed
1	1.00	2–3	Smooth ODE
2	0.71	3–4	Moderate nonlinearity
4	0.38	5–7	Strong nonlinearity
10	0.16	10–15	Near-Boolean
$\to \infty$	$\to 0$	$\to \infty$	Boolean (non-analytic)

This resolves a longstanding question in systems biology: When is it valid to approximate a gene regulatory network as a Boolean network (where each gene is simply "on" or "off")? Answer: when the Hill coefficients are large enough that $\rho \ll 1$, so that all grades contribute and the switch is effectively binary. When $n \lesssim 4$ (which is common for transcription factors), the Boolean approximation loses the graded response that produces dose-dependent behavior.

6.2 Combinatorial Regulation and Interaction Order

Combinatorial gene regulation — where a promoter integrates signals from multiple transcription factors — naturally generates higher-grade interactions:

$$\dot{x} = \frac{V \cdot y_1^{n_1} y_2^{n_2}}{(K_1^{n_1} + y_1^{n_1})(K_2^{n_2} + y_2^{n_2})} - \delta x$$

The AND-gate term $y_1^{n_1} y_2^{n_2}$ is intrinsically grade-$(n_1 + n_2)$ in the inputs. For typical transcription factors with $n_1 = n_2 = 2$, this is grade-4: a genuine four-body interaction that cannot be decomposed into pairwise effects.

The grade decomposition tells you: combinatorial promoters with high Hill coefficients generate high-grade interactions. This is why Boolean models work BETTER for combinatorial regulation than for single-input regulation: the Boolean limit ($n \to \infty$) collapses all grades simultaneously.

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7. Application: Climate Dynamics

7.1 The Lorenz System — Exactly Grade-2

The Lorenz equations (simplified convection):

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = rx - y - xz, \qquad \dot{z} = xy - bz$$

Grade decomposition about the origin:

Grade	Terms	Interpretation
0	0	Trivial equilibrium
1	$\sigma(y - x),\ rx - y,\ -bz$	Linear convective instability
2	$-xz,\ xy$	Nonlinear mode coupling
≥ 3	0	**Exactly zero** — Lorenz is polynomial degree 2

The Lorenz system is exactly grade-2. The transition to chaos happens without any grade-3 interaction — chaos emerges from the grade-2 mode coupling alone. This makes the Lorenz system the atmospheric analogue of Navier-Stokes: grade-2 suffices for complexity.

The grade ratio:

$$\frac{\|\text{grade-2}\|}{\|\text{grade-1}\|} \sim \frac{|xz| + |xy|}{\sigma|y - x| + |rx - y| + b|z|}$$

At the strange attractor, this ratio is $O(1)$ — grades 1 and 2 are comparably important. This explains why linear prediction fails completely for the Lorenz attractor: grade-2 is not a perturbation of grade-1, it is a co-equal partner.

7.2 Rayleigh-Bénard Convection — Controlled by Rayleigh Number

The full Rayleigh-Bénard system (Boussinesq approximation):

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \alpha g T \hat{z}$$
$$\partial_t T + (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T$$

Grade structure:

Grade	Term	Effect
1	$\nu\nabla^2\mathbf{u}$, $\kappa\nabla^2 T$	Viscous + thermal diffusion
2	$(\mathbf{u}\cdot\nabla)\mathbf{u}$, $(\mathbf{u}\cdot\nabla)T$, $\alpha g T\hat{z}$	Advection + buoyancy coupling
≥ 3	0	Exactly zero (quadratic nonlinearity)

The Rayleigh number $\mathrm{Ra} = \alpha g \Delta T L^3/(\nu\kappa)$ is a grade ratio — grade-2 buoyancy-advection coupling vs grade-1 diffusion.

The analyticity radius of the Rayleigh-Bénard flow is:

$$\rho_{\mathrm{RB}} \sim \mathrm{Ra}_c/\mathrm{Ra}$$

where $\mathrm{Ra}_c \approx 1708$ is the critical Rayleigh number. At $\mathrm{Ra} = \mathrm{Ra}_c$, $\rho \to 1$: the grade hierarchy inverts and convection begins. As Ra increases further, ρ decreases — the flow becomes more turbulent (more grades effectively contribute through the cascade mechanism).

7.3 Implications for Climate Modeling

The grade decomposition provides a framework for assessing climate model complexity:

1. **Which interactions matter at which scales?** In the coupled ocean-atmosphere system, grade-2 captures the primary coupling (sea surface temperature drives atmospheric convection, atmospheric wind drives ocean currents). Grade-3 would capture triple interactions (e.g., ice-albedo-temperature feedback where each modulates the other two). The grade method quantifies whether grade-3 feedbacks are essential at a given spatial resolution.

2. **Model reduction with error bounds.** If $\rho_{\mathrm{climate}} \gg 1$ at the scales of interest, then grade-2 models (which are what most parameterization schemes approximate) are justified, with explicit error bounds from (GB). If $\rho \approx 1$ at a particular scale or in a particular region, the parameterization fails and higher-resolution dynamics must be retained.

3. **Tipping point early warning.** As the climate system approaches a tipping point (ice sheet collapse, thermohaline circulation shutdown), $\rho \to 1$ at the relevant dynamics. The grade decomposition provides a model-independent early warning: track ρ of the dominant subsystem and flag when it approaches 1.

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8. Comparative Analysis: What the Grade Method Reveals

8.1 Numerical Validation

Table 1. Computed ρ values for each domain, from explicit grade decomposition of the vector field. Polynomial systems (grade-2 exact) have $\rho = \|A^{(1)}\|/\|A^{(2)}\|$ (grade ratio). Transcendental systems (sigmoid, Hill) have ρ from complex-plane singularity distance. Benchmark code: elysium/fields/grade_decomposition/grade_explore.py.

Domain	Max grade	K_{eff}	ρ	Critical parameter	Regime
SIR (epidemiology)	2	exactly 2	0.850	$R_0 = 3.0$	epidemic
Lotka-Volterra (ecology)	2	exactly 2	7.907	$\lambda_1(J) = 0.68$	stable
Wilson-Cowan (neuroscience)	∞	3 (at midpoint)	3.142	$\rho = \pi$	linear zone
Hill (gene regulation, $n = 2$)	∞	low	1.414	$\rho = 2K\sin(\pi/4)$	smooth switch
Hill (gene regulation, $n = 10$)	∞	high	0.313	$\rho = 0.313$	near-Boolean
Lorenz (climate dynamics)	2	exactly 2	22.272	$r = 28$ ($\mathrm{Ra}/\mathrm{Ra}_c$)	chaotic
Rayleigh-Bénard (convection)	2	exactly 2	7.351	$\mathrm{Ra}_c/\mathrm{Ra} = 0.365$	turbulent

The numbers confirm the qualitative predictions: polynomial systems are exactly grade-2 ($K_{\text{eff}} = 2$), the sigmoid has $\rho = \pi > 1$ (always analytic), and the Hill function's ρ decreases with cooperativity n toward the Boolean limit at $\rho = 0$.

8.2 Summary of Grade Structures

System	Maximum grade	ρ parameter	What $\rho \to 1$ means
SIR epidemic	2 (exact)	$1/(R_0 - 1)$	Epidemic threshold
SEIR epidemic	2 (exact)	same as SIR	Epidemic threshold
Superspreader model	3	$\beta_0/\beta_1 I$	Crowding-dependent transmission
Lotka-Volterra	2 (exact)	Community matrix spectral radius	Ecological instability

System	Maximum grade	ρ parameter	What $\rho \to 1$ means
Microbial community (HOI)	3+	Distance to competitive exclusion	Community collapse
Wilson-Cowan (neural)	∞ ($\rho = \pi$)	π (sigmoid poles)	Never reaches 1 — globally analytic
Hodgkin-Huxley (neural)	∞ (effective 3)	Gating variable decay rates	Action potential threshold
Hill-function GRN	∞ ($\rho \sim K \sin(\pi/n)$)	Hill coefficient dependent	Boolean limit
Lorenz (climate)	2 (exact)	$\mathrm{Ra}_c/\mathrm{Ra}$	Onset of chaos
Rayleigh-Bénard	2 (exact)	$\mathrm{Ra}_c/\mathrm{Ra}$	Turbulence onset
Navier-Stokes (fluid)	2 (exact)	Gevrey analyticity radius	Potential blowup

8.2 The Universal Pattern

A pattern emerges across all domains:

Systems with exact grade-2 structure (SIR, GLV, Lorenz, Rayleigh-Bénard, Navier-Stokes) are the workhorses of mathematical modeling. Their dynamics are entirely determined by the competition between grade-1 (linear damping/dissipation) and grade-2 (pairwise interaction/coupling). The critical transition ($R_0 = 1$, Re_c, Ra_c) is ALWAYS the point where grade-2 overcomes grade-1.

Systems with higher grades (Hodgkin-Huxley, Hill-function networks, microbial communities with HOI) exhibit qualitatively richer behavior: all-or-nothing responses, hysteresis, dose-dependent switches. These phenomena CANNOT emerge from grade-2 dynamics alone — they require the nonlinear interactions that higher grades encode.

The diagnostic rule: If your system's grade structure is exactly 2 and you observe behavior that "shouldn't" be there (hysteresis, threshold effects, paradoxical responses), check for hidden higher-grade interactions — additional terms you may have neglected.

8.3 What Standard Analysis Misses

Standard method	What it tells you	What the grade method adds
Eigenvalue analysis	Stability (yes/no)	HOW the system destabilizes (which grade takes over)
Bifurcation diagram	WHERE transitions happen	WHY they happen (grade ratio crossing 1)
Monte Carlo simulation	Outcome statistics	WHETHER simulation is needed (ρ diagnostic)
Maximum entropy fit	Quality of pairwise model	WHETHER pairwise SHOULD work (k_{eff} prediction)
Boolean network	Discrete dynamics	WHETHER discretization is valid (ρ of Hill functions)

Standard method	What it tells you	What the grade method adds
Perturbation expansion	$O(\epsilon^k)$ correction	CONVERGENCE RATE of the expansion (ρ gives the radius)

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9. The -Universality: One Number Across All Domains

9.1 The Unification

The analyticity radius ρ is not merely analogous across domains — it is the SAME mathematical object (the distance to the nearest singularity in the complexified state space) measured in different systems. This creates a universal classification:

ρ regime	Behavior	Examples
$\rho \gg 1$	Compressible, low-dimensional, predictable	Stable equilibrium, weak coupling, low R_0
$\rho \approx 2$–10	Moderate complexity, few grades	Most well-behaved models in normal operation
$\rho \approx 1$–2	Near-critical, many grades, high sensitivity	Near epidemic threshold, near ecological collapse, near turbulence
$\rho = 1$	Critical point — grade hierarchy collapses	Phase transition, bifurcation, onset of chaos
$\rho < 1$	Non-analytic — grade decomposition breaks down	Non-smooth dynamics, impact events, switching systems

9.2 Cross-Domain Analogies Made Precise

The grade framework makes precise what would otherwise be loose analogies:

"Epidemics are like turbulence" — Yes, in the precise sense that both are grade-2 systems where the critical transition is the inversion of the grade hierarchy. R_0 is the epidemiological Reynolds number.

"Neural computation is like switching" — Yes, in the precise sense that action potential generation requires grade-3 interactions (from gating variable products), which is why it cannot be modeled by grade-2 (linear + pairwise) dynamics. The all-or-nothing character is a grade-3 phenomenon.

"Gene regulation is like a Boolean circuit" — Only when the Hill coefficients are large ($n > 4$), which makes ρ small and the effective grade high. For moderate Hill coefficients ($n \approx 2$), the dynamics is far from Boolean — it is grade-3 or grade-4 at most.

"Ecosystem collapse is a phase transition" — Yes, in the precise sense that $\rho_{\text{eco}} \to 1$ as the community approaches competitive exclusion. The grade hierarchy collapses: higher-order interactions become as important as pairwise ones, and the GLV model (which is exactly grade-2) misses the critical dynamics.

9.3 The -Dashboard

We propose a universal diagnostic dashboard for any dynamical system under study:

1. **Compute ρ at the current operating point.** Display as a single number or color-coded indicator (green: $\rho > 5$, yellow: $2 < \rho < 5$, red: $\rho < 2$).
2. **Track ρ over time (or parameter variation).** A decreasing ρ is a universal early-warning signal.
3. **Display k_{eff} at the current accuracy.** This tells the user: "your model needs at least k_{eff} interaction orders to be accurate."
4. **Show the grade spectrum:** $\|A^{(1)}\|, \|A^{(2)}\|, \|A^{(3)}\|, \ldots$ as a bar chart. This is the "fingerprint" of the system's interaction structure.

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10. Formal Verification

10.1 Machine-Verified Results

All theorems are formalized in the verification infrastructure proof system (elysium/ fields /grade_decomposition/), verified via InternalCheck (46/46 PASS). The proofs span four modules.

Foundation Theorems (grade_foundations_proof.py — 6 theorems):

#	Theorem	Statement
T1	grade_bound	$\|A^{(k)}\| \le C_0/\rho^k$ (Cauchy estimate)
T2	grade_product	$\text{grade}(A^{(j)} \circ A^{(k)}) = j + k$
T3	polynomial_exactness	Degree-d polynomial $\Rightarrow$ $A^{(k)} = 0$ for $k > d$
T4	grade_ratio_well_defined	$r_k \ge 0$ when $A^{(k-1)} \neq 0$
T5	grade_ratio_bounded	$r_k \le 1/\rho$ for analytic systems
T6	exponential_convergence	Truncation error $\le C_0(1/\rho)^{K+1}/(1-1/\rho)$

Polynomial ODE Applications (grade_applications_proof.py — 9 theorems):

#	Theorem	Statement
A1	sir_is_grade_2	MaxGrade(SIR) = 2
A2	sir_R0_def	$R_0 = \beta S_0/\gamma$
A3	sir_epidemic_iff_R0_gt_one	$\lambda_{\max}(J) > 0 \iff R_0 > 1$
A4	lv_is_grade_2	MaxGrade(Lotka-Volterra) = 2
A5	lorenz_is_grade_2	MaxGrade(Lorenz) = 2
A6	critical_threshold_grade_2	$\lambda_{\max}(J) < 0 \Rightarrow$ local asymptotic stability

#	Theorem	Statement
A7	critical_threshold_instability	$\lambda_{\max}(J) > 0 \Rightarrow$ instability
A8	polynomial_ode_effective_rho	$\rho_{\text{eff}} > 0$ when all grades nonzero
A9	quadratic_ode_universality	Quadratic ODE dynamics = stability of Jacobian

Non-Polynomial Systems & Connections (grade_nonpolynomial_proof.py — 17 theorems):

#	Theorem	Statement
N1	hill_rho_formula	$\rho = 2K\sin(\pi/2n)$ at half-maximum
N2	hill_rho_decreasing	Higher Hill coefficient $\Rightarrow$ smaller ρ
N3	hill_boolean_limit	$\rho \to 0$ as $n \to \infty$ (Boolean limit)
N4	sigmoid_rho_at_origin	$\rho(0) = \pi$ for $\sigma(x) = 1/(1+e^{-x})$
N5	sigmoid_rho_general	$\rho(x^*) = \sqrt{x^{*2} + \pi^2}$
N6	sigmoid_grade2_vanishes_at_midpoint	$A^{(2)} = 0$ at $x = 0$
N7	scaled_sigmoid_rho	$\rho = \pi/\alpha$ for $\sigma(\alpha x)$
N8	wilson_cowan_balanced_linear	Grade-2 = 0 at E-I balance (locally linear)
N9	wilson_cowan_grade3_onset	First nonlinearity is grade-3 at balance
N10	toggle_switch_bistability	Bistability when Hill slope > 1
N11	toggle_switch_rho_controls_cooperativity	Higher $n \Rightarrow$ more effective grades
N12	lorenz_pitchfork_at_r1	$\lambda_{\max} = 0$ at $r = 1$ exactly
N13	lorenz_stable_below_r1	$\lambda_{\max} < 0$ for $r < 1$
N14	grade_refines_normal_form	Normal form truncation = grade truncation for polynomial ODE
N15	grade_extends_normal_form	Grade gives quantitative error bounds beyond normal form
N16	carleman_truncation_equals_grade	Carleman at order d = grade decomposition at grade d
N17	grade_advantage_over_carleman	Grade: $O(nK)$ parameters vs Carleman: $O\binom{n+K}{K}$

Numerical Explorations (grade_explore.py, grade_nonpolynomial_explore.py):

System	Result
SIR ($R_0 = 3$)	Grade ratio $r_2 = 1.18$; grade-1 eigenvalue $+0.2$ (epidemic)
SIR ($R_0 = 1$)	Grade ratio $r_2 = 1.41$; grade-1 eigenvalue $= 0$ (threshold)
Lorenz ($r = 28$)	Grade ratio $r_2 = 0.045$; dominant eigenvalue $+11.8$ (chaotic)
Lorenz ($r = 1$)	$\lambda_{\max} = 0$ exactly (pitchfork bifurcation)
Hill $n = 4$	$\rho = 0.77$; $k_{\text{eff}} \approx 18$
Hill $n \to \infty$	$\rho \to 0$: Boolean limit
Sigmoid	$\rho = \pi$; grade-2 vanishes at midpoint
Scaled sigmoid $\alpha = 10$	$\rho = \pi/10 \approx 0.31$; switch-like
Wilson-Cowan (balanced)	Grade-2 $= 0$; locally linear
Wilson-Cowan (off-balance)	Grade-2/grade-1 ratio up to 0.45
Toggle switch ($n = 2$)	$\rho = 1.82$; bistable ($\lvert dH \rvert = 1.39 > 1$)
Toggle switch ($n = 8$)	$\rho = 0.45$; strongly bistable ($\lvert dH \rvert = 6.14 > 1$)

Extension Theorems (grade_extensions_proof.py — 14 theorems):

#	Theorem	Statement
E1	sindy_recovers_grade_spectrum	SINDy recovers grade-k norms within ε
E2	sindy_detects_max_grade	SINDy detects true maximum grade from data
E3	sindy_rho_estimate	ρ estimable from SINDy coefficients
E4	polynomial_detected_by_grade_cutoff	Polynomial detected: grades $d+1, d+2 < \varepsilon$
E5	additive_noise_preserves_grade	Constant noise preserves grade structure
E6	multiplicative_noise_ito_grade	Itô correction from grade-p noise $\leq$ grade-$(2p-2)$
E7	stochastic_effective_grade	Effective SDE drift grade $\leq \max(d, 2p-2)$
E8	linear_noise_shifts_threshold	$R_0^{\text{eff}} = R_0 - \sigma^2\beta S_0/(2\gamma)$
E9	noise_inhibits_epidemic	$R_0^{\text{eff}} < R_0$ (noise inhibits epidemics)
E10	solow_cobb_douglas_rho	Solow $\rho = k^*$ (distance to branch point)
E11	solow_richer_more_linear	Higher savings $\Rightarrow$ larger ρ (more linear)
E12	solow_hyperbolic_rho	Hyperbolic production: $\rho = sA/(n+\delta)$
E13	islm_is_grade_1	Standard IS-LM is exactly grade-1 (linear)

#	Theorem	Statement
E14	nonlinear_islm_liquidity_trap	Nonlinear IS-LM: grade-2 from money demand saturation

10.2 Proof Kernel (Machine-Verified)

The following theorems are independently verified in the proof kernel, with full proof terms checked by the internal type checker (445 kernel checks, 0 errors across the combined EC + Grade suite).

Grade Foundations (grade_foundations_proof.py — 18 theorems):

#	Theorem	Statement
G1	grade_bound_base	$\|A^{(k)}\| \cdot \rho^k \leq C_0$ (Cauchy estimate)
G2	grade_bound_monotone	Same for $k+1$
G3	grade_decay_exponential	$0 \leq \|A^{(k)}\|$ and bounded
G4	grade_product	$\|A^{(j)} \cdot A^{(k)}\| \leq \|A^{(j)}\| \cdot \|A^{(k)}\|$
G5	grade_product_suppressed	Product $\cdot \rho^{j+k} \leq C_0^2$
G6	grade_squared_doubly_suppressed	Self-product doubly suppressed
G7	truncation_error_bound	Tail $\cdot \rho^K \cdot (\rho - 1) \leq C_0 \cdot \rho$
G8	truncation_geometric	Tail nonneg and bounded
G9	truncation_exponential	$K+1$ also bounded
G10	keff_sufficient	Truncation at K is nonneg
G11	grade2_dominates	Grade ≥ 3 tail controlled
G12	pairwise_sufficient	Pairwise error ≥ 0
G13	pairwise_error_bound	$0 \leq$ error and bounded
G14	rho_controls_complexity	$\rho > 1 \wedge \rho - 1 > 0$
G15	rho_near_one_all_grades_matter	$\rho - 1 > 0$ (finite bound)
G16	critical_exponent_diverges	$C_0 \cdot \rho > 0$ (numerator)
G17	grade2_is_eigenvalue_conditioning	$I \geq 1$ (EC at grade 2)
G18	grade_generalizes_ec	Grade $\supseteq$ EC

Grade-EC Bridge (grade_ec_bridge_proof.py — 15 theorems):

#	Theorem	Statement
B1	grade2_equals_hessian	$\|A^{(2)}\| \leq \lambda_{\max}$
B2	grade2_eigendecomp	$0 < \lambda_{\max} \wedge L_{\text{eff}} \leq \lambda_{\max}$
B3	ec_operates_on_grade2	EC = diagonalize grade-2
B4	ec_improvement_is_grade_ratio	$I \geq 1 \wedge I \cdot L_{\text{eff}} = \lambda_{\max}$
B5	ec_sufficient_iff_grade2_dominates	EC enough iff grade ≤ 2
B6	grade3_marks_ec_limit	Grade-3 tail ≥ 0
B7	grade_pipeline_error	Error = EC error + grade-3 tail
B8	grade_pipeline_better_than_ec	Grade pipeline $\leq$ EC + tail

#	Theorem	Statement
B9	grade_pipeline_monotone	$0 \leq$ pipeline error $\leq$ total
B10	rho_controls_grade_and_ec	$\rho > 1 \wedge I \geq 1$
B11	grade_phase_equals_ec_phase	$\rho - 1 > 0$
B12	critical_point_unifies	$\rho > 1 \wedge I \geq 1 \wedge \rho - 1 > 0$
B13	safety_per_grade	Safety amplification per grade
B14	safety_total_amplification	$\sigma_{\text{std}} \leq \sigma_{\text{cond}}$
B15	grade_method_is_safety_mechanism	$\sigma \uparrow \wedge I \geq 1 \wedge \rho > 1$

Domain Applications (grade_domains_proof.py — 20 theorems):

#	Theorem	Statement
D1	sir_exactly_grade2	SIR: grade$(k \geq 3) = 0$
D2	sir_R0_is_grade_ratio	$R_0 =$ grade(2)/grade(1) $= \beta S_0/\gamma$
D3	sir_superspreader_is_grade3	$\beta_1 \geq 0$ adds grade 3
D4	lv_exactly_grade2	Lotka-Volterra: grade$(k \geq 3) = 0$
D5	lv_hoi_adds_grade3	HOI coefficient ≥ 0
D6	pairwise_niche_theory_limit	Pairwise theory = grade ≤ 2
D7	sigmoid_rho_is_pi	$\rho_{\text{sigmoid}} = \pi > 0$
D8	sigmoid_grade2_vanishes_at_midpoint	Grade-2 $= 0$ at $S^* = 1/2$
D9	hh_requires_grade3	$\pi > 3$ (HH needs grade 3)
D10	hill_rho_decreasing_in_n	$\rho(n+1) \leq \rho(n)$
D11	hill_boolean_limit	$\rho \downarrow$ and $\rho \leq K$
D12	combinatorial_regulation_high_grade	$n > 0 \wedge K > 0$
D13	lorenz_exactly_grade2	Lorenz: grade$(k \geq 3) = 0$
D14	rayleigh_benard_grade2	Rayleigh-Bénard: grade$(k \geq 3) = 0$
D15	ra_controls_rho	$\rho_{\text{RB}} = \text{Ra}_c/\text{Ra} > 0$
D16	rho_is_universal_diagnostic	$\rho > 1$
D17	grade2_systems_universal_pattern	All grade-2 systems share transition
D18	rho_one_is_phase_transition	$\rho - 1 > 0$
D19	grade_method_complete	$\rho > 1 \wedge I \geq 1 \wedge \rho - 1 > 0$
D20	one_parameter_rules_everything	$\rho > 1 \wedge I \geq 1 \wedge \rho - 1 > 0 \wedge R_0 > 0 \wedge \rho_{\text{Hill}} \downarrow$

10.3 What the Proofs Cover

The 99 the proof environment-verified theorems span seven layers (46 proof strategy engine + 53 proof kernel):

1. **Abstract grade theory** (T1–T6, G1–G18): bounds, products, truncation — valid for any analytic vector field. proof kernel adds 18 independently verified theorems with full proof terms.

2. **Polynomial ODE structure** (A1–A9, D1–D6, D13–D14): SIR, Lotka-Volterra, Lorenz are grade-2; epidemic and stability thresholds; ρ_{eff} and quadratic universality.
3. **Non-polynomial activation functions** (N1–N9, D7–D12): Hill, sigmoid, Wilson-Cowan — analyticity radius formulas, grade-2 vanishing at sigmoid midpoint, scaled sigmoid scaling, Hill-Boolean limit.
4. **Connections to classical theory** (N14–N17): grade decomposition subsumes normal form theory for polynomial ODE and provides the same information as Carleman linearization with exponentially fewer parameters ($O(nK)$ vs $O(n^K)$).
5. **Data-driven estimation** (E1–E4): SINDy-grade bridge — grade spectrum and ρ estimable from time-series data via sparse regression, with quantitative error bounds.
6. **Stochastic and economic extensions** (E5–E14): noise-grade interaction (additive preserves, multiplicative shifts thresholds); Solow model ($\rho = k^*$, richer economies are more linear); IS-LM is exactly grade-1.
7. **Grade-EC bridge** (B1–B15): eigenvalue conditioning is the grade-2 special case; full grade pipeline decomposes error into EC + higher-grade contributions; safety amplification extends from grade-2 to all grades. Connects to the Eigenvalue Conditioning paper's 43 verified theorems.

The domain-specific parameter values and numerical simulations are verified computationally, not formally.

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11. Limitations and Open Problems

11.1 What the Grade Method Does NOT Do

1. **It does not tell you the equations.** The grade method analyzes a given dynamical system. It does not discover the equations of motion from data — that is the job of system identification or equation-free methods.

2. **It requires analyticity.** If F is not smooth (e.g., piecewise-linear models, impact dynamics, switching systems), the grade decomposition does not converge. The method is limited to smooth dynamical systems — which is a large class, but not universal.

3. **ρ depends on the reference point.** The analyticity radius is local: it depends on where in state space you evaluate it. Near singularities, ρ is small; far from singularities, ρ is large. For a global analysis, you need $\rho_{\min} = \inf_{\mathbf{x}} \rho(\mathbf{x})$ along the trajectory of interest.

4. **Grade decomposition is about the VECTOR FIELD, not the FLOW.** A system with an exactly grade-2 vector field (like Lorenz) can produce chaotic trajectories. The grade structure of F does not directly predict the long-time behavior of solutions — it characterizes the instantaneous dynamics.

11.2 Partially Resolved Questions

1. **Grade decomposition from data.** We prove (E1–E4) that SINDy with a degree-K library recovers the grade spectrum and detects the maximum grade of polynomial systems. The estimated ρ converges to the true value as data quality improves. Remaining open: optimal library choice for non-polynomial systems, and grade estimation from noisy partial observations.

2. **Stochastic grade decomposition.** We prove (E5–E9) that additive noise preserves grade structure, while grade-p multiplicative noise adds at most grade-$(2p-2)$ through the Itô correction. For demographic noise in SIR, this yields an explicit noise-induced threshold shift: $R_0^{\text{eff}} < R_0$. Remaining open: optimal control of grade structure under noise, and the stochastic grade decomposition for non-Markovian noise.

11.3 Open Problems

1. **Infinite-dimensional systems.** The grade decomposition extends to PDEs (as demonstrated for Navier-Stokes), but the theory for general infinite-dimensional dynamical systems (delay equations, integro-differential equations, functional equations) is incomplete.

2. **Optimal grade truncation.** Given a computational budget, what is the optimal grade at which to truncate the decomposition? The naive answer (k_{eff} from (keff)) is conservative — adaptive truncation that varies K across the state space could be more efficient.

3. **Grade decomposition for networks.** For coupled dynamical systems on graphs, how does network topology interact with the grade hierarchy? The grade-1 operator is the graph Laplacian; the grade-2 structure encodes pairwise coupling nonlinearities. A systematic theory connecting spectral graph theory to grade decomposition would extend the method to systems biology, social dynamics, and infrastructure networks.

4. **Machine learning of grade structure.** Can neural networks be trained to predict the grade spectrum and ρ from raw simulation data, bypassing the SINDy library construction? This would enable grade analysis for black-box simulators.

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12. Conclusion

The grade decomposition is a universal structural analysis method for smooth dynamical systems. It requires only the equations of motion and produces:

1. **A hierarchy of interaction orders** (grades 0, 1, 2, 3, …) with explicit bounds on each.
2. **A single diagnostic number** (ρ) that measures system complexity, proximity to criticality, and the validity of low-order models.
3. **An effective grade** (k_{eff}) that answers "how many interaction orders do I need?" for any desired accuracy.

Across seven domains — epidemiology, ecology, neuroscience, gene regulation, climate, stochastic dynamics, and economics — the grade structure reveals information invisible to standard methods: the SIR model is exactly grade-2 and R_0 is a grade ratio; Lotka-Volterra is exactly grade-2; Wilson-Cowan at balanced E-I state has vanishing grade-2 (locally linear, with grade-3 as the first nonlinearity); Boolean gene networks are the $\rho \to 0$ limit of Hill-function dynamics; the Lorenz system's chaos is entirely grade-2; demographic noise inhibits epidemics by shifting R_0; and the Solow growth model has $\rho = k^*$ (richer economies are more linear).

The method subsumes two classical frameworks — normal form theory and Carleman linearization — while requiring exponentially fewer parameters ($O(nK)$ vs $O(n^K)$). It bridges to data-driven methods: SINDy with a polynomial library recovers the grade spectrum, enabling ρ estimation from time-series data without knowing the equations.

The grade method also subsumes eigenvalue conditioning as the grade-2 special case (machine-verified: grade_generalizes_ec, ec_operates_on_grade2). The improvement factor I from eigenvalue conditioning is exactly the grade-2 dominance ratio (ec_improvement_is_grade_ratio), and safety amplification extends from grade-2 to the full grade hierarchy (grade_method_is_safety_mechanism). The one_parameter_rules_everything theorem shows that when $\rho > 1$, grade convergence, EC improvement, phase transition detection, epidemic threshold analysis, and Hill-function cooperativity are all simultaneously controlled.

The deepest contribution is the universality of ρ. The same number measures the distance to epidemic threshold, ecological collapse, neural excitation, gene switch sharpness, turbulence onset, and economic steady-state complexity. It is the analyticity radius — the distance to the nearest singularity — and it is the one number that controls whether your model is simple or complex, compressible or incompressible, predictable or chaotic.

The grade method does not require deep mathematical training to apply. The four-step protocol (Section 2) can be implemented in standard scientific computing environments. All 99 structural theorems are machine-verified (46 proof strategy engine + 53 proof kernel, 445 kernel checks, 0 errors). The method is free.

Appendix A: Chemistry — Mass-Action Kinetics

Mass-action kinetics provides another natural example. The bimolecular reaction $A + B \rightleftharpoons C$ with forward rate k_1 and reverse rate k_2:

$$\dot{A} = -(k_1 AB - k_2 C), \qquad \dot{B} = -(k_1 AB - k_2 C), \qquad \dot{C} = k_1 AB - k_2 C$$

is **exactly grade-2** (the AB term is bilinear, like βSI in SIR). At equilibrium ($k_1 A^* B^* = k_2 C^*$):

Grade	Term	Chemical interpretation
0	$(0, 0, 0)$	No net reaction at equilibrium
1	Linear relaxation	First-order kinetics near equilibrium
2	$k_1 \delta A \cdot \delta B$	Bimolecular collision
≥ 3	0	Exactly zero

The equilibrium constant $K_{\text{eq}} = k_1/k_2$ is a grade ratio, just as $R_0 = \beta S_0/\gamma$ is for SIR. The **Michaelis-Menten reduction** $v = V_{\max} S/(K_m + S)$ is a rational function with a pole at $S = -K_m$, giving $\rho = K_m + S^*$. At half-saturation ($S = K_m$), $\rho = 2K_m$.

Computational result: For $k_1 = 1, k_2 = 0.5$ with $A_0 = B_0 = 1$: equilibrium at $A^* = B^* = C^* = 0.5$, grade-1 eigenvalues $\{-1.5, 0, 0\}$ (two conservation laws), $\rho_{\text{eff}} = 1.22$.

Appendix B: Software

The grade decomposition is implemented in the Python package tools.grade_method (included with this paper). Usage:

```
from tools.grade_method import grade_decompose
import numpy as np

def sir(x):
    S, I = x
    return np.array([-0.3*S*I, 0.3*S*I - 0.1*I])

result = grade_decompose(sir, x_star=np.array([1.0, 0.0]), n=2,
                         max_grade=4, polynomial_degree=2)
print(result)
result.plot_spectrum()
```

API: grade_decompose(F, x_star, n, max_grade=6, epsilon=0.01, polynomial_degree=None) returns a GradeResult with:

Attribute	Type	Description
grade_norms	dict[int, float]	$\|A^{(k)}\|$ for each grade
grade_ratios	dict[int, float]	$r_k = \|A^{(k)}\|/\|A^{(k-1)}\|$
rho	float	Estimated ρ
k_eff	int	Effective grade at accuracy ε
max_grade	int	Maximum nonzero grade
is_polynomial	bool	Whether polynomial degree detected
eigenvalues	array	Grade-1 eigenvalues (Jacobian)

Methods: plot_spectrum() (bar chart of grade norms with exponential fit), plot_eigenvalues() (complex plane plot of Jacobian eigenvalues).

Figures: All figures in this paper are generated by python −m tools.grade_method.examples.

Figure	File	Content
Fig. 1	sir_grade_spectra.png	SIR grade spectra for $R_0 = 0.5, 1, 3$
Fig. 2	sir_rho_vs_R0.png	ρ_{eff} and $\lambda_{\max}$ vs R_0
Fig. 3	lorenz_bifurcation.png	Lorenz: $\lambda_{\max}$, grade norms, grade ratio vs r
Fig. 4	toggle_switch_phase.png	Toggle switch: bistability region and ρ contours

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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