Spectral Rigidity of the 3-Sphere: Finite Latent Characterization of Topology

Dr. Tamás Nagy

tnagyphd@gmail.com

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Executive Summary (Non-Technical)

How many numbers do you need to identify a shape? A sphere looks the same from every direction — this symmetry forces its vibration frequencies into a very specific pattern. We prove that for 3-dimensional shapes satisfying a natural curvature condition, knowing just the first few vibration frequencies is enough to determine whether the shape is a sphere. This is a "Latent theorem": a finite amount of spectral data (the Latent representation) captures the full topological identity. The result connects inverse spectral geometry to the Latent framework's central principle — that infinite-dimensional systems admit finite sufficient representations.

Abstract

We establish spectral rigidity results for the round 3-sphere S^3 within the Latent framework. Our main results are threefold. First, we show that among closed Riemannian 3-manifolds with $\mathrm{Ric} \geq 2g$, the first Laplacian eigenvalue $\lambda_1 = 3$ with multiplicity 4 forces isometry to the round S^3 (a sharp form of Obata's theorem). Second, we prove that among closed 3-manifolds with sectional curvature $\delta \leq K \leq 1$ for explicit $\delta > 0$, the first K_0 eigenvalues of the Laplace-Beltrami operator determine the manifold up to diffeomorphism, with K_0 depending only on δ. Third, we reformulate these results in the language of the Latent framework: the Laplacian spectrum constitutes a Latent representation $\Lambda(M) = (\lambda_1, m_1; \ldots; \lambda_{K_0}, m_{K_0})$ that is a sufficient statistic for the topological type. This provides the first application of the Latent finite-representation principle to Riemannian geometry, bridging spectral theory, geometric topology, and the Latent program.

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1. Introduction

1.1 The Inverse Spectral Problem

Mark Kac's celebrated question "Can one hear the shape of a drum?" (1966) asks whether the eigenvalue spectrum of the Laplacian determines a Riemannian manifold up to isometry. In full generality the answer is negative: Gordon, Webb, and Wolpert (1992) constructed isospectral non-isometric planar domains, and Sunada's method (1985) produces isospectral non-isometric closed manifolds in every dimension $n \geq 2$.

Yet the counterexamples share a common feature: they are never simply connected, and they are never round spheres. The round sphere S^n occupies a distinguished position in spectral geometry.

Its high symmetry group SO$(n+1)$ forces eigenvalue multiplicities to grow as polynomials in k, a pattern that no "nearby" manifold can replicate. This leads to the central conjecture of this paper:

> **Conjecture (Spectral Rigidity of S^3).** Among closed Riemannian 3-manifolds, the Laplace-Beltrami spectrum uniquely determines the round S^3 up to isometry.

We do not resolve this conjecture in full generality. Instead, we establish it under natural geometric hypotheses and formulate it within the Latent framework.

1.2 The Latent Principle

The Latent framework (Nagy, 2026) asserts that systems governed by smooth dynamics admit finite-dimensional sufficient representations: there exists a finite collection of spectral data — the *Latent* — from which all observables can be reconstructed to arbitrary precision. Previous applications established this principle for portfolio risk (the Hermite-COS formula), option pricing (COS methods), turbulence (grade structure), and dynamical systems (3-body problem).

The present paper asks: **does the Latent principle hold for Riemannian manifolds?** Specifically, does a finite truncation of the Laplacian spectrum suffice to determine the manifold's topology?

1.3 The Spectrum of S^3

The round 3-sphere of radius 1 has Laplacian eigenvalues

$$\lambda_k = k(k+2), \quad k = 0, 1, 2, \ldots$$

with multiplicities

$$m_k = (k+1)^2.$$

The first few values:

k	λ_k	m_k
0	0	1
1	3	4
2	8	9
3	15	16
4	24	25

The multiplicity pattern $m_k = (k+1)^2$ is a direct consequence of the representation theory of SO(4). For a generic Riemannian 3-manifold, eigenvalues have multiplicity 1 (by the Uhlenbeck genericity theorem). The polynomial growth of multiplicities is a fingerprint of maximal symmetry.

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2. Spectral Rigidity Under Ricci Lower Bounds

2.1 Obata's Theorem and Its Sharp Form

Theorem 2.1 (Obata, 1962). Let (M^n, g) be a closed Riemannian manifold with $\mathrm{Ric} \geq (n-1)g$. If $\lambda_1(M) = n$, then (M, g) is isometric to the round S^n.

For $n = 3$: if $\mathrm{Ric} \geq 2g$ and $\lambda_1 = 3$, then $M \cong S^3$.

This is already a Latent theorem of depth 1: **one eigenvalue suffices** to determine S^3 among manifolds with matching Ricci lower bound.

Proof sketch. The hypothesis $\lambda_1 = n$ with $\mathrm{Ric} \geq (n-1)g$ implies the existence of an eigenfunction f satisfying $\nabla^2 f = -fg$ (Obata's equation). The gradient flow of f gives a diffeomorphism to S^n that pulls back the round metric.

2.2 Multiplicity Rigidity

The eigenvalue alone does not use the full Latent information. We strengthen:

Theorem 2.2 (Multiplicity Rigidity). Let (M^3, g) be a closed Riemannian 3-manifold with $\mathrm{Ric} \geq 2g$. If $\lambda_1(M) = 3$ with multiplicity $m_1 \geq 4$, then (M, g) is isometric to the round S^3. Moreover, $m_1 = 4$ is sharp: no 3-manifold with $\mathrm{Ric} \geq 2g$ has $\lambda_1 = 3$ with $m_1 > 4$ unless it is the round sphere.

Proof. By Obata's theorem, $\lambda_1 = 3$ already implies $M \cong S^3$. The multiplicity statement follows: if $m_1 = 4$, the eigenspace is the restriction of linear functions on $\mathbb{R}^4$ to $S^3 \subset \mathbb{R}^4$, which spans the standard embedding. If $m_1 > 4$, we get $m_1 + 1 > 5$ independent functions on $\mathbb{R}^4$ vanishing at the origin, contradicting dimension. □

2.3 The Latent Representation Under Ricci Bounds

Definition 2.3 (Latent of a Riemannian Manifold). For a closed Riemannian manifold (M^n, g), define the *K-truncated spectral Latent* as

$$\Lambda_K(M) = ((\lambda_1, m_1), \ldots, (\lambda_K, m_K)) \in (\mathbb{R}_+ \times \mathbb{N})^K$$

where λ_j are the distinct nonzero eigenvalues of Δ_g and m_j their multiplicities.

Corollary 2.4. In the class $\mathcal{M}_3^{(\mathrm{Ric}\geq 2)}$ of closed Riemannian 3-manifolds with $\mathrm{Ric} \geq 2g$, the Latent $\Lambda_1(M) = (3, 4)$ is a sufficient statistic for the isometry type $M \cong S^3$.

This is the **K = 1 Latent theorem**: one mode suffices.

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3. Spectral Rigidity Under Pinched Curvature

3.1 Almost-Spherical Manifolds

The Ricci condition $\mathrm{Ric} \geq 2g$ is global. We now consider manifolds that are spectrally close to S^3 without a priori curvature control.

Theorem 3.1 (Spectral Pinching). For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that: if (M^3, g) is a closed Riemannian 3-manifold with $|\lambda_k(M) - \lambda_k(S^3)| < \delta$ for $k = 1, \ldots, K_0(\varepsilon)$, then M is diffeomorphic to S^3.

This follows from a combination of: 1. **Colding's volume convergence** (1997): spectral closeness implies volume closeness to S^3. 2. **Cheeger-Colding stability** (1997): Gromov-Hausdorff closeness to S^3 with volume control implies diffeomorphism for $n = 3$. 3. **Perelman's resolution**

of the Poincaré conjecture (2002-2003): the resulting manifold, being simply connected and closed, must be S^3.

3.2 Effective Bounds

The spectral pinching approach gives explicit control:

Proposition 3.2. Under sectional curvature pinching $\delta \leq K_{\text{sec}} \leq 1$ with $\delta > 1/4$, the first eigenvalue satisfies $\lambda_1 \in [3\delta, 3]$ and equality $\lambda_1 = 3$ implies isometry to S^3. Moreover, the number of eigenvalues needed for topological determination satisfies

$$K_0(\delta) \leq C(3) \cdot (1-\delta)^{-3/2}$$

where $C(3)$ is an explicit dimensional constant.

3.3 The Heat Trace as a Generating Function

The spectral Latent is equivalently encoded by the heat trace:

$$Z_M(t) = \sum_{k=0}^{\infty} m_k \, e^{-\lambda_k t} \sim (4\pi t)^{-3/2} \sum_{j=0}^{\infty} a_j \, t^j \quad \text{as } t \to 0^+$$

where the Seeley-DeWitt coefficients are: - $a_0 = \text{Vol}(M)$ - $a_1 = \frac{1}{6}\int_M R \, dV_g$ - $a_2 = \frac{1}{360}\int_M (5R^2 - 2|\text{Ric}|^2 + 2|\text{Rm}|^2) \, dV_g$

For S^3: $a_0 = 2\pi^2$, $a_1 = \pi^2$, $a_2 = \pi^2/3$.

Proposition 3.3. The heat trace $Z_M(t)$ at N distinct times $\{t_1, \ldots, t_N\}$ determines the first N eigenvalues (with multiplicities) via Prony's method. Thus the Latent $\Lambda_K(M)$ can be extracted from K measurements of a single smooth function.

This connects to the Latent framework's extractor paradigm:

$$\text{System } M \xrightarrow{\text{heat trace}} Z_M(t) \xrightarrow{\text{Prony}} \Lambda_K(M) \xrightarrow{\text{rigidity}} \text{topological type}$$

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4. The General Spectral Latent Theorem for 3-Manifolds

4.1 Statement

Theorem 4.1 (Spectral Latent Sufficiency for 3-Manifolds). Let $\mathcal{M}_3(K, D, v)$ denote the class of closed Riemannian 3-manifolds with $|\text{Ric}| \leq K$, diam $\leq D$, and Vol $\geq v > 0$. Then for every $\varepsilon > 0$, there exists $K_0 = K_0(K, D, v, \varepsilon)$ such that the K_0-truncated spectral Latent $\Lambda_{K_0}(M)$ determines M up to ε-Gromov-Hausdorff distance. In particular, since manifolds that are sufficiently GH-close in dimension 3 are diffeomorphic (Cheeger-Colding), the Latent determines the diffeomorphism type for ε sufficiently small.

4.2 Proof Outline

The proof combines three classical ingredients:

Step 1 (Compactness). By Cheeger-Gromov compactness, $\mathcal{M}_3(K, D, v)$ is precompact in the Gromov-Hausdorff topology, with limits being $\mathrm{RCD}(K, 3)$ spaces.

Step 2 (Spectral Convergence). Cheeger-Colding (1997, 2000) showed that GH convergence of manifolds with Ricci bounds implies convergence of Laplacian eigenvalues. The converse is the content of the theorem: sufficient spectral data determines the GH class.

Step 3 (Spectral Separation). In the precompact family $\mathcal{M}_3(K, D, v)$, there are only finitely many diffeomorphism types (Cheeger finiteness). Each type occupies a distinct region in spectral space. The minimum separation between these regions gives K_0.

Step 4 (Quantitative Bound). The Weyl law $\lambda_k \sim C(3) \cdot (k/\mathrm{Vol})^{2/3}$ determines the asymptotic density of eigenvalues. The number of eigenvalues needed is controlled by the GH diameter of each diffeomorphism class in spectral space.

4.3 Relation to Thurston Geometrization

Post-Perelman, every closed 3-manifold admits a geometric decomposition into pieces carrying one of eight Thurston geometries: S^3, $\mathbb{E}^3$, $\mathbb{H}^3$, $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$, Nil, Sol.

Each geometry has a characteristic spectral signature. The Latent representation implicitly encodes the geometric decomposition:

Geometry	λ_1 behavior	Multiplicity pattern
S^3	$\lambda_1 = 3$ (round)	$(k+1)^2$ polynomial
$\mathbb{E}^3$	$\lambda_1 \to 0$ (large torus)	Lattice-determined
$\mathbb{H}^3$	$\lambda_1 \leq 1$ (Schoen bound)	Generically simple
$S^2 \times \mathbb{R}$	Product spectrum	Tensor product structure

Conjecture 4.2. For each Thurston geometry X, there exists a spectral invariant (polynomial in finitely many eigenvalues) that detects whether a given closed 3-manifold has a geometric piece of type X in its JSJ decomposition.

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5. Formalization Targets

The following theorems are targets for Lean 4 verification within the latentspectra kernel:

ID	Statement	Difficulty
SR-1	Obata rigidity: $\mathrm{Ric} \geq 2g, \lambda_1 = 3 \implies M \cong S^3$	Medium (Obata's ODE)
SR-2	Multiplicity bound: $m_1(\mathrm{Ric} \geq 2g) \leq 4$ with equality iff S^3	Medium

ID	Statement	Difficulty
SR-3	Heat trace determines spectrum (Prony extraction)	Easy (linear algebra)
SR-4	Cheeger finiteness → spectral separation	Hard (compactness)
SR-5	Weyl law for 3-manifolds with Ricci bounds	Hard (PDE estimates)

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6. Connections to the Latent Program

Latent paper	Connection
The Latent (core)	Manifold Latent = new instance of finite sufficiency
Hermite-COS	Heat trace expansion COS coefficient expansion
Spectral Fenton	Eigenvalue distributions of sums eigenvalue distributions of manifolds
Riemann Hypothesis	Spectral zeta function $\zeta_M(s) = \sum \lambda_k^{-s}$ Riemann zeta
3-Body Problem	Singularity analysis in dynamical systems singularity analysis in Ricci flow
Turbulence	Grade structure of velocity field grade structure of curvature tensor

The spectral zeta connection deserves emphasis. For a closed Riemannian manifold:

$$\zeta_M(s) = \sum_{k=1}^{\infty} m_k \, \lambda_k^{-s}$$

This meromorphic function encodes the full spectrum. For S^3:

$$\zeta_{S^3}(s) = \sum_{k=1}^{\infty} (k+1)^2 \, [k(k+2)]^{-s} = \sum_{k=1}^{\infty} \frac{(k+1)^2}{[(k+1)^2 - 1]^s}$$

The analytic properties of $\zeta_M(s)$ — poles, residues, special values — are spectral invariants that the Latent framework can exploit.

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7. Conclusion

We have shown that the Latent principle extends to Riemannian geometry: a finite truncation of the Laplacian spectrum is a sufficient statistic for the topological type of a 3-manifold, under bounded geometry. For the 3-sphere specifically, a single eigenvalue-multiplicity pair $(3, 4)$ determines the isometry type under Ricci lower bounds — a K = 1 Latent theorem.

The broader implication is that the “shape” of a Riemannian manifold is a finite-dimensional object, just as the “risk” of a portfolio, the “state” of a turbulent flow, and the “dynamics” of a celestial system are all captured by finite Latent representations. The spectral characterization of topology is perhaps the most fundamental instance of this principle: the vibration frequencies of space itself encode its shape.

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