

# The Spectral Theory of Observation

How discrete outcomes arise from smooth latent structure

*Atomic measures, kernel lifts, and observer-relative chaos*

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Draft

## Executive Summary (Non-Technical)

Many scientific arguments are still organized around a false opposition: a system is treated as either **continuous** or **discrete**, either **smooth** or **jumping**, either **ordered** or **chaotic**. This paper argues that this opposition is often a property of the *observation layer*, not of the underlying state itself.

The key idea is simple. A discrete outcome does not force a discrete underlying reality. A four-state variable, a classifier output, a threshold event, or a phase label can often be understood as a **projection or quantization of a smoother latent object**. Once the latent object is written at the right level, the discrete observable can be embedded into continuous space, smoothed in a controlled way, and studied spectrally.

This changes the role of smoothness. Smoothness is not what makes spectral representation possible at all. It is what makes that representation **compressible, stable, and computationally useful**. A non-smooth object still has a spectrum, but its coefficients decay slowly and its approximation is fragile. A smooth latent object has rapidly decaying coefficients, so a small number of modes may already capture the real structure.

The paper does **not** claim that every discontinuity is fake, that every jump process is merely a linguistic mistake, or that classical chaos disappears once we use the right representation. **Smooth systems can still be chaotic**. The narrower claim is that many *observed* jumps, labels, catastrophes, and instabilities are intensified by projection, thresholding, coarse-graining, and finite measurement resolution.

This viewpoint creates one common language for several phenomena that are usually discussed separately: categorical variables, neural-network classification, threshold events, apparent phase transitions, and some forms of observed chaos. In each case the same pattern appears: a smooth latent state is pushed through an observation lens and a quantizer, and the output looks more discontinuous than the latent state really is.

The scientific contribution of the paper is therefore conceptual and mathematical rather than fully finished. It proposes a clean object language:

1. represent discrete observables as atomic measures on a continuous space,
2. lift them by kernel smoothing into a nearby smooth family,
3. study the spectral decay of the lifted object,
4. separate latent dynamics from observer-induced discontinuity.

If this program is right, then part of what we call “chaos” is not only a property of the world, but a property of how much of the latent smooth structure the observer can actually resolve.

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## Abstract

We propose a spectral framework for understanding how discrete outcomes arise from smooth latent structure. The core move is to represent a finite-valued or countable observable not as an object outside continuous analysis, but as an atomic measure

$$\mu = \sum_{j=1}^m p_j \delta_{x_j},$$

already living on a continuous state space. A smooth family is then generated by convolution with a resolution kernel  $K_\varepsilon$ ,

$$f_\varepsilon = \mu * K_\varepsilon = \sum_{j=1}^m p_j K_\varepsilon(\cdot - x_j),$$

with Fourier transform

$$\widehat{f}_\varepsilon(\omega) = \widehat{K}_\varepsilon(\omega) \sum_{j=1}^m p_j e^{-i\omega x_j}.$$

This factorization separates the discrete spectral content of the observable from the smoothing imposed by the observation lens. It makes precise the claim that discreteness and continuity are not mutually exclusive ontologies, but different resolutions of one state object.

We then place this measure-theoretic embedding inside a broader observation model. A latent state  $u_t$  evolves on a smooth space, an observation lens  $\kappa$  maps it into a measurement coordinate, and a quantizer  $Q$  produces the finite observable  $Y_t = Q(\kappa(u_t))$ . In this language, neural-network classification, threshold events, regime labels, and some apparent phase transitions share one structure: a smooth latent evolution followed by projection and quantization.

The paper makes three claims. First, discrete observables admit canonical continuous lifts, so spectral methods remain natural even when the recorded variable itself is finite-valued. Second, smoothness controls spectral compressibility rather than representability: smoother latent objects yield faster coefficient decay and more stable low-rank structure. Third, some observed discontinuity and some observed unpredictability are observer-relative: they arise because coarse-graining, thresholding, or finite resolution suppresses latent smooth structure.

We explicitly do **not** claim that all chaos is an observational illusion. Smooth deterministic systems can have genuine positive Lyapunov exponents. The narrower proposal is that a meaningful part of what appears as jump, collapse, catastrophe, or chaos at the measurement layer can be clarified by separating latent dynamics from observer-induced discretization. The paper is therefore programmatic: it defines the mathematical object language for a future theorem line connecting atomic measures, Fejer smoothing, coarse-grained dynamics, and the spectral representation of quantized observation.

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# 1. Introduction

One of the oldest bad habits in mathematical modeling is to treat the discrete and the continuous as different worlds. A variable is called “continuous” when it takes values in  $\mathbb{R}$  and “discrete” when it takes values in a finite or countable set. The distinction is useful at the level of recorded values, but it becomes misleading when it is promoted to an ontological claim about the underlying state.

The central intuition of this paper is that many apparently discrete phenomena are better understood as continuous latent states viewed through a finite-resolution observation lens. A classifier outputs one of  $m$  labels, but its logits vary continuously. A phase label says “solid” or “liquid”, but the order parameter moves continuously in finite systems. A jump event records whether a threshold was crossed, even when the latent state moved smoothly. A chaotic-looking symbolic trace may come from a smooth dynamics observed through a coarse partition.

This does not mean that discreteness is unreal. It means that **recorded discreteness does not by itself identify the right state object**.

The spectral viewpoint sharpens this distinction. In this repo’s core language, smoothness implies spectral compressibility:

$$\text{smooth latent object} \implies \text{fast spectral decay} \implies \text{small effective representation.}$$

The contrapositive is just as important: a non-smooth or singular object still has a spectral representation, but its coefficients decay slowly and low-rank approximation becomes unstable. The right question is therefore not “is the phenomenon discrete or continuous?” but:

what is the latent state, what is the observation lens, and where does the quantization enter?

This paper proposes one answer built from four ideas:

1. a discrete observable can be represented canonically as an atomic measure on a continuous space,
2. an observation lens  $\kappa$  maps the latent state into a measurable coordinate,
3. a quantizer  $Q$  creates the recorded discrete outcome,
4. a smoothing kernel  $K_\epsilon$  produces a nearby continuous family whose spectral structure can be studied directly.

Conceptually, this note extends an earlier kernel-and-lens line of thought by making the full observation stack explicit: a latent state is first mapped through an observation lens and only then quantized into a symbolic observable. The present paper’s distinctive addition is the third layer, namely the quantizer, together with the resulting symbolic-instability analysis.

The paper is conceptual and programmatic rather than final. Its contribution is a common object language for finite-valued variables, neural classification, threshold events, phase labels, and some coarse-grained manifestations of chaos. The main scientific claim is modest but sharp:

Many observed discrete outcomes are not primitive state objects. They are quantized projections of smoother latent structure.

An equivalent negative statement is just as important:

discrete observation  $\neq$  necessarily discrete latent dynamics. (0)

The strongest non-claim is equally important:

The paper does not claim that all discontinuity or all chaos is an observational artifact.

The goal is separation, not metaphysical elimination.

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## 2. Discrete Observables as Atomic Measures

### 2.1 The basic embedding

Let  $x_1, \dots, x_m \in \mathbb{R}^d$  be the support points of a finite-valued observable with probabilities  $p_1, \dots, p_m$ , where  $p_j \geq 0$  and  $\sum_j p_j = 1$ . The canonical continuous-space representation is the atomic measure

$$\mu = \sum_{j=1}^m p_j \delta_{x_j}. \quad (1)$$

This is already a measure on a continuous space. It is not a density with respect to Lebesgue measure, but it is not outside analysis. The discrete observable is therefore best viewed as **singular**, not as “non-continuous” in some absolute sense.

This simple move matters because it removes the false impression that a finite-valued random variable is in principle inaccessible to spectral or measure-theoretic methods. It is accessible immediately, but as an atomic object.

### 2.2 The categorical case

If the recorded values are 0, 1, 2, 3, then

$$\mu = p_0 \delta_0 + p_1 \delta_1 + p_2 \delta_2 + p_3 \delta_3. \quad (2)$$

Nothing prevents us from studying  $\mu$  on  $\mathbb{R}$ . The discrete law has simply become a measure concentrated on four points. The distinction between “four states” and “a measure on  $\mathbb{R}$ ” is not mathematical incompatibility; it is a choice of language.

### 2.3 Spectral consequences

For an atomic measure, the Fourier transform is explicit:

$$\hat{\mu}(\omega) = \sum_{j=1}^m p_j e^{-i\omega x_j}. \quad (3)$$

This is already a spectral object. The issue is not existence but regularity. Because  $\mu$  is singular, there is no reason to expect rapid spectral decay. The coefficients may oscillate forever and the approximation of  $\mu$  by naive truncation will display familiar spectral pathologies.

This is the right place to refine the repo’s broad intuition:

spectral representation exists very broadly,  
spectral compression appears only under regularity.

That distinction is central for everything that follows.

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### 3. Continuous Lifts by Kernel Smoothing

#### 3.1 The lift

Let  $K_\varepsilon$  be a smoothing kernel with unit mass, for example a Gaussian, a compactly supported mollifier, or on periodic domains a Fejer kernel. Define the lifted density

$$f_\varepsilon = \mu * K_\varepsilon = \sum_{j=1}^m p_j K_\varepsilon(\cdot - x_j). \tag{4}$$

The family  $\{f_\varepsilon\}_{\varepsilon>0}$  does not erase the original discrete law. It creates a nearby smooth family that converges back to  $\mu$  as  $\varepsilon \rightarrow 0$  in the weak sense.

This gives a very clean interpretation:

- $\mu$  is the exact recorded discrete law,
- $f_\varepsilon$  is the same law viewed at resolution  $\varepsilon$ ,
- $K_\varepsilon$  is the observer’s smoothing lens.

#### 3.2 Spectral factorization

The Fourier transform of the lift is

$$\widehat{f}_\varepsilon(\omega) = \widehat{K}_\varepsilon(\omega) \widehat{\mu}(\omega) = \widehat{K}_\varepsilon(\omega) \sum_{j=1}^m p_j e^{-i\omega x_j}. \tag{5}$$

This factorization is the paper’s first key equation. It splits the problem into two layers:

1. the **discrete spectral content** carried by the point masses  $x_j$  and weights  $p_j$ ,
2. the **resolution filter** imposed by the kernel.

The observation layer is therefore not external to the spectral representation. It acts inside it as a multiplicative frequency filter.

#### 3.3 Why Fejer matters here

For periodic or bounded-angle observables, Fejer smoothing is especially natural because it suppresses Gibbs-type oscillation while preserving the low-frequency structure. In the current repo vocabulary, this is the same moral already visible in the spectral-method line:

$$\text{regularize first, then compress.} \tag{6}$$

The point is not that smoothing magically creates truth. The point is that if the latent object really comes from a smooth family, then the unsmoothed atomic record may be a poor representational surface for that object.

### 3.4 Weak recovery

As  $\varepsilon \rightarrow 0$ ,

$$f_\varepsilon \rightarrow \mu. \tag{7}$$

So the smoothing is not a replacement of the discrete law by a different object. It is a controlled family of nearby observation scales. The relevant modeling question becomes:

which  $\varepsilon$  matches the real observation process?

This is a scientific question about sensors, aggregation, reporting, and ontology, not merely a numerical convenience.

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## 4. Observation as Lens Plus Quantizer

### 4.1 The latent state

Let  $u_t$  denote the latent state of a system at time  $t$ . This may be:

- a density,
- a field,
- a hidden state vector,
- a generator state,
- or a smooth order parameter on a manifold.

We do not assume that the recorded variable itself is smooth. We assume only that the **latent state object** may be smoother than the recorded one.

### 4.2 The observation lens

Let  $\kappa$  denote the observation lens. In the repo’s naming language,  $\kappa$  is the map that turns latent structure into an observable coordinate:

$$y_t = \kappa(u_t). \tag{8}$$

The key point is that  $\kappa$  can already lose information. It may project a high-dimensional smooth state onto one scalar, one feature, one energy level, one amplitude, or one score.

### 4.3 Quantization

The recorded discrete observable is then

$$Y_t = Q_\Delta(y_t), \tag{9}$$

where  $Q_\Delta$  is a quantizer, threshold map, partition map, or label assignment rule. This single equation covers several familiar cases:

- **classification:**  $Q = \text{argmax}$  on logits,
- **threshold events:**  $Q(y) = \mathbf{1}\{y \geq L\}$ ,
- **phase labels:**  $Q(y) \in \{\text{solid, liquid, gas}\}$ ,
- **symbolic dynamics:**  $Q$  records which partition cell the state occupies.

The discrete output is therefore not the whole state. It is a composed object:

$$u_t \xrightarrow{\kappa} y_t \xrightarrow{Q_\Delta} Y_t. \quad (10)$$

This decomposition is the paper’s second key equation.

#### 4.4 Neural networks as a clean example

Neural classification is almost embarrassingly clear in this language. A network produces a continuous logit vector

$$\ell(x) = (\ell_1(x), \dots, \ell_m(x)), \quad (11)$$

usually via compositions of smooth or piecewise-smooth maps. The recorded class label is then

$$\hat{c}(x) = \text{argmax}_j \ell_j(x). \quad (12)$$

The discrete class is real and operationally decisive, but the model’s internal state is not discrete. It is a continuous field on input space, then quantized at the last step. This is exactly the type of structure the paper wants to generalize.

The smoothness of the activation function matters here as a **regularity gate**: it helps determine which smoothness class the represented map can belong to. But the network is not “spectral” merely because it is smooth. The more careful statement is that neural models often generate a continuous or piecewise-smooth latent score field, while the final decision is discrete because of the output quantizer.

## 5. Smoothness and Spectral Compressibility

### 5.1 What smoothness actually buys

Smoothness does not create spectral representation from nothing. It improves the decay of spectral coefficients. For a latent object  $u$  with spectral coefficients  $A_k$ , the rough hierarchy is:

- rough or singular: slow decay,
- $C^r$ : algebraic decay,
- analytic: exponential decay.

So the correct implication is

$$\text{smoothness} \implies \text{spectral compressibility}. \quad (13)$$

This matters because the observation problem is not just metaphysical. It is computational. If the right latent object is smooth, then a small number of modes may already capture the meaningful state. If we work directly on a discontinuous or quantized surface, we may be forcing ourselves into a spectrally inefficient description.

## 5.2 The role of the latent lift

The lifted family  $f_\varepsilon$  is not introduced to pretend the atoms were never there. It is introduced because the latent state may be smooth even when the report is atomic. The lift therefore turns an observationally sharp record into a spectrally analyzable family.

This is exactly the setting where the repo’s broader spectral program becomes relevant. If the latent state has spectral quality  $\rho > 1$ , then a stable low-rank representation may exist even though the measured output looks discrete. If the latent state itself is non-smooth, the representation will be much less compressible. The distinction is empirical and mathematical, not ideological.

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# 6. Phase Transition and Apparent Discontinuity

## 6.1 Why this question appears naturally

The thought that “what looks discontinuous may in fact be smooth underneath” is especially tempting in phase-transition language. We speak as if a system truly jumps from one state to another. But finite systems often soften that picture. Sharp non-analytic behavior is frequently tied to limiting procedures: infinite volume, zero noise, infinite time, or vanishing smoothing scale.

## 6.2 The narrow claim

This paper does **not** claim that phase transitions are merely artifacts. It makes the narrower claim that:

$$\text{some observed discontinuity belongs to the observation scale or limiting procedure.} \quad (14)$$

At finite size, many order parameters change rapidly but continuously. Once one bins the order parameter into labels, the representation becomes strictly discrete. The apparent “jump” may therefore mix three different things:

1. a real change in latent geometry,
2. a limiting singularity,
3. a quantized reporting rule.

The framework of Sections 2–4 gives a way to separate them.

## 6.3 Thresholds are honest but partial

Threshold observables are not mistakes. They are useful operational summaries. A barrier breach, a classifier label, or a regime trigger often matters more in action than the continuous latent score. The problem appears only when one forgets that a threshold variable is a **summary statistic** of a richer state.

## 6.4 Jumps and hidden smooth structure

The same caution applies to jump-like observations. A recorded jump does not automatically imply that the underlying state law is itself primitive, discrete, or non-smooth. In some systems a jump

variable is a thresholded view of a smoother latent process, a projection of a higher-dimensional continuous state, or an event-time summary of dynamics that remain smooth between observations.

This does **not** justify the opposite extreme claim that every jump process is only a representational mistake. Some systems are genuinely modeled at the jump level. The point is narrower:

jump-like observation does not by itself identify the true regularity class of the latent state. (14a)

That is why kernel lifts and latent-state reconstructions are not cosmetic in this setting. They are a way to test whether the observational discontinuity belongs to the underlying dynamics or to the measurement surface.

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## 7. Chaos: What This View Explains and What It Does Not

### 7.1 What it does not explain away

Classical chaos does not require a non-smooth world. Poincare’s lesson was precisely that smooth deterministic systems can exhibit sensitive dependence on initial conditions. In the standard form,

$$\|\delta x_t\| \approx e^{\lambda t} \|\delta x_0\|, \quad \lambda > 0, \quad (15)$$

and nothing in this paper denies that.

So the strong claim

“chaos exists only because we measure badly”

is too strong and should be rejected.

### 7.2 The claim we do make

The weaker and more interesting claim is this:

observed chaos can be amplified by projection, thresholding, and coarse-graining. (16)

A smooth latent trajectory may be pushed through a partition map, a threshold detector, or a finite alphabet encoder. The symbolic output can look violently discontinuous even when the latent path varies smoothly. In that sense, part of what the observer experiences as chaos belongs to the observation layer.

### 7.3 Observer-relative chaos

This suggests a useful distinction.

- **latent chaos:** instability of the underlying dynamics,
- **observed chaos:** instability of the recorded symbolic or quantized trace,
- **observer-relative chaos:** the instability created or intensified by the chosen lens  $\kappa$  and quantizer  $Q_\Delta$ .

The third notion is the paper’s intended contribution. It does not replace Lyapunov theory. It sits one layer above it.

## 7.4 A spectral reading

Suppose a latent smooth state has spectral expansion

$$u = \sum_{k \geq 0} A_k \varphi_k. \tag{17}$$

If only the first  $N$  modes are robustly resolved under the observer’s lens, then the observer sees

$$u^{(\kappa, N)} = \sum_{k=0}^N A_k^{(\kappa)} \varphi_k^{(\kappa)}. \tag{18}$$

When the retained low-mode picture changes sharply under small perturbations of the lens, threshold, or binning rule, the observer experiences instability that is at least partly representational.

That is the sense in which chaos may become “more understandable” under a smoother latent model: not by disappearing, but by being decomposed into

1. genuine dynamical sensitivity,
2. observer-induced symbolic discontinuity.

## 8. Relation to Quantum Measurement

The quantum-measurement theme remains an important special case of the broader thesis. A wavefunction or density operator evolves on a continuous or Hilbert-space object, while recorded outcomes are discrete detector events or binned pointer values. The paper does not solve the measurement problem, but it does clarify one point:

$$\text{discrete detection is compatible with a smoother latent state representation.} \tag{19}$$

In that sense, the quantum case is not unique. It is one member of a larger class in which observation selects, bins, or thresholds a richer state object.

The value of the present framework is therefore not that it replaces quantum foundations. It is that it puts quantum measurement, neural classification, threshold events, and coarse-grained dynamics into one formal family.

## 9. Mini Formal Core

This section isolates the paper’s formal core in a theorem-queue-compatible form. The goal is not to pretend that the full theory is already proved, but to separate the object language from the surrounding motivation.

## 9.1 The state-observation stack

Let

- $X$  be the latent state space,
- $Y$  be the observation-coordinate space,
- $C$  be a finite or countable code space,
- $\kappa : X \rightarrow Y$  be the observation lens,
- $Q : Y \rightarrow C$  be the quantizer.

The basic observation architecture is

$$u_t \in X \xrightarrow{\kappa} y_t \in Y \xrightarrow{Q} c_t \in C. \quad (20)$$

Equivalently, the recorded variable is

$$c_t = (Q \circ \kappa)(u_t). \quad (21)$$

When the latent state is probabilistic, we write its law as  $\nu_t$  on  $X$ , its pushed-forward law on  $Y$  as

$$\mu_t := \kappa_{\#} \nu_t, \quad (22)$$

and its coded observation law on  $C$  as

$$\pi_t := Q_{\#} \mu_t. \quad (23)$$

This stack is the paper's formal skeleton.

## 9.2 Definitions: discreteness

**Definition 9.1 (Latent discreteness).** The system is *latently discrete* if the relevant state object itself lives on a finite or countable space, or equivalently if the latent law  $\nu_t$  is supported on a finite or countable state set in the modeling layer treated as fundamental.

**Definition 9.2 (Observational discreteness).** The system is *observationally discrete* if the recorded output space  $C$  is finite or countable, regardless of whether the latent state space  $X$  is discrete or continuous.

These notions are intentionally separated. Observational discreteness is a statement about the codomain of  $Q$ . Latent discreteness is a statement about the ontology or effective state object chosen for the model.

## 9.3 Definitions: jumps

Assume  $t \mapsto z_t$  is a cadlag process in some metric space.

**Definition 9.3 (Latent jump).** The system has a *latent jump* at time  $t$  if

$$\Delta u_t := u_t - u_{t-} \neq 0 \quad (24)$$

in the latent state representation.

**Definition 9.4 (Observational jump).** The system has an *observational jump* at time  $t$  if the recorded process  $c_t$  or the intermediate observation coordinate  $y_t$  has a discontinuity at  $t$ , even if  $u_t$  remains continuous.

The two notions need not coincide. For example, if  $u_t$  is continuous and

$$c_t = \mathbf{1}\{\kappa(u_t) \geq L\}, \quad (25)$$

then  $c_t$  may jump when the threshold is crossed, although the latent path is continuous.

#### 9.4 Definitions: chaos

**Definition 9.5 (Latent chaos).** The latent dynamics are *chaotic* if the flow or semigroup on  $X$  exhibits sensitive dependence on initial conditions in the usual dynamical sense, e.g.

$$\|\delta u_t\| \approx e^{\lambda t} \|\delta u_0\|, \quad \lambda > 0. \quad (26)$$

**Definition 9.6 (Observed chaos).** The observed process is *chaotic* if the recorded dynamics on  $Y$  or  $C$  displays instability, unpredictability, or rapidly growing symbolic divergence under the chosen observation scheme.

**Definition 9.7 (Observer-relative chaos).** Observer-relative chaos is the component of observed instability that is created or amplified by the choice of observation lens  $\kappa$ , quantizer  $Q$ , partition, threshold, or finite-resolution reporting rule.

This definition is deliberately comparative. It is not meant to replace latent chaos, but to measure the instability introduced by the state-observation stack itself.

#### 9.5 The atomic lift

Let the recorded observation law at a fixed time be

$$\pi = \sum_{j=1}^m p_j \delta_{c_j} \quad (27)$$

on a finite code space embedded into a continuous ambient space. Let  $K_\varepsilon$  be a smoothing kernel. Define the lifted family

$$f_\varepsilon = \pi * K_\varepsilon = \sum_{j=1}^m p_j K_\varepsilon(\cdot - c_j). \quad (28)$$

This lift does not assert that the code space was the true latent state. It only constructs a nearby smooth family at resolution  $\varepsilon$ . The paper's standing interpretation is:

- $\pi$  = exact recorded code law,
- $f_\varepsilon$  = the same law at finite observation scale,
- $K_\varepsilon$  = the smoothing or resolution operator.

#### 9.6 Proposition candidates

The following statements are the natural first theorem candidates for the program.

**Proposition Candidate 9.1 (Observational discreteness does not imply latent discreteness).** There exist continuous latent processes  $u_t$  and observation stacks  $(\kappa, Q)$  such that the recorded process  $c_t = Q(\kappa(u_t))$  is finite-valued, while  $u_t$  evolves on a continuous state space.

**Proposition Candidate 9.2 (Observational jump without latent jump).** There exist continuous latent trajectories  $u_t$  and threshold quantizers  $Q$  such that  $c_t$  has jump discontinuities even though  $u_t$  is continuous.

**Proposition Candidate 9.3 (Atomic lift convergence).** If  $K_\varepsilon$  is an approximate identity and  $\pi$  is an atomic measure, then

$$f_\varepsilon = \pi * K_\varepsilon \rightarrow \pi \quad \text{as } \varepsilon \rightarrow 0. \quad (29)$$

**Proposition Candidate 9.4 (Spectral factorization under lift).** The Fourier transform of the lifted law satisfies

$$\widehat{f}_\varepsilon(\omega) = \widehat{K}_\varepsilon(\omega)\widehat{\pi}(\omega), \quad (30)$$

so the observation-scale effect is multiplicative in frequency space.

**Proposition Candidate 9.5 (Quantizer-induced instability).** Under suitable regularity assumptions on  $u_t$  and  $\kappa$ , there exist quantizers  $Q$  such that small perturbations in the latent state produce bounded perturbations in  $y_t = \kappa(u_t)$  but large symbolic divergence in  $c_t = Q(y_t)$ .

**Proposition Candidate 9.6 (Observer-relative chaos decomposition).** For a suitable class of systems, observed instability can be decomposed into

$$\text{observed instability} = \text{latent instability} + \text{quantization} / \text{coarse-graining instability} + \text{residual interaction term}. \quad (31)$$

This decomposition is still heuristic at the current stage, but it states the paper's intended theorem direction very cleanly.

Two of the candidate statements above already admit theorem-ready elementary forms.

## 9.6A Two elementary propositions

**Proposition 9.8 (Observational discreteness without latent discreteness).** There exists a system whose recorded output is finite-valued while the latent state evolves on a continuous state space.

**Proof.** Take

$$X = \mathbb{R}, \quad Y = \mathbb{R}, \quad C = \{0, 1\}. \quad (31a)$$

Let the latent trajectory be

$$u_t = t, \quad t \in \mathbb{R}. \quad (31b)$$

Let the observation lens be the identity,

$$\kappa(y) = y, \quad (31c)$$

and let the quantizer be the threshold map

$$Q(y) = \mathbf{1}\{y \geq 0\}. \quad (31d)$$

Then

$$c_t = Q(\kappa(u_t)) = \mathbf{1}\{t \geq 0\} \in \{0, 1\}. \quad (31e)$$

So the recorded process is observationally discrete, while the latent state  $u_t$  evolves on the continuous state space  $\mathbb{R}$ . Therefore observational discreteness does not imply latent discreteness.  $\square$

This proposition is elementary, but it is conceptually decisive: the discreteness of the output lives in the codomain of  $Q$ , not in the topology of the latent state space.

**Proposition 9.9 (Threshold crossing creates observational jump without latent jump).** Let  $u_t$  be continuous in a neighborhood of  $\tau$ , let  $\kappa : X \rightarrow \mathbb{R}$  be continuous, and define

$$c_t = \mathbf{1}\{\kappa(u_t) \geq L\}. \quad (31f)$$

Assume there exists  $\eta > 0$  such that

$$\kappa(u_t) < L \quad \text{for } t \in (\tau - \eta, \tau), \quad \kappa(u_t) \geq L \quad \text{for } t \in [\tau, \tau + \eta). \quad (31g)$$

Then  $u_t$  has no latent jump at  $\tau$ , but  $c_t$  has an observational jump at  $\tau$ .

**Proof.** By assumption,  $u_t$  is continuous near  $\tau$ , so in particular

$$\Delta u_\tau = u_\tau - u_{\tau-} = 0. \quad (31h)$$

But from the sign conditions in (31g) we have

$$c_t = 0 \quad \text{for } t \in (\tau - \eta, \tau), \quad c_t = 1 \quad \text{for } t \in [\tau, \tau + \eta), \quad (31i)$$

hence

$$c_{\tau-} = 0, \quad c_\tau = 1, \quad \Delta c_\tau = 1. \quad (31j)$$

So the recorded process jumps at  $\tau$  even though the latent trajectory remains continuous.  $\square$

This proposition identifies the exact layer at which the discontinuity appears: not in the latent path, but in the threshold quantizer.

**Proposition 9.10 (Near-threshold quantizer instability).** Let

$$Q(y) = \mathbf{1}\{y \geq 0\}, \quad y \in \mathbb{R}. \quad (31k)$$

Then for every  $\varepsilon > 0$  there exist  $y_-, y_+ \in \mathbb{R}$  such that

$$|y_+ - y_-| < \varepsilon \quad (31l)$$

but

$$|Q(y_+) - Q(y_-)| = 1. \quad (31m)$$

In particular,  $Q$  is not continuous at the threshold and cannot be locally Lipschitz there.

**Proof.** Fix  $\varepsilon > 0$  and choose

$$y_- = -\varepsilon/3, \quad y_+ = \varepsilon/3. \quad (31n)$$

Then

$$|y_+ - y_-| = 2\varepsilon/3 < \varepsilon, \quad (31o)$$

while

$$Q(y_-) = 0, \quad Q(y_+) = 1. \quad (31p)$$

Hence

$$|Q(y_+) - Q(y_-)| = 1. \quad (31q)$$

Since this holds for every  $\varepsilon > 0$ , arbitrarily small perturbations in the observation coordinate can produce maximal code disagreement at the threshold. Therefore  $Q$  is discontinuous at 0 and not locally Lipschitz there.  $\square$

**Corollary 9.11 (Latent perturbation can be symbolically amplified).** Let  $\kappa : X \rightarrow \mathbb{R}$  be continuous and let  $Q(y) = \mathbf{1}\{y \geq 0\}$ . Suppose  $u_*, v_* \in X$  satisfy

$$|\kappa(u_*) - \kappa(v_*)| < \varepsilon \quad (31r)$$

for arbitrarily small  $\varepsilon > 0$ , while

$$\kappa(u_*) < 0 \leq \kappa(v_*). \quad (31s)$$

Then

$$|(Q \circ \kappa)(u_*) - (Q \circ \kappa)(v_*)| = 1. \quad (31t)$$

This is the paper's simplest formal instance of **observer-relative amplification**: the latent or observation-coordinate discrepancy can be arbitrarily small, while the coded discrepancy is already maximal.

**Proposition 9.12 (Crossing-time shift creates interval-scale symbolic divergence).** For  $a \in \mathbb{R}$ , define the step process

$$c_t^{(a)} = \mathbf{1}\{t \geq a\}, \quad t \in \mathbb{R}. \quad (31u)$$

Let  $T > 0$ , and assume  $0 \leq a < b \leq T$ . Then

$$\int_0^T |c_t^{(a)} - c_t^{(b)}| dt = b - a. \quad (31v)$$

**Proof.** For  $t < a$ , both processes are 0. For  $t \in [a, b)$ , we have

$$c_t^{(a)} = 1, \quad c_t^{(b)} = 0. \quad (31w)$$

For  $t \geq b$ , both processes are 1. Hence

$$|c_t^{(a)} - c_t^{(b)}| = \mathbf{1}_{[a,b)}(t), \quad (31x)$$

and therefore

$$\int_0^T |c_t^{(a)} - c_t^{(b)}| dt = \int_0^T \mathbf{1}_{[a,b)}(t) dt = b - a. \quad (31y)$$

$\square$

This proposition shows that a small displacement in crossing time is not merely an instantaneous coding flip. It generates a whole interval of symbolic disagreement whose size is exactly the crossing-time shift.

**Corollary 9.13 (Dynamic observer-relative amplification).** Let  $u_t^{(1)}, u_t^{(2)}$  be two latent trajectories and let

$$c_t^{(i)} = \mathbf{1}\{\kappa(u_t^{(i)}) \geq L\}, \quad i = 1, 2. \quad (31z)$$

Assume both trajectories have a unique threshold-crossing time in  $[0, T]$ , denoted  $\tau_1, \tau_2$ , with  $0 \leq \tau_1 < \tau_2 \leq T$ , and satisfy

$$\kappa(u_t^{(1)}) < L \text{ for } t < \tau_1, \quad \kappa(u_t^{(1)}) \geq L \text{ for } t \geq \tau_1, \quad (31aa)$$

$$\kappa(u_t^{(2)}) < L \text{ for } t < \tau_2, \quad \kappa(u_t^{(2)}) \geq L \text{ for } t \geq \tau_2. \quad (31ab)$$

Then

$$\int_0^T |c_t^{(1)} - c_t^{(2)}| dt = |\tau_2 - \tau_1|. \quad (31ac)$$

In particular, any mechanism that makes the crossing-time map sensitive to the latent state produces interval-scale symbolic divergence even when the instantaneous latent discrepancy remains small.

**Proposition 9.14 (Inverse-slope amplification in the linear crossing model).** Fix  $a > 0$ ,  $\tau \in \mathbb{R}$ , and  $\delta > 0$ . Define two observation-coordinate paths

$$y_t^{(1)} = a(t - \tau), \quad y_t^{(2)} = a(t - \tau) - \delta, \quad (31ad)$$

and let

$$c_t^{(i)} = \mathbf{1}\{y_t^{(i)} \geq 0\}, \quad i = 1, 2. \quad (31ae)$$

Then the crossing times are

$$\tau_1 = \tau, \quad \tau_2 = \tau + \delta/a, \quad (31af)$$

and for every  $T \geq \tau_2$ ,

$$\int_0^T |c_t^{(1)} - c_t^{(2)}| dt = \frac{\delta}{a}. \quad (31ag)$$

**Proof.** The first path crosses the threshold when  $a(t - \tau) = 0$ , hence at  $t = \tau$ . The second crosses when

$$a(t - \tau) - \delta = 0, \quad (31ah)$$

that is,

$$t = \tau + \delta/a. \quad (31ai)$$

Applying Proposition 9.12 to the step processes with crossing times  $\tau_1$  and  $\tau_2$  gives

$$\int_0^T |c_t^{(1)} - c_t^{(2)}| dt = \tau_2 - \tau_1 = \delta/a. \quad (31aj)$$

□

This formula isolates the key dynamic amplification factor:

$$\text{symbolic disagreement scale} \sim \frac{\text{observation-coordinate error}}{\text{crossing slope}}. \quad (31ak)$$

So near-grazing crossings ( $a$  small) are intrinsically fragile: even tiny perturbations in the observation coordinate can produce long intervals of symbolic disagreement.

**Corollary 9.15 (Near-grazing observer-relative instability).** In the linear crossing model above, if  $\delta$  is fixed and  $a \downarrow 0$ , then

$$\frac{\delta}{a} \rightarrow \infty. \quad (31al)$$

Hence the symbolic disagreement window can become arbitrarily large even though the raw observation perturbation  $\delta$  is held fixed and small.

This is the paper's first explicit bridge from threshold quantization to a dynamic form of observer-relative chaos: the instability is not only due to the discontinuity of  $Q$ , but also to the small transverse velocity with which the latent path approaches the threshold.

**Proposition 9.16 (Local transversality controls crossing-time sensitivity).** Let  $I \subset \mathbb{R}$  be an interval, and let  $y^{(1)}, y^{(2)} \in C^1(I)$ . Assume there exist crossing times  $\tau_1, \tau_2 \in I$  such that

$$y^{(1)}(\tau_1) = 0, \quad y^{(2)}(\tau_2) = 0, \quad (31am)$$

and suppose both paths cross transversally with the same positive lower slope bound:

$$(y^{(i)})'(t) \geq m > 0, \quad t \in I, \quad i = 1, 2. \quad (31an)$$

If

$$\sup_{t \in I} |y^{(1)}(t) - y^{(2)}(t)| \leq \varepsilon, \quad (31ao)$$

then

$$|\tau_1 - \tau_2| \leq \frac{\varepsilon}{m}. \quad (31ap)$$

**Proof.** Assume first that  $\tau_1 \leq \tau_2$ . Since  $(y^{(1)})'(t) \geq m$  on  $I$ , the fundamental theorem of calculus gives

$$y^{(1)}(\tau_2) - y^{(1)}(\tau_1) = \int_{\tau_1}^{\tau_2} (y^{(1)})'(s) ds \geq m(\tau_2 - \tau_1). \quad (31aq)$$

Using  $y^{(1)}(\tau_1) = 0$ , this becomes

$$y^{(1)}(\tau_2) \geq m(\tau_2 - \tau_1). \quad (31ar)$$

But  $y^{(2)}(\tau_2) = 0$ , so the uniform perturbation bound yields

$$|y^{(1)}(\tau_2)| = |y^{(1)}(\tau_2) - y^{(2)}(\tau_2)| \leq \varepsilon. \quad (31as)$$

Combining the last two inequalities gives

$$m(\tau_2 - \tau_1) \leq \varepsilon, \quad (31at)$$

hence

$$\tau_2 - \tau_1 \leq \frac{\varepsilon}{m}. \quad (31au)$$

The case  $\tau_2 \leq \tau_1$  is identical after swapping the roles of the two paths. Therefore

$$|\tau_1 - \tau_2| \leq \frac{\varepsilon}{m}. \quad (31av)$$

□

This is the nonlinear local version of Proposition 9.14. The same inverse-slope mechanism survives beyond the exactly linear model: away from grazing, crossing-time sensitivity is controlled by

$$\text{crossing-time shift} \lesssim \frac{\text{pathwise observation error}}{\text{transverse slope}}. \quad (31aw)$$

**Corollary 9.17 (Local latent-to-symbolic stability away from grazing).** Let

$$y^{(i)}(t) = \kappa(u_t^{(i)}) - L, \quad c_t^{(i)} = \mathbf{1}\{\kappa(u_t^{(i)}) \geq L\}, \quad i = 1, 2, \quad (31ax)$$

and assume the hypotheses of Proposition 9.16 hold on an interval  $I$  containing both crossing times. Then for every  $T$  with  $I \subset [0, T]$ ,

$$\int_0^T |c_t^{(1)} - c_t^{(2)}| dt = |\tau_1 - \tau_2| \leq \frac{\varepsilon}{m}. \quad (31ay)$$

So the symbolic disagreement window is locally Lipschitz in the observation-coordinate perturbation as long as the crossing remains uniformly transversal. The divergence becomes large only when one approaches the grazing regime  $m \downarrow 0$ .

**Proposition 9.18 (Implicit crossing-time derivative formula).** Let  $J, I \subset \mathbb{R}$  be intervals, and let  $Y : J \times I \rightarrow \mathbb{R}$  be  $C^1$ . Suppose that for some  $(\alpha_0, \tau_0) \in J \times I$ ,

$$Y(\alpha_0, \tau_0) = 0, \quad \partial_t Y(\alpha_0, \tau_0) \neq 0. \quad (31az)$$

Then there exist neighborhoods  $U \subset J$  of  $\alpha_0$  and  $V \subset I$  of  $\tau_0$ , and a unique  $C^1$  function  $\tau : U \rightarrow V$ , such that

$$Y(\alpha, \tau(\alpha)) = 0 \quad \text{for all } \alpha \in U. \quad (31ba)$$

Moreover,

$$\tau'(\alpha) = -\frac{\partial_\alpha Y(\alpha, \tau(\alpha))}{\partial_t Y(\alpha, \tau(\alpha))}. \quad (31bb)$$

**Proof.** The existence and uniqueness of the local function  $\tau(\alpha)$  follow from the implicit function theorem, since  $\partial_t Y(\alpha_0, \tau_0) \neq 0$ . Differentiating the identity

$$Y(\alpha, \tau(\alpha)) = 0 \quad (31bc)$$

with respect to  $\alpha$  gives

$$\partial_\alpha Y(\alpha, \tau(\alpha)) + \partial_t Y(\alpha, \tau(\alpha)) \tau'(\alpha) = 0. \quad (31bd)$$

Solving for  $\tau'(\alpha)$  yields

$$\tau'(\alpha) = -\frac{\partial_\alpha Y(\alpha, \tau(\alpha))}{\partial_t Y(\alpha, \tau(\alpha))}. \quad (31be)$$

□

This is the formal bridge from latent sensitivity to crossing-time sensitivity. The numerator measures how strongly the observed coordinate changes with the latent parameter  $\alpha$ , while the denominator measures the transverse speed of the crossing itself.

**Corollary 9.19 (Parameter sensitivity of the symbolic disagreement window).** Under the hypotheses of Proposition 9.18, assume further that on some interval  $U_0 \subset U$ ,

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq \lambda > 0, \quad 0 < \partial_t Y(\alpha, \tau(\alpha)) \leq M \quad (31bf)$$

for all  $\alpha \in U_0$ . Then

$$\tau'(\alpha) \leq -\frac{\lambda}{M}, \quad \alpha \in U_0. \quad (31bg)$$

Consequently, for any  $\alpha, \beta \in U_0$ ,

$$|\tau(\alpha) - \tau(\beta)| \geq \frac{\lambda}{M} |\alpha - \beta|. \quad (31bh)$$

If the associated symbolic processes are

$$c_t^{(\alpha)} = \mathbf{1}\{Y(\alpha, t) \geq 0\}, \quad c_t^{(\beta)} = \mathbf{1}\{Y(\beta, t) \geq 0\}, \quad (31bi)$$

and both have unique crossing times in  $[0, T]$ , then

$$\int_0^T |c_t^{(\alpha)} - c_t^{(\beta)}| dt = |\tau(\alpha) - \tau(\beta)| \geq \frac{\lambda}{M} |\alpha - \beta|. \quad (31bj)$$

**Proof.** The derivative formula (31bb) gives

$$\tau'(\alpha) = -\frac{\partial_\alpha Y(\alpha, \tau(\alpha))}{\partial_t Y(\alpha, \tau(\alpha))} \leq -\frac{\lambda}{M}. \quad (31bk)$$

If  $\alpha < \beta$ , integrating the derivative bound over  $[\alpha, \beta]$  gives

$$\tau(\beta) - \tau(\alpha) = \int_\alpha^\beta \tau'(s) ds \leq -\frac{\lambda}{M} (\beta - \alpha). \quad (31bl)$$

Hence

$$\tau(\alpha) - \tau(\beta) \geq \frac{\lambda}{M} (\beta - \alpha), \quad (31bm)$$

which is equivalent to (31bh). The symbolic identity then follows from Proposition 9.12.  $\square$

**Proposition 9.19A (Certified crossing window from visible margin and perturbation budget).** Let  $J, I \subset \mathbb{R}$  be intervals, and let  $Y, \tilde{Y} : J \times I \rightarrow \mathbb{R}$  be  $C^1$ . Assume there exist  $C^1$  crossing branches

$$\tau, \tilde{\tau} : U_0 \rightarrow I \quad (31bm1)$$

on an interval  $U_0 \subset J$  such that

$$Y(\alpha, \tau(\alpha)) = 0, \quad \tilde{Y}(\alpha, \tilde{\tau}(\alpha)) = 0 \quad \text{for all } \alpha \in U_0. \quad (31bm2)$$

Assume further that there exist constants  $\lambda, m, M, \varepsilon > 0$  such that for all  $\alpha \in U_0$ ,

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq \lambda, \quad 0 < \partial_t Y(\alpha, \tau(\alpha)) \leq M, \quad (31bm3)$$

and for all  $t \in I$ ,

$$\partial_t Y(\alpha, t) \geq m, \quad \partial_t \tilde{Y}(\alpha, t) \geq m, \quad (31bm4)$$

$$\sup_{t \in I} |Y(\alpha, t) - \tilde{Y}(\alpha, t)| \leq \varepsilon. \quad (31bm5)$$

Define the symbolic processes

$$c_t^{(\alpha)} = \mathbf{1}\{Y(\alpha, t) \geq 0\}, \quad \tilde{c}_t^{(\alpha)} = \mathbf{1}\{\tilde{Y}(\alpha, t) \geq 0\}. \quad (31bm6)$$

Then:

1. for any  $\alpha, \beta \in U_0$ ,

$$|\tau(\alpha) - \tau(\beta)| \geq \frac{\lambda}{M} |\alpha - \beta|, \quad (31bm7)$$

2. for every  $\alpha \in U_0$ ,

$$|\tau(\alpha) - \tilde{\tau}(\alpha)| \leq \frac{\varepsilon}{m}, \quad (31bm8)$$

3. if, for a fixed  $\alpha \in U_0$ , the symbolic processes  $c^{(\alpha)}$  and  $\tilde{c}^{(\alpha)}$  satisfy the unique-crossing hypotheses of Corollary 9.13 on  $[0, T]$ , then

$$\int_0^T |c_t^{(\alpha)} - \tilde{c}_t^{(\alpha)}| dt = |\tau(\alpha) - \tilde{\tau}(\alpha)| \leq \frac{\varepsilon}{m}, \quad (31bm9)$$

and in particular the disagreement set is contained in the certified window

$$\{t \in [0, T] : c_t^{(\alpha)} \neq \tilde{c}_t^{(\alpha)}\} \subseteq \left[ \tau(\alpha) - \frac{\varepsilon}{m}, \tau(\alpha) + \frac{\varepsilon}{m} \right] \cap [0, T]. \quad (31bma)$$

**Proof.** The parameter-separation estimate (31bm7) is exactly Corollary 9.19.

Fix  $\alpha \in U_0$ . Apply Proposition 9.16 on the time interval  $I$  to the pair of paths

$$y^{(1)}(t) := Y(\alpha, t), \quad y^{(2)}(t) := \tilde{Y}(\alpha, t). \quad (31bmb)$$

The crossing identities (31bm2), the uniform slope lower bound (31bm4), and the perturbation estimate (31bm5) yield

$$|\tau(\alpha) - \tilde{\tau}(\alpha)| \leq \frac{\varepsilon}{m}, \quad (31bmc)$$

which is (31bm8).

If the unique-crossing hypotheses of Corollary 9.13 hold, then Proposition 9.12 gives

$$\int_0^T |c_t^{(\alpha)} - \tilde{c}_t^{(\alpha)}| dt = |\tau(\alpha) - \tilde{\tau}(\alpha)|. \quad (31bmd)$$

Combining this with (31bmc) proves (31bm9).

Finally, the same unique-crossing description shows that the disagreement set is precisely the interval between the two crossing times. Since both crossing times lie within  $\varepsilon/m$  of  $\tau(\alpha)$  by (31bmc), the containment (31bma) follows.  $\square$

This proposition is the bookkeeping half of the larger theorem program. Once a visible margin  $\partial_\alpha Y \geq \lambda$  and a uniform perturbation budget  $\varepsilon$  are available, the rest of the diagnosis is geometric: the symbolic disagreement is forced into a certified crossing window of size  $\varepsilon/m$ , while parameter separation is inherited at rate  $\lambda/M$ .

At this point the bookkeeping layer can be treated as operationally closed. Its proof content separates into three one-dimensional components:

crossing-time shift + certified-window membership + symbolic disagreement geometry. (31bme)

Those components admit a stable Lean surface: the current formal spine already covers the crossing-time shift bound, mutual certified-window containment, symbolic disagreement localization, and the ordered disagreement-volume identity on finite windows. So the active frontier of the paper is no longer the window bookkeeping itself. It is the production of the visible margin that this proposition consumes.

**Corollary 9.20 (Flow-derivative bridge to observer-relative chaos).** Let  $X \subset \mathbb{R}^d$ , let  $\Phi_t : X \rightarrow X$  be a  $C^1$  flow generated by a vector field  $F$ , and let  $x : J \rightarrow X$  be a  $C^1$  curve of initial conditions. Define

$$u_t(\alpha) = \Phi_t(x(\alpha)), \quad Y(\alpha, t) = \kappa(u_t(\alpha)) - L. \quad (31bn)$$

Assume the threshold crossing time  $\tau(\alpha)$  is locally well-defined and transversal. Then

$$\partial_\alpha Y(\alpha, t) = D\kappa(u_t(\alpha)) D_x \Phi_t(x(\alpha)) x'(\alpha), \quad (31bo)$$

and therefore

$$\tau'(\alpha) = -\frac{D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) x'(\alpha)}{D\kappa(u_{\tau(\alpha)}(\alpha)) F(u_{\tau(\alpha)}(\alpha))}. \quad (31bp)$$

**Proof.** The chain rule gives

$$\partial_\alpha Y(\alpha, t) = D\kappa(u_t(\alpha)) \partial_\alpha u_t(\alpha). \quad (31bq)$$

Since  $u_t(\alpha) = \Phi_t(x(\alpha))$ , differentiating with respect to  $\alpha$  yields

$$\partial_\alpha u_t(\alpha) = D_x \Phi_t(x(\alpha)) x'(\alpha), \quad (31br)$$

which proves (31bo). Likewise, differentiation with respect to time gives

$$\partial_t Y(\alpha, t) = D\kappa(u_t(\alpha)) \partial_t u_t(\alpha) = D\kappa(u_t(\alpha)) F(u_t(\alpha)). \quad (31bs)$$

Substituting these expressions into Proposition 9.18 yields the formula (31bp).  $\square$

In particular, if the observed tangent amplification in the numerator becomes large while the denominator stays bounded away from 0, then the crossing-time map becomes highly sensitive. By Proposition 9.12, this produces long symbolic disagreement intervals from small initial-condition perturbations.

This is the paper's clearest local formula for **observer-relative chaos**: the symbolic instability inherits a factor from the latent flow derivative, but it is modulated by the observation lens and blown up further when the threshold crossing is nearly grazing.

**Proposition 9.21 (One-sided observable growth implies symbolic amplification).** Let  $X \subset \mathbb{R}^d$ , let  $\Phi_t : X \rightarrow X$  be a  $C^1$  flow, let  $\kappa \in C^1(X)$ , and fix a threshold  $L \in \mathbb{R}$ . Let  $x : U \rightarrow X$  be a  $C^1$  curve of initial conditions, where  $U$  is an interval containing 0. Define

$$u_t(\alpha) = \Phi_t(x(\alpha)), \quad Y(\alpha, t) = \kappa(u_t(\alpha)) - L. \quad (31bt)$$

Assume there exists a  $C^1$  crossing-time branch  $\tau : U \rightarrow (0, \infty)$  such that

$$Y(\alpha, \tau(\alpha)) = 0 \quad \text{for all } \alpha \in U. \quad (31bu)$$

Suppose further that there exist constants  $C > 0$ ,  $\lambda > 0$ , and  $M > 0$  such that for all  $\alpha \in U$ ,

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq C e^{\lambda \tau(\alpha)}, \quad (31bv)$$

and

$$0 < \partial_t Y(\alpha, \tau(\alpha)) \leq M. \quad (31bw)$$

Then

$$\tau'(\alpha) \leq -\frac{C}{M} e^{\lambda \tau(\alpha)}. \quad (31bx)$$

In particular, for each  $\alpha_0 \in U$ , there exists a neighborhood  $U_{\alpha_0}$  and a constant  $c_{\alpha_0} > 0$  such that

$$|\tau(\alpha) - \tau(\beta)| \geq c_{\alpha_0} e^{\lambda \tau(\alpha_0)} |\alpha - \beta|, \quad \alpha, \beta \in U_{\alpha_0}. \quad (31by)$$

If, in addition, the symbolic processes

$$c_t^{(\alpha)} = \mathbf{1}\{\kappa(\Phi_t(x(\alpha))) \geq L\} \quad (31bz)$$

have unique threshold crossings in  $[0, T]$ , then

$$\int_0^T |c_t^{(\alpha)} - c_t^{(\beta)}| dt \geq c_{\alpha_0} e^{\lambda\tau(\alpha_0)} |\alpha - \beta|, \quad \alpha, \beta \in U_{\alpha_0}. \quad (31ca)$$

**Proof.** By Proposition 9.18,

$$\tau'(\alpha) = -\frac{\partial_\alpha Y(\alpha, \tau(\alpha))}{\partial_t Y(\alpha, \tau(\alpha))} \leq -\frac{C}{M} e^{\lambda\tau(\alpha)}. \quad (31cb)$$

Fix  $\alpha_0 \in U$ . Since  $\tau$  is continuous, there exists a neighborhood  $U_{\alpha_0}$  and a constant  $b_{\alpha_0} > 0$  such that

$$e^{\lambda\tau(\alpha)} \geq b_{\alpha_0} e^{\lambda\tau(\alpha_0)}, \quad \alpha \in U_{\alpha_0}. \quad (31cc)$$

Hence

$$\tau'(\alpha) \leq -\frac{C b_{\alpha_0}}{M} e^{\lambda\tau(\alpha_0)}, \quad \alpha \in U_{\alpha_0}. \quad (31cd)$$

Setting

$$c_{\alpha_0} := \frac{C b_{\alpha_0}}{M}, \quad (31ce)$$

integrating over an interval between  $\alpha$  and  $\beta$  gives (31by), and the symbolic lower bound then follows from Proposition 9.12.  $\square$

**Proposition 9.22 (Aligned visible unstable direction implies symbolic amplification).**

In the setting of Proposition 9.21, assume there exists a  $C^1$  unit vector field  $e(\alpha)$  along the initial curve and constants  $a_0 > 0$ ,  $C_u > 0$ ,  $\lambda > 0$ ,  $c_o > 0$ , and  $M > 0$  such that

$$x'(\alpha) = a(\alpha)e(\alpha), \quad a(\alpha) \geq a_0 \quad \text{for all } \alpha \in U, \quad (31cf)$$

$$\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e(\alpha)\| \geq C_u e^{\lambda\tau(\alpha)}, \quad (31cg)$$

$$\hat{e}_\tau(\alpha) := \frac{D_x \Phi_{\tau(\alpha)}(x(\alpha)) e(\alpha)}{\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e(\alpha)\|}, \quad (31ch)$$

$$D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_\tau(\alpha) \geq c_o, \quad (31ci)$$

and

$$0 < D\kappa(u_{\tau(\alpha)}(\alpha)) F(u_{\tau(\alpha)}(\alpha)) \leq M. \quad (31cj)$$

Then the conclusions of Proposition 9.21 hold with  $C = a_0 c_o C_u$ .

**Proof.** Using Corollary 9.20 and the alignment assumption,

$$\partial_\alpha Y(\alpha, \tau(\alpha)) = D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) x'(\alpha) \quad (31cj)$$

$$= a(\alpha) D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) e(\alpha). \quad (31ck)$$

Writing the transported vector in normalized form gives

$$\partial_\alpha Y(\alpha, \tau(\alpha)) = a(\alpha) (D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_\tau(\alpha)) \|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e(\alpha)\|. \quad (31cl)$$

Applying (31cf), (31cg), and (31ci), we obtain

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq a_0 c_o C_u e^{\lambda\tau(\alpha)}. \quad (31cm)$$

Thus Proposition 9.21 applies with  $C = a_0 c_o C_u$ .  $\square$

**Proposition 9.23 (One-dimensional unstable projection with controlled remainder).** In the setting of Proposition 9.21, assume  $X \subset \mathbb{R}^d$  and there exists a  $C^1$  unit vector field  $e_u(\alpha)$  along the initial curve and a decomposition

$$x'(\alpha) = a_u(\alpha) e_u(\alpha) + r(\alpha), \quad a_u(\alpha) \geq a_0 > 0, \quad (31cn)$$

for all  $\alpha \in U$ . Assume further that there exist constants  $C_u > 0$ ,  $\lambda > 0$ ,  $c_o > 0$ ,  $B \geq 0$ , and  $M > 0$  such that

$$\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\| \geq C_u e^{\lambda\tau(\alpha)}, \quad (31co)$$

$$\hat{e}_{u,\tau}(\alpha) := \frac{D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)}{\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\|}, \quad (31cp)$$

$$D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_{u,\tau}(\alpha) \geq c_o, \quad (31cq)$$

$$D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) r(\alpha) \geq -B e^{\lambda\tau(\alpha)}, \quad (31cr)$$

and

$$0 < D\kappa(u_{\tau(\alpha)}(\alpha)) F(u_{\tau(\alpha)}(\alpha)) \leq M. \quad (31cs)$$

If

$$C_* := a_0 c_o C_u - B > 0, \quad (31ct)$$

then the conclusions of Proposition 9.21 hold with  $C = C_*$ .

**Proof.** Using Corollary 9.20 and the decomposition of  $x'(\alpha)$ , we obtain

$$\partial_\alpha Y(\alpha, \tau(\alpha)) = D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) x'(\alpha) \quad (31cu)$$

$$= a_u(\alpha) D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha) + D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) r(\alpha). \quad (31cv)$$

Writing the transported unstable direction in normalized form gives

$$a_u(\alpha) D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha) = a_u(\alpha) (D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_{u,\tau}(\alpha)) \|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\|. \quad (31cw)$$

Hence, by (31cn), (31co), and (31cq),

$$a_u(\alpha) (D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_{u,\tau}(\alpha)) \|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\| \geq a_0 c_o C_u e^{\lambda\tau(\alpha)}. \quad (31cx)$$

Combining this with the remainder bound (31cr), we find

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq (a_0 c_o C_u - B) e^{\lambda\tau(\alpha)} = C_* e^{\lambda\tau(\alpha)}. \quad (31cy)$$

Therefore Proposition 9.21 applies with  $C = C_*$ .  $\square$

When  $\|x'(\alpha)\| = 1$  and the decomposition in (31cn) is orthogonal relative to the one-dimensional unstable bundle  $E^u(\alpha) = \text{span}\{e_u(\alpha)\}$ , the lower bound  $a_u(\alpha) \geq a_0$  is equivalent to a uniform angle condition between the parameter direction and the unstable bundle.

**Proposition 9.24 (Bundle domination yields eventual symbolic amplification).** In the setting of Proposition 9.21, assume  $X \subset \mathbb{R}^d$ , and let

$$x'(\alpha) = a_u(\alpha)e_u(\alpha) + r_c(\alpha), \quad a_u(\alpha) \geq a_0 > 0, \quad (31ea)$$

where  $e_u(\alpha)$  is a  $C^1$  unit vector field and  $r_c(\alpha)$  lies in a complementary subspace  $E^c(\alpha)$ . Assume also that

$$\|r_c(\alpha)\| \leq R_0 \quad \text{for all } \alpha \in U. \quad (31eb)$$

Suppose there exist constants  $C_u > 0$ ,  $C_c > 0$ ,  $\lambda > \mu \geq 0$ ,  $c_o > 0$ ,  $L_\kappa > 0$ , and  $M > 0$  such that

$$\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\| \geq C_u e^{\lambda\tau(\alpha)}, \quad (31ec)$$

$$\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) v\| \leq C_c e^{\mu\tau(\alpha)} \|v\| \quad \text{for all } v \in E^c(\alpha), \quad (31ed)$$

$$\hat{e}_{u,\tau}(\alpha) := \frac{D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)}{\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\|}, \quad (31ee)$$

$$D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_{u,\tau}(\alpha) \geq c_o, \quad (31ef)$$

$$\|D\kappa(u_{\tau(\alpha)}(\alpha))\|_{\text{op}} \leq L_\kappa, \quad (31eg)$$

and

$$0 < D\kappa(u_{\tau(\alpha)}(\alpha)) F(u_{\tau(\alpha)}(\alpha)) \leq M. \quad (31eh)$$

Assume further that there exists  $\tau_* \geq 0$  such that

$$\tau(\alpha) \geq \tau_* \quad \text{for all } \alpha \in U, \quad (31ei)$$

and

$$L_\kappa C_c R_0 e^{\mu\tau_*} \leq \frac{1}{2} a_0 c_o C_u e^{\lambda\tau_*}. \quad (31ej)$$

Then the conclusions of Proposition 9.21 hold with

$$C = \frac{1}{2} a_0 c_o C_u. \quad (31ek)$$

**Proof.** By Corollary 9.20,

$$\partial_\alpha Y(\alpha, \tau(\alpha)) = D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) x'(\alpha). \quad (31dk)$$

Using the decomposition (31ea), this becomes

$$\partial_\alpha Y(\alpha, \tau(\alpha)) = a_u(\alpha) D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha) + D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) r_c(\alpha). \quad (31dl)$$

The unstable contribution satisfies

$$a_u(\alpha) D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha) \geq a_0 c_o C_u e^{\lambda\tau(\alpha)} \quad (31dm)$$

by the same normalized-direction argument used in Proposition 9.23 together with (31ec) and (31ef). For the complementary term, (31ed), (31eg), and (31eb) give

$$\left| D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) r_c(\alpha) \right| \leq L_\kappa C_c R_0 e^{\mu\tau(\alpha)}. \quad (31dn)$$

Hence

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq a_0 c_o C_u e^{\lambda\tau(\alpha)} - L_\kappa C_c R_0 e^{\mu\tau(\alpha)}. \quad (31do)$$

Since  $\lambda > \mu$ , the ratio

$$e^{-(\lambda-\mu)\tau} \quad (31dp)$$

decreases with  $\tau$ . Therefore (31ei) and (31ej) imply

$$L_\kappa C_c R_0 e^{\mu\tau(\alpha)} \leq \frac{1}{2} a_0 c_o C_u e^{\lambda\tau(\alpha)}. \quad (31dq)$$

Substituting this into (31do) yields

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq \frac{1}{2} a_0 c_o C_u e^{\lambda\tau(\alpha)}. \quad (31dr)$$

Thus Proposition 9.21 applies with  $C = \frac{1}{2} a_0 c_o C_u$ .  $\square$

This proposition shows how the controlled-remainder condition in Proposition 9.23 can be replaced by a more dynamical domination mechanism: the unstable component grows at rate  $e^{\lambda\tau}$ , the complementary component grows only at rate  $e^{\mu\tau}$ , and beyond a sufficiently late crossing time the unstable contribution necessarily dominates.

**Proposition 9.25 (Dominated splitting with unstable projection implies eventual amplification).** In the setting of Proposition 9.21, assume  $X \subset \mathbb{R}^d$ , and for each  $\alpha \in U$  there is a direct-sum decomposition

$$T_{x(\alpha)}X = E^u(\alpha) \oplus E^c(\alpha), \quad \dim E^u(\alpha) = 1, \quad (31et)$$

with associated projections  $P_\alpha^u$  and  $P_\alpha^c$ . Assume there exists a constant  $P_* \geq 1$  such that

$$\|P_\alpha^u\|_{\text{op}} \leq P_*, \quad \|P_\alpha^c\|_{\text{op}} \leq P_* \quad \text{for all } \alpha \in U. \quad (31eu)$$

Let  $e_u(\alpha)$  be a  $C^1$  unit vector field spanning  $E^u(\alpha)$ , and assume

$$P_\alpha^u x'(\alpha) = a_u(\alpha) e_u(\alpha), \quad a_u(\alpha) \geq a_0 > 0, \quad (31ev)$$

$$\|x'(\alpha)\| \leq X_0 \quad \text{for all } \alpha \in U. \quad (31ew)$$

Suppose there exist constants  $C_u > 0$ ,  $C_c > 0$ ,  $\lambda > \mu \geq 0$ ,  $c_o > 0$ ,  $L_\kappa > 0$ , and  $M > 0$  such that

$$\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\| \geq C_u e^{\lambda\tau(\alpha)}, \quad (31ex)$$

$$\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) v\| \leq C_c e^{\mu\tau(\alpha)} \|v\| \quad \text{for all } v \in E^c(\alpha), \quad (31ey)$$

$$\hat{e}_{u,\tau}(\alpha) := \frac{D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)}{\|D_x \Phi_{\tau(\alpha)}(x(\alpha)) e_u(\alpha)\|}, \quad (31ez)$$

$$D\kappa(u_{\tau(\alpha)}(\alpha)) \hat{e}_{u,\tau}(\alpha) \geq c_o, \quad (31fa)$$

$$\|D\kappa(u_{\tau(\alpha)}(\alpha))\|_{\text{op}} \leq L_\kappa, \quad (31fb)$$

$$0 < D\kappa(u_{\tau(\alpha)}(\alpha)) F(u_{\tau(\alpha)}(\alpha)) \leq M, \quad (31fc)$$

and

$$\tau(\alpha) \geq \tau_* \quad \text{for all } \alpha \in U. \quad (31fd)$$

If

$$L_\kappa C_c P_* X_0 e^{\mu\tau_*} \leq \frac{1}{2} a_0 c_o C_u e^{\lambda\tau_*}, \quad (31fe)$$

then the conclusions of Proposition 9.21 hold with

$$C = \frac{1}{2} a_0 c_o C_u. \quad (31ff)$$

**Proof.** Define

$$r_c(\alpha) := P_\alpha^c x'(\alpha). \quad (31fg)$$

Then

$$x'(\alpha) = P_\alpha^u x'(\alpha) + P_\alpha^c x'(\alpha) = a_u(\alpha) e_u(\alpha) + r_c(\alpha). \quad (31fh)$$

By (31eu) and (31ew),

$$\|r_c(\alpha)\| \leq \|P_\alpha^c\|_{\text{op}} \|x'(\alpha)\| \leq P_* X_0. \quad (31fi)$$

Thus Proposition 9.24 applies with

$$R_0 := P_* X_0, \quad (31fj)$$

and the hypothesis (31fe) is exactly the domination condition required there. Therefore the conclusions of Proposition 9.21 follow with  $C = \frac{1}{2} a_0 c_o C_u$ .  $\square$

This formulation is closer to the real bundle geometry of the problem: the unstable projection of the parameter direction is assumed directly, while the complementary contribution is controlled via the bundle projection norm and the slower complementary growth rate.

**Corollary 9.26 (Derived domination time).** In the setting of Proposition 9.25, define

$$T_{\text{dom}} := \max \left\{ 0, \frac{1}{\lambda - \mu} \log \left( \frac{2L_\kappa C_c P_* X_0}{a_0 c_o C_u} \right) \right\}. \quad (31ga)$$

Then for every  $\tau \geq T_{\text{dom}}$ ,

$$L_\kappa C_c P_* X_0 e^{\mu\tau} \leq \frac{1}{2} a_0 c_o C_u e^{\lambda\tau}. \quad (31gb)$$

Consequently, if the crossing branch satisfies

$$\tau(\alpha) \geq T_{\text{dom}} \quad \text{for all } \alpha \in U, \quad (31gc)$$

then the conclusions of Proposition 9.21 hold with

$$C = \frac{1}{2} a_0 c_o C_u. \quad (31gd)$$

**Proof.** By definition of  $T_{\text{dom}}$ , we have

$$e^{(\lambda-\mu)T_{\text{dom}}} \geq \frac{2L_\kappa C_c P_* X_0}{a_0 c_o C_u}. \quad (31ge)$$

Therefore, for every  $\tau \geq T_{\text{dom}}$ ,

$$e^{(\lambda-\mu)\tau} \geq e^{(\lambda-\mu)T_{\text{dom}}} \geq \frac{2L_\kappa C_c P_* X_0}{a_0 c_o C_u}. \quad (31gf)$$

Rearranging gives (31gb), and Proposition 9.25 then yields the claimed amplification bound.  $\square$

This corollary makes the late-time threshold explicit. The domination time is not an ad hoc human choice; it is the scale at which unstable growth overtakes the maximal complementary contribution allowed by the bundle geometry and the observation lens.

**Corollary 9.27 (Orthogonal bundle splitting with angle criterion).** In the setting of Proposition 9.25, assume in addition that the splitting

$$T_{x(\alpha)}X = E^u(\alpha) \oplus E^c(\alpha) \quad (31gh)$$

is orthogonal for each  $\alpha$ , and that there exist constants  $x_* > 0$  and  $0 \leq \theta_* < \pi/2$  such that

$$\|x'(\alpha)\| \geq x_* \quad \text{for all } \alpha \in U, \quad (31gi)$$

and the angle between  $x'(\alpha)$  and  $E^u(\alpha)$  satisfies

$$\angle(x'(\alpha), E^u(\alpha)) \leq \theta_* \quad \text{for all } \alpha \in U. \quad (31gj)$$

Then Proposition 9.25 applies with

$$a_0 = x_* \cos \theta_*. \quad (31gk)$$

**Proof.** Since the splitting is orthogonal and  $e_u(\alpha)$  spans  $E^u(\alpha)$ , the unstable projection has magnitude

$$a_u(\alpha) = \|P_\alpha^u x'(\alpha)\| = \|x'(\alpha)\| \cos(\angle(x'(\alpha), E^u(\alpha))). \quad (31gl)$$

Using (31gi) and (31gj), we obtain

$$a_u(\alpha) \geq x_* \cos \theta_*. \quad (31gm)$$

Thus the lower-bound hypothesis in Proposition 9.25 holds with  $a_0 = x_* \cos \theta_*$ .  $\square$

This corollary translates the bundle criterion into a directly geometric statement: if the parameter direction is never too small and never too close to the complementary bundle, then its unstable component is uniformly nontrivial.

**Corollary 9.28 (Derived domination time under angle criterion).** Under the hypotheses of Corollaries 9.26 and 9.27, define

$$T_{\text{ang}} := \max \left\{ 0, \frac{1}{\lambda - \mu} \log \left( \frac{2L_\kappa C_c P_* X_0}{x_* \cos \theta_* c_o C_u} \right) \right\}. \quad (31gn)$$

If

$$\tau(\alpha) \geq T_{\text{ang}} \quad \text{for all } \alpha \in U, \quad (31go)$$

then the conclusions of Proposition 9.21 hold with

$$C = \frac{1}{2} x_* \cos \theta_* c_o C_u. \quad (31gp)$$

**Proof.** By Corollary 9.27, Proposition 9.25 applies with  $a_0 = x_* \cos \theta_*$ . Substituting this value into Corollary 9.26 gives the stated domination time and amplification constant.  $\square$

This is the cleanest geometric packaging so far: angle, unstable growth, slower complementary growth, and lens visibility together imply symbolic amplification once the crossing time passes a derived threshold.

**Proposition 9.29 (Pointwise coefficient-gap criterion).** In the setting of Proposition 9.25, but without assuming the late-time hypotheses (31fd) and (31fe), define

$$G(\alpha) := a_u(\alpha)c_oC_u - L_\kappa C_c \|P_\alpha^c x'(\alpha)\| e^{-(\lambda-\mu)\tau(\alpha)}. \quad (31gq)$$

If there exists a constant  $G_* > 0$  such that

$$G(\alpha) \geq G_* \quad \text{for all } \alpha \in U, \quad (31gr)$$

then the conclusions of Proposition 9.21 hold with

$$C = G_*. \quad (31gs)$$

**Proof.** As in the proof of Proposition 9.25, write

$$x'(\alpha) = a_u(\alpha)e_u(\alpha) + P_\alpha^c x'(\alpha). \quad (31gt)$$

Then Corollary 9.20 gives

$$\partial_\alpha Y(\alpha, \tau(\alpha)) = D\kappa(u_{\tau(\alpha)}(\alpha)) D_x \Phi_{\tau(\alpha)}(x(\alpha)) x'(\alpha). \quad (31gu)$$

Using (31gt), the unstable-growth bound (31ex), the visibility bound (31fa), the complementary-growth bound (31ey), and the lens norm bound (31fb), we obtain

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq a_u(\alpha)c_oC_u e^{\lambda\tau(\alpha)} - L_\kappa C_c \|P_\alpha^c x'(\alpha)\| e^{\mu\tau(\alpha)}. \quad (31gv)$$

Factoring out  $e^{\lambda\tau(\alpha)}$  yields

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq G(\alpha) e^{\lambda\tau(\alpha)}. \quad (31gw)$$

By (31gr),

$$\partial_\alpha Y(\alpha, \tau(\alpha)) \geq G_* e^{\lambda\tau(\alpha)}. \quad (31gx)$$

Therefore Proposition 9.21 applies with  $C = G_*$ .  $\square$

This proposition isolates the genuinely relevant coefficient: not the raw crossing time by itself, but the observable unstable contribution minus the exponentially discounted complementary contribution.

**Corollary 9.30 (Compact positive gap yields uniform amplification).** In the setting of Proposition 9.29, let  $I \Subset U$  be a compact interval. Assume  $G$  is continuous on  $I$  and satisfies

$$G(\alpha) > 0 \quad \text{for all } \alpha \in I. \quad (31gy)$$

Then

$$G_I := \min_{\alpha \in I} G(\alpha) > 0, \quad (31gz)$$

and the conclusions of Proposition 9.21 hold on the interior interval  $I^\circ$  with

$$C = G_I. \quad (31ha)$$

**Proof.** Since  $G$  is continuous on the compact interval  $I$ , it attains its minimum there. The positivity assumption (31gy) implies that this minimum is strictly positive, giving (31gz). Applying Proposition 9.29 on  $I^\circ$  yields the stated conclusion with  $C = G_I$ .  $\square$

This compactness step removes the need to hand-insert a single global time threshold into the statement. What remains is the real dynamical problem: show that the intrinsic coefficient gap  $G(\alpha)$  is positive along the relevant crossing branch for structural reasons, not by external tuning.

**Proposition 9.31 (Delayed crossing branch yields local positive gap).** In the setting of Proposition 9.25, but without assuming the late-time hypotheses (31fd) and (31fe), assume there exists a subinterval  $J \subset U$  and a boundary point

$$\alpha_{\partial} \in \bar{J} \setminus J \quad (31hb)$$

such that

$$\tau(\alpha) \rightarrow \infty \quad \text{as } \alpha \rightarrow \alpha_{\partial} \text{ within } J. \quad (31hc)$$

Then there exists a subinterval  $J' \subset J$  with  $\alpha_{\partial} \in \bar{J}'$  such that

$$G(\alpha) \geq \frac{1}{2} a_0 c_o C_u \quad \text{for all } \alpha \in J'. \quad (31hd)$$

Consequently, the conclusions of Proposition 9.21 hold on  $J'$  with

$$C = \frac{1}{2} a_0 c_o C_u. \quad (31he)$$

**Proof.** By Proposition 9.25, the unstable projection satisfies  $a_u(\alpha) \geq a_0$ . Moreover, by (31eu) and (31ew),

$$\|P_{\alpha}^c x'(\alpha)\| \leq \|P_{\alpha}^c\|_{\text{op}} \|x'(\alpha)\| \leq P_* X_0 \quad \text{for all } \alpha \in U. \quad (31hf)$$

Hence

$$G(\alpha) \geq a_0 c_o C_u - L_{\kappa} C_c P_* X_0 e^{-(\lambda-\mu)\tau(\alpha)}. \quad (31hg)$$

Since  $\lambda > \mu$ , the factor  $e^{-(\lambda-\mu)\tau(\alpha)}$  tends to 0 as  $\alpha \rightarrow \alpha_{\partial}$  within  $J$  by (31hc). Therefore there exists a subinterval  $J' \subset J$  with  $\alpha_{\partial} \in \bar{J}'$  such that

$$L_{\kappa} C_c P_* X_0 e^{-(\lambda-\mu)\tau(\alpha)} \leq \frac{1}{2} a_0 c_o C_u \quad \text{for all } \alpha \in J'. \quad (31hh)$$

Combining (31hg) and (31hh) gives (31hd). Proposition 9.29 then yields the claimed amplification bound on  $J'$ .  $\square$

This proposition is the first threshold-free local step in the ladder. No explicit lower bound on  $\tau(\alpha)$  is assumed in advance; the delayed branch itself forces the discounted complementary term to become negligible near the boundary point where the crossing time diverges.

**Corollary 9.32 (Delayed crossing branch under angle criterion).** In the setting of Corollary 9.27, assume there exists a subinterval  $J \subset U$  and a boundary point

$$\alpha_{\partial} \in \bar{J} \setminus J \quad (31hi)$$

such that

$$\tau(\alpha) \rightarrow \infty \quad \text{as } \alpha \rightarrow \alpha_{\partial} \text{ within } J. \quad (31hj)$$

Then there exists a subinterval  $J' \subset J$  with  $\alpha_{\partial} \in \bar{J}'$  such that the conclusions of Proposition 9.21 hold on  $J'$  with

$$C = \frac{1}{2} x_* \cos \theta_* c_o C_u. \quad (31hk)$$

**Proof.** By Corollary 9.27, Proposition 9.25 applies with  $a_0 = x_* \cos \theta_*$ . Proposition 9.31 therefore gives the conclusion on a sufficiently small delayed-branch subinterval  $J'$ .  $\square$

This is the cleanest local no-threshold statement obtained so far: orthogonal bundle geometry plus a delayed crossing branch already produce symbolic amplification near the delayed endpoint, without inserting an external domination time into the assumptions.

**Proposition 9.33 (Visible unstable cone domination implies positive gap).** In the setting of Proposition 9.25, but without assuming the late-time hypotheses (31fd) and (31fe), assume there exists a constant  $\eta \geq 0$  such that

$$\|P_\alpha^c x'(\alpha)\| \leq \eta a_u(\alpha) \quad \text{for all } \alpha \in U. \quad (31hl)$$

If

$$L_\kappa C_c \eta < c_o C_u, \quad (31hm)$$

then

$$G(\alpha) \geq a_0(c_o C_u - L_\kappa C_c \eta) > 0 \quad \text{for all } \alpha \in U, \quad (31hn)$$

and the conclusions of Proposition 9.21 hold with

$$C = a_0(c_o C_u - L_\kappa C_c \eta). \quad (31ho)$$

**Proof.** By definition of  $G(\alpha)$ ,

$$G(\alpha) = a_u(\alpha)c_o C_u - L_\kappa C_c \|P_\alpha^c x'(\alpha)\| e^{-(\lambda-\mu)\tau(\alpha)}. \quad (31hp)$$

Since  $\lambda > \mu$  and  $\tau(\alpha) > 0$ , we have

$$e^{-(\lambda-\mu)\tau(\alpha)} \leq 1. \quad (31hq)$$

Using (31hl), we therefore obtain

$$G(\alpha) \geq a_u(\alpha)c_o C_u - L_\kappa C_c \eta a_u(\alpha) = a_u(\alpha)(c_o C_u - L_\kappa C_c \eta). \quad (31hr)$$

By (31ev),

$$G(\alpha) \geq a_0(c_o C_u - L_\kappa C_c \eta). \quad (31hs)$$

The strict positivity follows from (31hm), giving (31hn). Proposition 9.29 now yields the claimed amplification constant.  $\square$

This is the first genuinely structural positivity criterion for  $G(\alpha)$ : the complementary component is not made small by waiting for a delayed crossing, but by requiring that the parameter direction remain inside a visible unstable cone whose complementary leakage is dominated at the coefficient level.

**Corollary 9.34 (Orthogonal angle cone criterion).** In the setting of Corollary 9.27, but without assuming the late-time hypotheses of Corollary 9.28, assume

$$\angle(x'(\alpha), E^u(\alpha)) \leq \theta_* < \frac{\pi}{2} \quad \text{for all } \alpha \in U, \quad (31ht)$$

and

$$L_\kappa C_c \tan \theta_* < c_o C_u. \quad (31hu)$$

Then the conclusions of Proposition 9.21 hold with

$$C = x_* \cos \theta_* (c_o C_u - L_\kappa C_c \tan \theta_*). \quad (31hv)$$

**Proof.** By Corollary 9.27, Proposition 9.25 applies with  $a_0 = x_* \cos \theta_*$ . Because the splitting is orthogonal,

$$\frac{\|P_\alpha^c x'(\alpha)\|}{a_u(\alpha)} = \tan(\angle(x'(\alpha), E^u(\alpha))) \leq \tan \theta_* \quad \text{for all } \alpha \in U. \quad (31hw)$$

Thus Proposition 9.33 applies with  $\eta = \tan \theta_*$ , and its conclusion becomes exactly (31hv).  $\square$

This corollary is the cleanest threshold-free global statement obtained so far: orthogonal bundle geometry, uniform angle control, and visible unstable dominance imply symbolic amplification without any late-time cutoff and without any delayed-endpoint asymptotic.

**Proposition 9.35 (Bundle-preserving transport of a seed cone implies positive gap).** In the setting of Proposition 9.25, but without assuming the late-time hypotheses (31fd) and (31fe), assume there exists a normed vector space  $V = V^u \oplus V^c$  with  $\dim V^u = 1$ , a unit vector  $e_u^0 \in V^u$ , and for each  $\alpha \in U$  a linear map

$$T_\alpha : V \rightarrow T_{x(\alpha)} X \quad (31ia)$$

such that

$$T_\alpha(V^u) = E^u(\alpha), \quad T_\alpha(V^c) = E^c(\alpha). \quad (31ib)$$

Assume there exist constants  $m_u > 0$ ,  $M_c > 0$ ,  $b_0 > 0$ , and  $R_0 \geq 0$  such that

$$\|T_\alpha e_u^0\| \geq m_u \quad \text{for all } \alpha \in U, \quad (31ic)$$

$$\|T_\alpha v\| \leq M_c \|v\| \quad \text{for all } v \in V^c \text{ and } \alpha \in U, \quad (31id)$$

and

$$x'(\alpha) = T_\alpha \xi(\alpha), \quad \xi(\alpha) = b_u(\alpha) e_u^0 + \xi_c(\alpha), \quad (31ie)$$

with

$$b_u(\alpha) \geq b_0 > 0, \quad \|\xi_c(\alpha)\| \leq R_0 \quad \text{for all } \alpha \in U. \quad (31if)$$

If

$$L_\kappa C_c M_c R_0 < b_0 m_u c_o C_u, \quad (31ig)$$

then

$$G(\alpha) \geq b_0 m_u c_o C_u - L_\kappa C_c M_c R_0 > 0 \quad \text{for all } \alpha \in U, \quad (31ih)$$

and the conclusions of Proposition 9.21 hold with

$$C = b_0 m_u c_o C_u - L_\kappa C_c M_c R_0. \quad (31ii)$$

**Proof.** Because  $T_\alpha$  preserves the splitting, (31ie) gives the decomposition

$$x'(\alpha) = b_u(\alpha) T_\alpha e_u^0 + T_\alpha \xi_c(\alpha), \quad (31ij)$$

where the first term lies in  $E^u(\alpha)$  and the second in  $E^c(\alpha)$ . Let

$$e_u(\alpha) := \frac{T_\alpha e_u^0}{\|T_\alpha e_u^0\|}. \quad (31ik)$$

Then

$$P_\alpha^u x'(\alpha) = b_u(\alpha) \|T_\alpha e_u^0\| e_u(\alpha), \quad (31il)$$

so the unstable coefficient satisfies

$$a_u(\alpha) = b_u(\alpha) \|T_\alpha e_u^0\| \geq b_0 m_u. \quad (31im)$$

For the complementary part,

$$\|P_\alpha^c x'(\alpha)\| = \|T_\alpha \xi_c(\alpha)\| \leq M_c R_0. \quad (31in)$$

Therefore Proposition 9.33 applies with

$$a_0 := b_0 m_u, \quad \eta := \frac{M_c R_0}{b_0 m_u}, \quad (31io)$$

and its positivity condition is exactly (31ig). Substituting these values into Proposition 9.33 yields (31ih) and (31ii).  $\square$

This proposition turns the cone criterion into a transport criterion: positivity of the observable gap is now derived from how a seed unstable direction and a seed complementary remainder are transported into the moving bundle splitting along the initial-condition curve.

**Corollary 9.36 (Orthogonal seed-angle transport criterion).** In the setting of Proposition 9.35, assume  $V = V^u \oplus V^c$  is an orthogonal splitting and there exist constants  $\xi_* > 0$ ,  $\Xi^* > 0$ , and  $0 \leq \theta_0 < \pi/2$  such that

$$\xi_* \leq \|\xi(\alpha)\| \leq \Xi^* \quad \text{for all } \alpha \in U, \quad (31ip)$$

and

$$\angle(\xi(\alpha), V^u) \leq \theta_0 \quad \text{for all } \alpha \in U. \quad (31iq)$$

If

$$L_\kappa C_c M_c \Xi^* \sin \theta_0 < \xi_* m_u \cos \theta_0 c_o C_u, \quad (31ir)$$

then the conclusions of Proposition 9.21 hold with

$$C = \xi_* m_u \cos \theta_0 c_o C_u - L_\kappa C_c M_c \Xi^* \sin \theta_0. \quad (31is)$$

**Proof.** Orthogonality gives

$$b_u(\alpha) = \|\xi(\alpha)\| \cos(\angle(\xi(\alpha), V^u)) \geq \xi_* \cos \theta_0, \quad (31it)$$

and

$$\|\xi_c(\alpha)\| = \|\xi(\alpha)\| \sin(\angle(\xi(\alpha), V^u)) \leq \Xi^* \sin \theta_0. \quad (31iu)$$

Thus Proposition 9.35 applies with

$$b_0 := \xi_* \cos \theta_0, \quad R_0 := \Xi^* \sin \theta_0, \quad (31iv)$$

and its conclusion becomes exactly (31is).  $\square$

This is the cleanest transport-based structural criterion so far: a seed cone around the unstable direction, carried through a bundle-preserving transport with controlled complementary distortion, already forces positivity of  $G(\alpha)$ .

**Proposition 9.37 (Orthogonal bundle trivialization on a compact interval).** In the setting of Proposition 9.25, assume in addition that the splitting

$$T_{x(\alpha)}X = E^u(\alpha) \oplus E^c(\alpha) \quad (31iw)$$

is orthogonal and depends continuously on  $\alpha$  along a compact interval  $I \Subset U$ . Then there exist:

- a Euclidean vector space  $V = V^u \oplus V^c$  with  $\dim V^u = 1$ ,
- a unit vector  $e_u^0 \in V^u$ ,
- and a continuous family of linear isometries

$$T_\alpha : V \rightarrow T_{x(\alpha)}X, \quad \alpha \in I, \quad (31ix)$$

such that

$$T_\alpha(V^u) = E^u(\alpha), \quad T_\alpha(V^c) = E^c(\alpha) \quad \text{for all } \alpha \in I. \quad (31iy)$$

Consequently, Proposition 9.35 applies on  $I$  with

$$m_u = 1, \quad M_c = 1. \quad (31iz)$$

**Proof.** Because  $E^u(\alpha)$  is a continuous one-dimensional bundle over the compact interval  $I$ , there exists a continuous unit vector field  $e_u(\alpha)$  spanning  $E^u(\alpha)$ . Since  $E^c(\alpha)$  is a continuous orthogonal complementary bundle over the contractible base  $I$ , it admits a continuous orthonormal frame  $e_{c,1}(\alpha), \dots, e_{c,m}(\alpha)$ , where  $m = \dim E^c(\alpha)$ . Let

$$V := \mathbb{R}e_u^0 \oplus \mathbb{R}^m, \quad (31ja)$$

with its standard orthogonal splitting  $V^u \oplus V^c$ , and define  $T_\alpha$  by

$$T_\alpha e_u^0 := e_u(\alpha), \quad T_\alpha \varepsilon_j := e_{c,j}(\alpha), \quad 1 \leq j \leq m, \quad (31jb)$$

where  $\varepsilon_1, \dots, \varepsilon_m$  is the standard basis of  $V^c = \mathbb{R}^m$ . Then  $T_\alpha$  is a linear isometry onto  $T_{x(\alpha)}X$ , and (31iy) holds by construction. Therefore

$$\|T_\alpha e_u^0\| = 1, \quad \|T_\alpha v\| = \|v\| \quad \text{for all } v \in V^c, \quad (31jc)$$

which gives (31iz).  $\square$

This proposition removes a major source of arbitrariness from Proposition 9.35: on a compact interval with orthogonal continuous splitting, the bundle-preserving transport family is not extra structure but an automatic consequence of bundle triviality.

**Corollary 9.38 (Pointwise seed margin yields local positive gap).** In the setting of Proposition 9.37, define

$$\xi(\alpha) := T_\alpha^{-1}x'(\alpha), \quad \alpha \in I, \quad (31jd)$$

and write

$$\xi(\alpha) = b_u(\alpha)e_u^0 + \xi_c(\alpha), \quad b_u(\alpha) \in \mathbb{R}, \quad \xi_c(\alpha) \in V^c. \quad (31je)$$

Fix  $\alpha_0 \in I^\circ$ . If

$$b_u(\alpha_0) > 0 \quad (31jf)$$

and

$$L_\kappa C_c \|\xi_c(\alpha_0)\| < b_u(\alpha_0)c_o C_u, \quad (31jg)$$

then there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31jh)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** By continuity of  $T_\alpha$  and  $x'(\alpha)$ , the map  $\xi(\alpha) = T_\alpha^{-1}x'(\alpha)$  is continuous on  $I$ . Hence  $b_u(\alpha)$  and  $\|\xi_c(\alpha)\|$  are continuous at  $\alpha_0$ . Choose  $\varepsilon > 0$  such that

$$L_\kappa C_c (\|\xi_c(\alpha_0)\| + \varepsilon) < (b_u(\alpha_0) - \varepsilon) c_o C_u, \quad (31ji)$$

with  $b_u(\alpha_0) - \varepsilon > 0$ . By continuity, there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$b_u(\alpha) \geq b_u(\alpha_0) - \varepsilon, \quad \|\xi_c(\alpha)\| \leq \|\xi_c(\alpha_0)\| + \varepsilon \quad \text{for all } \alpha \in J. \quad (31jj)$$

Applying Proposition 9.35 on  $J$  with

$$b_0 := b_u(\alpha_0) - \varepsilon, \quad R_0 := \|\xi_c(\alpha_0)\| + \varepsilon, \quad (31jk)$$

and  $m_u = M_c = 1$ , we obtain (31jh) and the stated symbolic-amplification conclusion.  $\square$

This corollary sharply localizes the remaining obstruction: once the bundle is trivialized intrinsically, the entire theorem frontier reduces to producing a strict unstable-versus-complementary margin at a single point and showing that the dynamics propagate it.

**Corollary 9.38A (Pointwise intrinsic projection-ratio criterion).** In the setting of Proposition 9.37, fix  $\alpha_0 \in I^\circ$ . If

$$L_\kappa C_c \|P_{\alpha_0}^c x'(\alpha_0)\| < a_u(\alpha_0) c_o C_u, \quad (31jk0)$$

then there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31jk1)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Let

$$\xi(\alpha_0) = T_{\alpha_0}^{-1}x'(\alpha_0) := b_u(\alpha_0)e_u^0 + \xi_c(\alpha_0). \quad (31jk2)$$

Because  $T_{\alpha_0}$  is an isometry and preserves the splitting, the proof of Proposition 9.35 gives

$$a_u(\alpha_0) = b_u(\alpha_0), \quad \|P_{\alpha_0}^c x'(\alpha_0)\| = \|\xi_c(\alpha_0)\|. \quad (31jk3)$$

So the hypothesis (31jk0) is exactly the seed-margin inequality from Corollary 9.38. Applying that corollary gives (31jk1) and the stated conclusion.  $\square$

This is the cleanest one-point intrinsic reformulation of the local seed step in bundle language. The remaining local input is not the transported coordinate pair itself, but the coefficient comparison between unstable projection and complementary leakage at one point.

**Corollary 9.38B (Pointwise intrinsic visible-cone criterion).** In the setting of Proposition 9.37, fix  $\alpha_0 \in I^\circ$ . Assume

$$x'(\alpha_0) \neq 0, \quad \angle(x'(\alpha_0), E^u(\alpha_0)) = \vartheta_0 < \frac{\pi}{2}. \quad (31jk4)$$

If

$$L_\kappa C_c \tan \vartheta_0 < c_o C_u, \quad (31jk5)$$

then there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31jk6)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Let

$$\xi(\alpha_0) = T_{\alpha_0}^{-1}x'(\alpha_0) = b_u(\alpha_0)e_u^0 + \xi_c(\alpha_0). \quad (31jk7)$$

Because  $T_{\alpha_0}$  is an isometry carrying  $V^u$  onto  $E^u(\alpha_0)$  and  $V^c$  onto  $E^c(\alpha_0)$ , orthogonality gives

$$b_u(\alpha_0) = \|x'(\alpha_0)\| \cos \vartheta_0, \quad \|\xi_c(\alpha_0)\| = \|x'(\alpha_0)\| \sin \vartheta_0. \quad (31jk8)$$

Since  $x'(\alpha_0) \neq 0$  and  $\vartheta_0 < \pi/2$ , we have  $b_u(\alpha_0) > 0$ . Moreover,

$$L_\kappa C_c \|\xi_c(\alpha_0)\| = L_\kappa C_c \|x'(\alpha_0)\| \sin \vartheta_0 < \|x'(\alpha_0)\| \cos \vartheta_0 c_o C_u = b_u(\alpha_0) c_o C_u. \quad (31jk9)$$

So the hypotheses of Corollary 9.38 are satisfied, which yields (31jk6) and the claimed local amplification conclusion.  $\square$

This removes the last coordinate artifact from the local seed step. The decisive local object is not a chosen transported coefficient pair  $(b_u, \xi_c)$  by itself, but a one-point entry of the parameter direction into a visible unstable cone. Equivalently: once the parameter direction at some crossing lies strictly inside the observable unstable cone determined by

$$L_\kappa C_c \tan \vartheta < c_o C_u, \quad (31jk6a)$$

the local positive-gap neighborhood follows automatically.

**Proposition 9.38C (Equivalent local frontier forms).** In the setting of Proposition 9.37, fix  $\alpha_0 \in I^\circ$ , and let

$$\xi(\alpha_0) = b_u(\alpha_0)e_u^0 + \xi_c(\alpha_0), \quad \vartheta_0 := \angle(x'(\alpha_0), E^u(\alpha_0)). \quad (31jk6b)$$

Then the following are equivalent:

1. the seed-margin inequality

$$b_u(\alpha_0) > 0, \quad L_\kappa C_c \|\xi_c(\alpha_0)\| < b_u(\alpha_0) c_o C_u, \quad (31jk6c)$$

2. the intrinsic projection-ratio inequality

$$L_\kappa C_c \|P_{\alpha_0}^c x'(\alpha_0)\| < a_u(\alpha_0) c_o C_u, \quad (31jk6d)$$

3. the intrinsic visible-cone condition

$$x'(\alpha_0) \neq 0, \quad L_\kappa C_c \tan \vartheta_0 < c_o C_u. \quad (31jk6e)$$

**Proof.** The equivalence of (1) and (2) is exactly the identity pair from Corollary 9.38A. Under orthogonality and one-dimensionality of  $E^u(\alpha_0)$ , we also have

$$a_u(\alpha_0) = \|x'(\alpha_0)\| \cos \vartheta_0, \quad \|P_{\alpha_0}^c x'(\alpha_0)\| = \|x'(\alpha_0)\| \sin \vartheta_0. \quad (31jk6f)$$

Hence (2) is equivalent to

$$L_\kappa C_c \|x'(\alpha_0)\| \sin \vartheta_0 < \|x'(\alpha_0)\| \cos \vartheta_0 c_o C_u. \quad (31jk6g)$$

If (3) holds, then  $\cos \vartheta_0 > 0$  and multiplying by  $\|x'(\alpha_0)\| > 0$  gives (31jk6g). Conversely, if (31jk6g) holds then its right-hand side is positive, so  $x'(\alpha_0) \neq 0$  and  $\cos \vartheta_0 > 0$ ; dividing by  $\|x'(\alpha_0)\| \cos \vartheta_0$  yields (3).  $\square$

This proposition shows that the local frontier has three exactly equivalent one-point forms:

- transported seed-margin form,
- intrinsic projection-ratio form,
- and intrinsic visible-cone form.

So the remaining work is not to choose among local formulations, but to derive any one of them from genuine dynamics.

**Lemma Candidate 9.38D (Intrinsic seed-margin criterion).** In the setting of Proposition 9.37, keep the notation

$$\xi(\alpha) = b_u(\alpha)e_u^0 + \xi_c(\alpha). \quad (31jkb)$$

The real missing lemma is to derive, from intrinsic dynamics rather than from a hand-inserted local hypothesis, the existence of a point  $\alpha_0 \in I^\circ$  such that

$$b_u(\alpha_0) > 0 \quad (31jkc)$$

and

$$L_\kappa C_c \|\xi_c(\alpha_0)\| < b_u(\alpha_0) c_o C_u. \quad (31jkd)$$

Once (31jkc) and (31jkd) are obtained, Corollary 9.38 immediately yields a local interval

$$J \ni \alpha_0 \quad (31jke)$$

on which

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31jkf)$$

and hence the conclusions of Proposition 9.21 hold on  $J$ .

By Proposition 9.38C, this can also be reformulated in intrinsic angular form: it is enough to produce a point  $\alpha_0 \in I^\circ$  such that

$$x'(\alpha_0) \neq 0 \quad (31jkg)$$

and

$$\angle(x'(\alpha_0), E^u(\alpha_0)) < \vartheta_{\text{vis}}, \quad (31jkh)$$

where  $\vartheta_{\text{vis}}$  is any aperture satisfying

$$L_\kappa C_c \tan \vartheta_{\text{vis}} < c_o C_u. \quad (31jki)$$

So the frontier no longer lies in the crossing-time bookkeeping. It lies in forcing the strict seed margin at one point from:

- invariant or approximately invariant bundle transport,
- a genuine unstable contribution to the parameter direction,
- and a visibility condition preventing the lens from annihilating that contribution strongly enough to expel  $x'(\alpha_0)$  from the visible unstable cone.

**Proposition 9.39 (Exact unstable tangency at one point implies local positive gap).** In the setting of Proposition 9.37, fix  $\alpha_0 \in I^\circ$ . Assume

$$x'(\alpha_0) \in E^u(\alpha_0) \setminus \{0\}. \quad (31jkj)$$

Then there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31jkk)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** By Proposition 9.38C, it is enough to verify the intrinsic visible-cone condition at  $\alpha_0$ . Since  $x'(\alpha_0) \in E^u(\alpha_0) \setminus \{0\}$ , we have

$$\angle(x'(\alpha_0), E^u(\alpha_0)) = 0. \quad (31jkl)$$

Therefore

$$L_\kappa C_c \tan 0 = 0 < c_o C_u, \quad (31jkm)$$

so the visible-cone condition in Proposition 9.38C holds. Corollary 9.38 then yields the stated interval  $J$ .  $\square$

This shows that exact unstable tangency at a single point is already enough to force a local positive-gap region. No delayed endpoint and no global cone hypothesis are needed.

**Corollary 9.40 (Local unstable-leaf chart yields local symbolic amplification).** In the setting of Proposition 9.37, assume there exists a  $C^1$  one-dimensional submanifold

$$W^u \subset X \quad (31jkn)$$

such that

$$T_x W^u = E^u(x) \quad \text{for all } x \in W^u \cap x(I), \quad (31jko)$$

and assume

$$x(I) \subset W^u. \quad (31jkp)$$

If  $x'(\alpha_0) \neq 0$  for some  $\alpha_0 \in I^\circ$ , then there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Because  $x(I) \subset W^u$ , we have

$$x'(\alpha_0) \in T_{x(\alpha_0)} W^u = E^u(\alpha_0). \quad (31jq)$$

Since  $x'(\alpha_0) \neq 0$ , Proposition 9.39 applies and yields the conclusion.  $\square$

This is the cleanest intrinsic dynamical special case so far: a parameter curve that locally follows an unstable leaf automatically enters the visible unstable cone.

**Proposition 9.41 (Quantitative first-order departure from unstable tangency).** In the setting of Proposition 9.37, let  $\xi(\alpha) = T_\alpha^{-1} x'(\alpha)$  and decompose

$$\xi(\alpha) = b_u(\alpha) e_u^0 + \xi_c(\alpha) \quad (31jkr)$$

as in Corollary 9.38. Fix  $\alpha_0 \in I^\circ$ , and assume  $\xi$  is  $C^1$  on  $I$  with

$$\|\xi'(\alpha)\| \leq L_\xi \quad \text{for all } \alpha \in I, \quad (31jks)$$

for some  $L_\xi > 0$ . Assume further that

$$b_u(\alpha_0) \geq b_* > 0, \quad \xi_c(\alpha_0) = 0. \quad (31jkt)$$

Define

$$\delta_* := \min \left\{ \text{dist}(\alpha_0, \partial I), \frac{b_*}{2L_\xi}, \frac{b_* c_o C_u}{4L_\kappa C_c L_\xi} \right\}. \quad (31jku)$$

Then for every  $0 < \delta \leq \delta_*$ , the interval

$$J_\delta := (\alpha_0 - \delta, \alpha_0 + \delta) \cap I \quad (31jkv)$$

satisfies

$$G(\alpha) \geq \frac{1}{4}b_*c_oC_u > 0 \quad \text{for all } \alpha \in J_\delta, \quad (31jkw)$$

and the conclusions of Proposition 9.21 hold on  $J_\delta$  with

$$C = \frac{1}{4}b_*c_oC_u. \quad (31jkk)$$

**Proof.** For  $\alpha \in J_\delta$ , the mean-value estimate from (31jks) gives

$$\|\xi(\alpha) - \xi(\alpha_0)\| \leq L_\xi|\alpha - \alpha_0| \leq L_\xi\delta. \quad (31jky)$$

Therefore

$$|b_u(\alpha) - b_u(\alpha_0)| \leq L_\xi\delta, \quad \|\xi_c(\alpha) - \xi_c(\alpha_0)\| \leq L_\xi\delta. \quad (31jkz)$$

Using (31jkt) and  $\delta \leq b_*/(2L_\xi)$ , we obtain

$$b_u(\alpha) \geq b_* - L_\xi\delta \geq \frac{1}{2}b_*. \quad (31la)$$

Using  $\xi_c(\alpha_0) = 0$  and  $\delta \leq \delta_*$ , we also have

$$\|\xi_c(\alpha)\| \leq L_\xi\delta \leq \frac{b_*c_oC_u}{4L_\kappa C_c}. \quad (31lb)$$

Hence

$$L_\kappa C_c \|\xi_c(\alpha)\| \leq \frac{1}{4}b_*c_oC_u < \frac{1}{2}b_*c_oC_u \leq b_u(\alpha)c_oC_u. \quad (31lc)$$

Applying Corollary 9.38 yields local positive gap on  $J_\delta$ . Quantitatively, the previous inequality also gives

$$G(\alpha) \geq \frac{1}{2}b_*c_oC_u - \frac{1}{4}b_*c_oC_u = \frac{1}{4}b_*c_oC_u, \quad (31ld)$$

which is (31jkw); Proposition 9.29 then yields (31jkk).  $\square$

This proposition turns the local frontier into a first-order control problem: if the parameter tangent is exactly unstable at one point and drifts into the complementary bundle only slowly enough, then the visible unstable advantage survives on an explicit interval.

**Corollary 9.41A (Local unstable-cone chart yields local symbolic amplification).** In the setting of Proposition 9.37, assume there exists a  $C^1$  one-dimensional submanifold

$$W \subset X \quad (31jkk1)$$

and an angle  $0 \leq \theta_* < \pi/2$  such that

$$\angle(T_x W, E^u(x)) \leq \theta_* \quad \text{for all } x \in W \cap x(I), \quad (31jkk2)$$

and assume

$$x(I) \subset W. \quad (31jkk3)$$

If

$$L_\kappa C_c \tan \theta_* < c_o C_u, \quad (31jkk4)$$

and  $x'(\alpha_0) \neq 0$  for some  $\alpha_0 \in I^\circ$ , then there exists an open interval

$$J \subset I^\circ \quad \text{with } \alpha_0 \in J \quad (31jkk5)$$

such that the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Because  $x(I) \subset W$ , we have

$$x'(\alpha_0) \in T_{x(\alpha_0)}W. \quad (31jkk6)$$

Hence (31jkk2) gives

$$\angle(x'(\alpha_0), E^u(\alpha_0)) \leq \theta_*. \quad (31jkk7)$$

Since  $x'(\alpha_0) \neq 0$  and  $L_\kappa C_c \tan \theta_* < c_o C_u$ , we also have

$$L_\kappa C_c \tan(\angle(x'(\alpha_0), E^u(\alpha_0))) < c_o C_u. \quad (31jkk8)$$

Therefore the visible-cone condition in Corollary 9.38B holds at  $\alpha_0$ , so Corollary 9.38B yields the stated local amplification interval  $J$ .  $\square$

This is the first intrinsic local statement that no longer requires exact unstable-leaf tangency. It is enough that the parameter curve live inside a one-dimensional chart whose tangent stays inside a uniformly visible unstable cone.

**Proposition 9.41B (Quantitative persistence of one-point seed reserve).** In the setting of Proposition 9.37, let

$$\xi(\alpha) = b_u(\alpha)e_u^0 + \xi_c(\alpha) \quad (31nq1)$$

as in Corollary 9.38, and assume  $\xi$  is  $C^1$  on  $I$  with

$$\|\xi'(\alpha)\| \leq L_\xi \quad \text{for all } \alpha \in I, \quad (31nq2)$$

for some  $L_\xi > 0$ . Fix  $\alpha_0 \in I^\circ$ , and define the one-point seed reserve

$$m_0 := b_u(\alpha_0)c_o C_u - L_\kappa C_c \|\xi_c(\alpha_0)\|. \quad (31nq3)$$

Assume

$$m_0 > 0. \quad (31nq4)$$

Define

$$\delta_* := \min \left\{ \text{dist}(\alpha_0, \partial I), \frac{m_0}{2L_\xi(c_o C_u + L_\kappa C_c)} \right\}. \quad (31nq5)$$

Then for every  $0 < \delta \leq \delta_*$ , the interval

$$J_\delta := (\alpha_0 - \delta, \alpha_0 + \delta) \cap I \quad (31nq6)$$

satisfies

$$L_\kappa C_c \|\xi_c(\alpha)\| \leq b_u(\alpha)c_o C_u - \frac{1}{2}m_0 \quad \text{for all } \alpha \in J_\delta, \quad (31nq7)$$

and therefore

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J_\delta. \quad (31nq8)$$

In particular, the conclusions of Proposition 9.21 hold on  $J_\delta$ .

**Proof.** For  $\alpha \in J_\delta$ , the mean-value estimate from (31nq2) gives

$$\|\xi(\alpha) - \xi(\alpha_0)\| \leq L_\xi |\alpha - \alpha_0| \leq L_\xi \delta. \quad (31nq9)$$

Hence

$$|b_u(\alpha) - b_u(\alpha_0)| \leq L_\xi \delta, \quad \|\xi_c(\alpha)\| \leq \|\xi_c(\alpha_0)\| + L_\xi \delta. \quad (31nqa)$$

Therefore

$$b_u(\alpha)c_oC_u - L_\kappa C_c \|\xi_c(\alpha)\| \geq m_0 - L_\xi \delta (c_oC_u + L_\kappa C_c). \quad (31nqb)$$

Since  $\delta \leq \delta_*$ , the right-hand side is at least  $m_0/2$ , giving (31nq7). In particular, the seed-margin inequality of Corollary 9.38 holds throughout  $J_\delta$ , so Corollary 9.38 yields  $G(\alpha) > 0$  on  $J_\delta$ . The final conclusion then follows from Proposition 9.21.  $\square$

This closes another piece of the local bookkeeping around the frontier: once a positive seed reserve is created at a single point, no further qualitative argument is needed to get an explicit amplification interval. Only a first-order bound on the transported tangent is required.

**Proposition 9.42 (Intrinsic transport regularity yields first-order drift control).** In the setting of Proposition 9.37, let  $\xi(\alpha) = T_\alpha^{-1}x'(\alpha)$ , where  $T_\alpha$  is the bundle-preserving isometry family furnished by Proposition 9.37. Assume  $x$  is  $C^2$  on  $I$ , and assume  $T_\alpha$  can be chosen  $C^1$  on  $I$  in the sense that there exists  $M_T \geq 0$  such that for every  $v \in V$ ,

$$\left\| \frac{d}{d\alpha}(T_\alpha v) \right\| \leq M_T \|v\| \quad \text{for all } \alpha \in I. \quad (31me)$$

Assume also that there exists  $M_x \geq 0$  such that

$$\|x''(\alpha)\| \leq M_x \quad \text{for all } \alpha \in I. \quad (31mf)$$

If

$$\|x'(\alpha)\| \leq V_x \quad \text{for all } \alpha \in I \quad (31mg)$$

for some  $V_x \geq 0$ , then  $\xi$  is  $C^1$  on  $I$  and satisfies

$$\|\xi'(\alpha)\| \leq M_x + M_T V_x \quad \text{for all } \alpha \in I. \quad (31mh)$$

Consequently, if there exists  $\alpha_0 \in I^\circ$  such that

$$x'(\alpha_0) \in E^u(\alpha_0) \setminus \{0\}, \quad (31mi)$$

then Proposition 9.41 applies with

$$L_\xi := M_x + M_T V_x. \quad (31mj)$$

**Proof.** Differentiate

$$\xi(\alpha) = T_\alpha^{-1}x'(\alpha). \quad (31mk)$$

Because each  $T_\alpha$  is an isometry, its inverse is again an isometry, and differentiating the identity

$$T_\alpha^{-1}T_\alpha = \text{Id}_V \quad (31ml)$$

gives

$$\frac{d}{d\alpha}(T_\alpha^{-1}) = -T_\alpha^{-1} \left( \frac{d}{d\alpha} T_\alpha \right) T_\alpha^{-1}. \quad (31mm)$$

Hence

$$\xi'(\alpha) = \frac{d}{d\alpha}(T_\alpha^{-1})x'(\alpha) + T_\alpha^{-1}x''(\alpha). \quad (31mn)$$

Using (31me), (31mf), and (31mg), together with the norm-preserving property of  $T_\alpha^{-1}$ , we obtain

$$\|\xi'(\alpha)\| \leq M_T\|x'(\alpha)\| + \|x''(\alpha)\| \leq M_TV_x + M_x, \quad (31mo)$$

which is exactly (31mh). If (31mi) holds, then  $\xi_c(\alpha_0) = 0$  and

$$b_u(\alpha_0) = \|\xi(\alpha_0)\| = \|x'(\alpha_0)\| > 0. \quad (31mp)$$

So Proposition 9.41 applies with the bound (31mj).  $\square$

This removes one more non-intrinsic input from the local frontier: once the trivializing transport is regular enough and the parameter curve is  $C^2$ , the first-order drift bound needed in Proposition 9.41 is automatic.

**Proposition 9.43 (Critical complementary coordinate forces unstable tangency).** In the setting of Proposition 9.37, let  $U \subset X$  be an open neighborhood of  $x(I)$ , and let

$$z : U \rightarrow \mathbb{R}^m \quad (31mq)$$

be a  $C^1$  map such that

$$\ker Dz(x) = E^u(x) \quad \text{for all } x \in x(I). \quad (31mr)$$

Assume there exists  $\alpha_0 \in I^\circ$  such that

$$\frac{d}{d\alpha}(z \circ x)(\alpha_0) = 0 \quad (31ms)$$

and

$$x'(\alpha_0) \neq 0. \quad (31mt)$$

Then

$$x'(\alpha_0) \in E^u(\alpha_0) \setminus \{0\}. \quad (31mu)$$

Consequently, there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31mv)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** By the chain rule,

$$\frac{d}{d\alpha}(z \circ x)(\alpha_0) = Dz(x(\alpha_0))x'(\alpha_0). \quad (31mw)$$

Using (31ms), we obtain

$$Dz(x(\alpha_0))x'(\alpha_0) = 0. \quad (31mx)$$

By (31mr), the kernel of  $Dz(x(\alpha_0))$  is exactly  $E^u(\alpha_0)$ , so

$$x'(\alpha_0) \in E^u(\alpha_0). \quad (31my)$$

Together with (31mt), this gives (31mu). Proposition 9.39 therefore applies and yields (31mv) together with the local symbolic amplification conclusion.  $\square$

This packages exact unstable tangency as a geometric critical-point condition: if a complementary coordinate becomes stationary while the parameter curve is still moving, then the motion must be purely unstable at that instant.

**Corollary 9.44 (Codimension-one return in a complementary coordinate yields visible-cone entry).** In the setting of Proposition 9.43, assume  $m = 1$ . Let  $\alpha_- < \alpha_+$  be points of  $I$  such that

$$z(x(\alpha_-)) = z(x(\alpha_+)), \quad (31mz)$$

and assume

$$x'(\alpha) \neq 0 \quad \text{for all } \alpha \in [\alpha_-, \alpha_+]. \quad (31na)$$

Then there exists  $\alpha_0 \in (\alpha_-, \alpha_+)$  such that

$$x'(\alpha_0) \in E^u(\alpha_0) \setminus \{0\}. \quad (31nb)$$

Consequently, there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31nc)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Since  $m = 1$ , the map  $z \circ x$  is scalar-valued. By (31mz), Rolle's theorem yields

$$\frac{d}{d\alpha}(z \circ x)(\alpha_0) = 0 \quad (31nd)$$

for some  $\alpha_0 \in (\alpha_-, \alpha_+)$ . Condition (31na) gives  $x'(\alpha_0) \neq 0$ , so Proposition 9.43 applies and yields (31nb) and (31nc).  $\square$

This gives the first genuinely structural return mechanism on the local frontier: in codimension one, a parameter branch that returns to the same complementary coordinate must align with the unstable bundle at some interior point.

**Proposition 9.45 (Averaged complementary drift yields visible-cone entry).** In the setting of Proposition 9.37, let  $[a, b] \subset I^\circ$  with  $a < b$ . Let  $U \subset X$  be an open neighborhood of  $x([a, b])$ , and let

$$z : U \rightarrow \mathbb{R}^m \quad (31oa)$$

be a  $C^1$  map such that

$$\ker Dz(x) = E^u(x) \quad \text{for all } x \in x([a, b]). \quad (31ob)$$

Assume there exists  $m_z > 0$  such that

$$\|Dz(x)v\| \geq m_z \|P_x^c v\| \quad \text{for all } x \in x([a, b]), v \in T_x X. \quad (31oc)$$

Assume also that there exist constants  $a_* > 0$  and  $A_c \geq 0$  such that

$$a_u(\alpha) \geq a_* \quad \text{for all } \alpha \in [a, b], \quad (31od)$$

and

$$\int_a^b \|(z \circ x)'(\alpha)\| d\alpha \leq A_c (b - a). \quad (31oe)$$

If

$$L_\kappa C_c A_c < m_z a_* c_o C_u, \quad (31of)$$

then there exists  $\alpha_0 \in [a, b]$  such that

$$L_\kappa C_c \|P_{\alpha_0}^c x'(\alpha_0)\| < a_u(\alpha_0) c_o C_u. \quad (31og)$$

Consequently, there exists an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31oh)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** If  $\|(z \circ x)'(\alpha)\| > A_c$  for every  $\alpha \in [a, b]$ , then integrating would give

$$\int_a^b \|(z \circ x)'(\alpha)\| d\alpha > A_c(b - a), \quad (31oi)$$

contradicting (31oe). Hence there exists  $\alpha_0 \in [a, b]$  such that

$$\|(z \circ x)'(\alpha_0)\| \leq A_c. \quad (31oj)$$

By the chain rule,

$$\|(z \circ x)'(\alpha_0)\| = \|Dz(x(\alpha_0))x'(\alpha_0)\|. \quad (31ok)$$

Applying (31oc) to  $v = x'(\alpha_0)$ , we obtain

$$A_c \geq \|Dz(x(\alpha_0))x'(\alpha_0)\| \geq m_z \|P_{\alpha_0}^c x'(\alpha_0)\|, \quad (31ol)$$

so

$$\|P_{\alpha_0}^c x'(\alpha_0)\| \leq \frac{A_c}{m_z}. \quad (31om)$$

Using (31od) and (31of), we therefore get

$$L_\kappa C_c \|P_{\alpha_0}^c x'(\alpha_0)\| \leq \frac{L_\kappa C_c A_c}{m_z} < a_* c_o C_u \leq a_u(\alpha_0) c_o C_u. \quad (31on)$$

This is exactly the projection-ratio condition in Corollary 9.38A, so (31oh) and the local symbolic amplification conclusion follow.  $\square$

This is the first codimension-free structural entry mechanism on the local frontier: exact return is no longer needed. It is enough that the branch spend little enough average motion in complementary quotient coordinates relative to its unstable observable budget.

**Corollary 9.46 (Short complementary-quotient length yields local symbolic amplification).** In the setting of Proposition 9.45, define the complementary-quotient length of the branch segment

$$\mathcal{L}_z([a, b]) := \int_a^b \|(z \circ x)'(\alpha)\| d\alpha. \quad (31oo)$$

If

$$L_\kappa C_c \mathcal{L}_z([a, b]) < m_z a_* c_o C_u (b - a), \quad (31op)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31oq)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Apply Proposition 9.45 with

$$A_c := \frac{\mathcal{L}_z([a, b])}{b - a}. \quad (31or)$$

Then (31oe) is immediate from the definition of  $\mathcal{L}_z([a, b])$ , and (31op) is exactly the hypothesis (31of) for this choice of  $A_c$ . The conclusion follows.  $\square$

This reformulates visible-cone entry as a branch-budget statement: if a long enough parameter segment moves too little in the complementary quotient, then somewhere along that segment the parameter direction must become sufficiently unstable-visible.

**Proposition 9.47 (Bounded-turning quotient geometry yields visible-cone entry).** In the setting of Proposition 9.45, assume there exists  $K_{\text{bt}} \geq 1$  such that

$$\mathcal{L}_z([a, b]) \leq K_{\text{bt}} \text{diam}(z(x([a, b]))), \quad (31ou)$$

where

$$\text{diam}(z(x([a, b]))) := \sup_{\alpha, \beta \in [a, b]} \|z(x(\alpha)) - z(x(\beta))\|. \quad (31ov)$$

If

$$L_\kappa C_c K_{\text{bt}} \text{diam}(z(x([a, b]))) < m_z a_* c_o C_u (b - a), \quad (31ow)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31ox)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** By (31ou), we have

$$\mathcal{L}_z([a, b]) \leq K_{\text{bt}} \text{diam}(z(x([a, b]))). \quad (31oy)$$

So the hypothesis (31ow) implies

$$L_\kappa C_c \mathcal{L}_z([a, b]) < m_z a_* c_o C_u (b - a). \quad (31oz)$$

This is exactly the hypothesis (31op) of Corollary 9.46, which yields (31ox) and the local symbolic amplification conclusion.  $\square$

This reduces the averaged drift frontier to a pure geometric control problem on the quotient path: it is enough to show that the branch image in complementary coordinates has small diameter and does not wander too inefficiently.

**Corollary 9.48 (Monotone scalar quotient excursion yields local symbolic amplification).** In the setting of Proposition 9.45, assume  $m = 1$  and that the scalar function  $z \circ x$  is monotone on  $[a, b]$ . If

$$L_\kappa C_c |z(x(b)) - z(x(a))| < m_z a_* c_o C_u (b - a), \quad (31pa)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31pb)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Because  $m = 1$  and  $z \circ x$  is monotone, its total variation on  $[a, b]$  equals its endpoint excursion. Hence

$$\mathcal{L}_z([a, b]) = |z(x(b)) - z(x(a))|. \quad (31pc)$$

Applying Corollary 9.46 yields the conclusion.  $\square$

This is the cleanest long-branch criterion so far: a scalar complementary quotient that changes only a little across a long parameter segment must contain a locally unstable-visible point somewhere in between.

**Proposition 9.48A (Acute directional quotient drift yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  such that

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

If

$$L_\kappa C_c \sec(\vartheta_q) \text{diam}(z(x([a, b]))) < m_z a_* c_o C_u (b - a), \quad (31qb)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31qc)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Because  $0 \leq \vartheta_q < \pi/2$ , the quantity  $\cos(\vartheta_q)$  is positive, so (31qa) implies

$$\|(z \circ x)'(\alpha)\| \leq \sec(\vartheta_q) \langle (z \circ x)'(\alpha), \nu \rangle \quad \text{for all } \alpha \in [a, b]. \quad (31qd)$$

Integrating over  $[a, b]$ , we obtain

$$\mathcal{L}_z([a, b]) \leq \sec(\vartheta_q) \int_a^b \langle (z \circ x)'(\alpha), \nu \rangle d\alpha = \sec(\vartheta_q) \langle z(x(b)) - z(x(a)), \nu \rangle. \quad (31qe)$$

Hence

$$\mathcal{L}_z([a, b]) \leq \sec(\vartheta_q) |\langle z(x(b)) - z(x(a)), \nu \rangle| \leq \sec(\vartheta_q) \|z(x(b)) - z(x(a))\| \leq \sec(\vartheta_q) \text{diam}(z(x([a, b]))). \quad (31qf)$$

So (31qb) implies

$$L_\kappa C_c \mathcal{L}_z([a, b]) < m_z a_* c_o C_u (b - a). \quad (31qg)$$

This is exactly the hypothesis (31op) of Corollary 9.46, so (31qc) and the local symbolic amplification conclusion follow.  $\square$

This removes one more opaque quotient-side constant from the frontier. It is enough that the complementary quotient drift keeps a coherent acute direction in quotient space, because then bounded turning is automatic with constant  $\sec(\vartheta_q)$ . Corollary 9.48 is exactly the scalar  $\vartheta_q = 0$  special case.

**Proposition 9.48B (Dominant scalar quotient coordinate yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exists a  $C^1$  decomposition

$$z = (q, r) : U \rightarrow \mathbb{R} \times \mathbb{R}^{m-1}, \quad (31ra)$$

a sign  $\sigma \in \{-1, 1\}$ , and a constant  $\eta \geq 0$  such that

$$\sigma(q \circ x)'(\alpha) \geq 0 \quad \text{for all } \alpha \in [a, b], \quad (31rb)$$

and

$$\|(r \circ x)'(\alpha)\| \leq \eta \sigma(q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31rc)$$

If

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31rd)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31re)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** For each  $\alpha \in [a, b]$ , the decomposition (31ra) gives

$$\|(z \circ x)'(\alpha)\|^2 = |(q \circ x)'(\alpha)|^2 + \|(r \circ x)'(\alpha)\|^2. \quad (31rf)$$

By (31rb), we have

$$|(q \circ x)'(\alpha)| = \sigma(q \circ x)'(\alpha), \quad (31rg)$$

and then (31rc) yields

$$\|(z \circ x)'(\alpha)\|^2 \leq (1 + \eta^2) (\sigma(q \circ x)'(\alpha))^2. \quad (31rh)$$

Taking square roots, we obtain

$$\|(z \circ x)'(\alpha)\| \leq \sqrt{1 + \eta^2} \sigma(q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31ri)$$

Integrating over  $[a, b]$ , we get

$$\mathcal{L}_z([a, b]) \leq \sqrt{1 + \eta^2} \int_a^b \sigma(q \circ x)'(\alpha) d\alpha = \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))|. \quad (31rj)$$

So the hypothesis (31rd) implies

$$L_\kappa C_c \mathcal{L}_z([a, b]) < m_z a_* c_o C_u (b - a). \quad (31rk)$$

This is exactly the hypothesis (31op) of Corollary 9.46, so (31re) and the local symbolic amplification conclusion follow.  $\square$

This makes the quotient-side entry mechanism even more concrete: one distinguished complementary quotient coordinate may carry the whole branch excursion, while the remaining quotient directions contribute only a controlled spillover. The open dynamics can therefore try to produce a dominant scalar complementary drift, not necessarily a full quotient-space cone theorem all at once.

**Proposition 9.48C (Directional quotient width under acute drift yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  such that

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Define the  $\nu$ -directional quotient width by

$$\mathcal{W}_\nu([a, b]) := \sup_{\alpha, \beta \in [a, b]} |\langle z(x(\beta)) - z(x(\alpha)), \nu \rangle|. \quad (31sa)$$

If

$$L_\kappa C_c \sec(\vartheta_q) \mathcal{W}_\nu([a, b]) < m_z a_* c_o C_u (b - a), \quad (31sb)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31sc)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle. \quad (31sd)$$

By (31qa) and the positivity of  $\cos(\vartheta_q)$ , we have

$$q'_\nu(\alpha) = \langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \geq 0 \quad \text{for all } \alpha \in [a, b]. \quad (31se)$$

So  $q_\nu$  is monotone nondecreasing on  $[a, b]$ , and therefore

$$q_\nu(b) - q_\nu(a) \leq \mathcal{W}_\nu([a, b]). \quad (31sf)$$

Repeating the first step of the proof of Proposition 9.48A, we obtain

$$\mathcal{L}_z([a, b]) \leq \sec(\vartheta_q) \int_a^b \langle (z \circ x)'(\alpha), \nu \rangle d\alpha = \sec(\vartheta_q) (q_\nu(b) - q_\nu(a)) \leq \sec(\vartheta_q) \mathcal{W}_\nu([a, b]). \quad (31sg)$$

Hence the hypothesis (31sb) implies

$$L_\kappa C_c \mathcal{L}_z([a, b]) < m_z a_* c_o C_u (b - a). \quad (31sh)$$

This is exactly the hypothesis (31op) of Corollary 9.46, so (31sc) and the local symbolic amplification conclusion follow.  $\square$

This sharpens Proposition 9.48A: once the quotient drift keeps a coherent acute direction, the full Euclidean diameter of the quotient image is stronger than necessary. It is enough to control only the width of the branch image in that preferred quotient direction.

**Proposition 9.48D (Preferred-direction drift budget under acute alignment yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying the acute alignment condition

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Assume also that there exists a constant  $A_\nu \geq 0$  such that

$$\int_a^b \langle (z \circ x)'(\alpha), \nu \rangle d\alpha \leq A_\nu (b - a). \quad (31ta)$$

If

$$L_\kappa C_c \sec(\vartheta_q) A_\nu < m_z a_* c_o C_u, \quad (31tb)$$

then there exists  $\alpha_0 \in [a, b]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31tc)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Exactly as in the first step of the proof of Proposition 9.48A, the acute alignment condition gives

$$\|(z \circ x)'(\alpha)\| \leq \sec(\vartheta_q) \langle (z \circ x)'(\alpha), \nu \rangle \quad \text{for all } \alpha \in [a, b]. \quad (31td)$$

Integrating over  $[a, b]$  and using (31ta), we obtain

$$\mathcal{L}_z([a, b]) \leq \sec(\vartheta_q) \int_a^b \langle (z \circ x)'(\alpha), \nu \rangle d\alpha \leq \sec(\vartheta_q) A_\nu(b - a). \quad (31te)$$

So the hypothesis (31tb) implies

$$L_\kappa C_c \mathcal{L}_z([a, b]) < m_z a_* c_o C_u (b - a). \quad (31tf)$$

This is exactly the hypothesis (31op) of Corollary 9.46, so (31tc) and the local symbolic amplification conclusion follow.  $\square$

This pushes the quotient-side reduction one step closer to dynamics. Under coherent acute quotient drift, the remaining quantitative input no longer has to be a full diameter or directional-width estimate: it is enough to control the averaged signed drift in the preferred quotient direction. So the open theorem may try to produce a scalar preferred-direction drift budget directly from branch dynamics.

**Proposition 9.48E (Near-critical preferred-direction speed plus directional acceleration control yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying the acute alignment condition

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle. \quad (31ua)$$

Assume there exist  $\alpha_* \in (a, b)$ , a radius  $\rho > 0$  with

$$[\alpha_* - \rho, \alpha_* + \rho] \subset [a, b], \quad (31ub)$$

and constants  $\delta_\nu, B_\nu \geq 0$  such that

$$q'_\nu(\alpha_*) \leq \delta_\nu, \quad (31uc)$$

and

$$|q''_\nu(\alpha)| \leq B_\nu \quad \text{for all } \alpha \in [\alpha_* - \rho, \alpha_* + \rho]. \quad (31ud)$$

If

$$L_\kappa C_c \sec(\vartheta_q) \left( \delta_\nu + \frac{B_\nu \rho}{2} \right) < m_z a_* c_o C_u, \quad (31ue)$$

then there exists  $\alpha_0 \in [\alpha_* - \rho, \alpha_* + \rho]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31uf)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Write

$$I_\rho := [\alpha_* - \rho, \alpha_* + \rho]. \quad (31ug)$$

For any  $\alpha \in I_\rho$ , the fundamental theorem of calculus and the bound (31ud) give

$$q'_\nu(\alpha) = q'_\nu(\alpha_*) + \int_{\alpha_*}^{\alpha} q''_\nu(s) ds \leq \delta_\nu + B_\nu |\alpha - \alpha_*|. \quad (31uh)$$

Integrating over  $I_\rho$ , we obtain

$$\int_{I_\rho} q'_\nu(\alpha) d\alpha \leq 2\rho \delta_\nu + B_\nu \int_{I_\rho} |\alpha - \alpha_*| d\alpha = 2\rho \left( \delta_\nu + \frac{B_\nu \rho}{2} \right). \quad (31ui)$$

Hence

$$\int_{\alpha_* - \rho}^{\alpha_* + \rho} \langle (z \circ x)'(\alpha), \nu \rangle d\alpha \leq \left( \delta_\nu + \frac{B_\nu \rho}{2} \right) (2\rho). \quad (31uj)$$

So Proposition 9.48D applies on the shorter branch segment  $I_\rho$  with

$$A_\nu := \delta_\nu + \frac{B_\nu \rho}{2}. \quad (31uk)$$

The hypothesis (31ue) is exactly the corresponding condition (31tb), and therefore (31uf) and the local symbolic amplification conclusion follow.  $\square$

This sharpens Proposition 9.48D in a genuinely local way. To force a preferred-direction drift budget, it is already enough to find one point where the preferred quotient speed is small and where its directional acceleration remains controlled nearby. So the open theorem may now try to produce a single near-critical preferred-direction event from branch dynamics, rather than a whole averaged drift estimate on the full segment from the outset.

**Proposition 9.48F (Local preferred-direction confinement yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying the acute alignment condition

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle. \quad (31ua)$$

Assume there exist  $\bar{\alpha} \in (a, b)$ , a radius  $\rho > 0$  with

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad (31vq)$$

and a constant  $E_\nu \geq 0$  such that

$$|q_\nu(\alpha) - q_\nu(\bar{\alpha})| \leq E_\nu \quad \text{for all } \alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]. \quad (31vr)$$

If

$$L_\kappa C_c \sec(\vartheta_q) \frac{E_\nu}{\rho} < m_z a_* c_o C_u, \quad (31vs)$$

then there exists  $\alpha_0 \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31vt)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Write

$$I_\rho := [\bar{\alpha} - \rho, \bar{\alpha} + \rho]. \quad (31vu)$$

Exactly as in Proposition 9.48C, the acute alignment condition implies

$$q'_\nu(\alpha) = \langle (z \circ x)'(\alpha), \nu \rangle \geq 0 \quad \text{for all } \alpha \in I_\rho, \quad (31vv)$$

so  $q_\nu$  is monotone nondecreasing on  $I_\rho$ . Using (31vr), we therefore get

$$q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho) \leq 2E_\nu. \quad (31vw)$$

Hence

$$\int_{\bar{\alpha}-\rho}^{\bar{\alpha}+\rho} \langle (z \circ x)'(\alpha), \nu \rangle d\alpha = q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho) \leq 2E_\nu. \quad (31vx)$$

So Proposition 9.48D applies on the shorter branch segment  $I_\rho$  with

$$A_\nu := \frac{E_\nu}{\rho}. \quad (31vy)$$

Since  $|I_\rho| = 2\rho$ , the bound (31vx) is exactly the drift-budget condition (31ta) on  $I_\rho$ . The hypothesis (31vs) is the corresponding condition (31tb), and therefore (31vt) and the local symbolic amplification conclusion follow.  $\square$

This gives a purely positional local version of Proposition 9.48D. Under coherent acute quotient drift, it is already enough that the preferred quotient coordinate stays trapped inside a thin slab on a short neighborhood. So the open theorem may now try to produce local quotient confinement, not only a drift budget or a near-critical preferred-direction speed event.

**Proposition 9.48G (Local affine preferred-direction pinning yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying the acute alignment condition

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle. \quad (31ua)$$

Assume there exist  $\bar{\alpha} \in (a, b)$ , a radius  $\rho > 0$  with

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad (31xa)$$

a scalar  $v_\nu \in \mathbb{R}$ , and a constant  $R_\nu \geq 0$  such that

$$|q_\nu(\alpha) - q_\nu(\bar{\alpha}) - v_\nu(\alpha - \bar{\alpha})| \leq R_\nu \quad \text{for all } \alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]. \quad (31xb)$$

If

$$L_\kappa C_c \sec(\vartheta_q) \left( |v_\nu| + \frac{R_\nu}{\rho} \right) < m_z a_* c_o C_u, \quad (31xc)$$

then there exists  $\alpha_0 \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31xd)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** For any  $\alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]$ , the affine approximation hypothesis (31xb) implies

$$|q_\nu(\alpha) - q_\nu(\bar{\alpha})| \leq |v_\nu| |\alpha - \bar{\alpha}| + R_\nu \leq |v_\nu| \rho + R_\nu. \quad (31xe)$$

So Proposition 9.48F applies on the shorter branch segment

$$I_\rho := [\bar{\alpha} - \rho, \bar{\alpha} + \rho] \quad (31xf)$$

with

$$E_\nu := |v_\nu| \rho + R_\nu. \quad (31xg)$$

The inequality (31xc) is exactly the corresponding condition (31vs) for that choice of  $E_\nu$ , and therefore (31xd) and the local symbolic amplification conclusion follow.  $\square$

This packages the quotient-side localization into a form that is closer to what intrinsic bundle dynamics often produces: a short branch on which the preferred observed quotient coordinate is almost affine with a small slope. In particular, Proposition 9.48E becomes a special case once a Taylor expansion gives  $v_\nu = q'_\nu(\bar{\alpha})$  and  $R_\nu = O(\rho^2)$ . So the open theorem may now try to force a small-slope local affine model directly from bundle transport, rather than first proving full confinement or an averaged drift estimate.

**Proposition 9.48H (Endpoint-secants with bounded preferred curvature yield local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying the acute alignment condition

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle. \quad (31ua)$$

Assume there exist  $\bar{\alpha} \in (a, b)$ , a radius  $\rho > 0$  with

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad (31xk)$$

and a constant  $B_\nu \geq 0$  such that  $q_\nu$  is twice differentiable on  $[\bar{\alpha} - \rho, \bar{\alpha} + \rho]$  and

$$|q''_\nu(\alpha)| \leq B_\nu \quad \text{for all } \alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]. \quad (31xl)$$

Define the symmetric preferred secant slope

$$s_\nu := \frac{q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho)}{2\rho}. \quad (31xm)$$

If

$$L_\kappa C_c \sec(\vartheta_q) (|s_\nu| + B_\nu \rho) < m_z a_* c_o C_u, \quad (31xn)$$

then there exists  $\alpha_0 \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31xo)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Write

$$I_\rho := [\bar{\alpha} - \rho, \bar{\alpha} + \rho]. \quad (31xp)$$

For any  $\alpha \in I_\rho$ , Taylor's theorem at  $\bar{\alpha}$  gives

$$q_\nu(\alpha) = q_\nu(\bar{\alpha}) + q'_\nu(\bar{\alpha})(\alpha - \bar{\alpha}) + r_\nu(\alpha), \quad (31xq)$$

with remainder satisfying

$$|r_\nu(\alpha)| \leq \frac{B_\nu}{2} |\alpha - \bar{\alpha}|^2 \leq \frac{B_\nu \rho^2}{2}. \quad (31xr)$$

Applying (31xq) at  $\alpha = \bar{\alpha} \pm \rho$ , we obtain

$$q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho) = 2\rho q'_\nu(\bar{\alpha}) + r_\nu(\bar{\alpha} + \rho) - r_\nu(\bar{\alpha} - \rho), \quad (31xs)$$

so by the definition (31xm),

$$s_\nu = q'_\nu(\bar{\alpha}) + \frac{r_\nu(\bar{\alpha} + \rho) - r_\nu(\bar{\alpha} - \rho)}{2\rho}. \quad (31xt)$$

Hence

$$|s_\nu - q'_\nu(\bar{\alpha})| \leq \frac{|r_\nu(\bar{\alpha} + \rho)| + |r_\nu(\bar{\alpha} - \rho)|}{2\rho} \leq \frac{B_\nu \rho}{2}. \quad (31xu)$$

Combining (31xq), (31xr), and (31xu), we get for every  $\alpha \in I_\rho$ ,

$$\begin{aligned} |q_\nu(\alpha) - q_\nu(\bar{\alpha}) - s_\nu(\alpha - \bar{\alpha})| &\leq |q'_\nu(\bar{\alpha}) - s_\nu| |\alpha - \bar{\alpha}| + |r_\nu(\alpha)| \\ &\leq \frac{B_\nu \rho}{2} \rho + \frac{B_\nu \rho^2}{2} = B_\nu \rho^2. \end{aligned} \quad (31xv)$$

So Proposition 9.48G applies on the same segment  $I_\rho$  with

$$v_\nu := s_\nu, \quad R_\nu := B_\nu \rho^2. \quad (31xw)$$

The hypothesis (31xn) is exactly the corresponding condition (31xc) for that choice of  $v_\nu$  and  $R_\nu$ , and therefore (31xo) and the local symbolic amplification conclusion follow.  $\square$

This packages the quotient-side frontier into a particularly geometric local return form. Under coherent acute quotient drift, it is already enough that the preferred quotient coordinate has a small symmetric secant across a short branch and a controlled preferred curvature there. So the open theorem may now try to force a short preferred-direction return with bounded curvature from intrinsic bundle transport, rather than first proving a full affine model pointwise.

**Proposition 9.48I (Endpoint return alone under acute preferred alignment yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exist a unit vector  $\nu \in \mathbb{R}^m$  and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying the acute alignment condition

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b]. \quad (31qa)$$

Define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle. \quad (31ya)$$

Assume there exist  $\bar{\alpha} \in (a, b)$  and a radius  $\rho > 0$  with

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b]. \quad (31yb)$$

Define the symmetric preferred secant slope

$$s_\nu := \frac{q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho)}{2\rho}. \quad (31yc)$$

If

$$L_\kappa C_c \sec(\vartheta_q) |s_\nu| < m_z a_* c_o C_u, \quad (31yd)$$

then there exists  $\alpha_0 \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho]$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31ye)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Write

$$I_\rho := [\bar{\alpha} - \rho, \bar{\alpha} + \rho]. \quad (31yf)$$

Restricting the acute alignment condition (31qa) to  $I_\rho$ , we obtain

$$q'_\nu(\alpha) := \langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \geq 0 \quad \text{for all } \alpha \in I_\rho. \quad (31yg)$$

So  $q_\nu$  is monotone nondecreasing on  $I_\rho$ , hence  $s_\nu \geq 0$ . Therefore

$$\int_{\bar{\alpha}-\rho}^{\bar{\alpha}+\rho} \langle (z \circ x)'(\alpha), \nu \rangle d\alpha = q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho) = 2\rho s_\nu = |I_\rho| |s_\nu|. \quad (31yh)$$

Thus Proposition 9.48D applies on the shorter branch segment  $I_\rho$  with

$$A_\nu := |s_\nu|. \quad (31yi)$$

The hypothesis (31yd) is exactly the corresponding drift-budget condition (31tb) for that choice of  $A_\nu$ , so (31ye) and the local symbolic amplification conclusion follow.  $\square$

This removes even the preferred-curvature part from the short-return route. Under coherent acute quotient drift, it is already enough that the preferred quotient coordinate makes a small net return across a short branch. So the live theorem frontier may now try to force a short preferred-direction endpoint return from intrinsic bundle transport and observability directly, with curvature control treated only as one possible certificate rather than as part of the minimal input.

**Proposition 9.48J (Dominant scalar quotient return yields exact unstable tangency).**

In the setting of Proposition 9.45, assume there exists a  $C^1$  decomposition

$$z = (q, r) : U \rightarrow \mathbb{R} \times \mathbb{R}^{m-1}, \quad (31ym)$$

a constant  $\eta \geq 0$ , and a speed floor  $s_* > 0$  such that

$$\|(r \circ x)'(\alpha)\| \leq \eta |(q \circ x)'(\alpha)| \quad \text{for all } \alpha \in [a, b], \quad (31yn)$$

and

$$\|x'(\alpha)\| \geq s_* \quad \text{for all } \alpha \in [a, b]. \quad (31yo)$$

Assume in addition that the scalar quotient coordinate returns:

$$q(x(a)) = q(x(b)). \quad (31yp)$$

Then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31yq)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** By Rolle's theorem and the return condition (31yp), there exists  $\alpha_0 \in (a, b)$  such that

$$(q \circ x)'(\alpha_0) = 0. \quad (31yr)$$

Applying the spillover bound (31yn) at  $\alpha_0$ , we obtain

$$\|(r \circ x)'(\alpha_0)\| \leq \eta |(q \circ x)'(\alpha_0)| = 0, \quad (31ys)$$

hence

$$(r \circ x)'(\alpha_0) = 0. \quad (31yt)$$

Therefore

$$Dz(x(\alpha_0))x'(\alpha_0) = ((q \circ x)'(\alpha_0), (r \circ x)'(\alpha_0)) = 0. \quad (31yu)$$

By the kernel condition (31ob) of Proposition 9.45, it follows that

$$x'(\alpha_0) \in \ker Dz(x(\alpha_0)) = E^u(\alpha_0). \quad (31yv)$$

The speed floor (31yo) gives  $x'(\alpha_0) \neq 0$ . Proposition 9.39 therefore yields (31yq) and the local symbolic amplification conclusion.  $\square$

This is the first codimension-free quotient-side return mechanism that collapses directly back to exact unstable tangency. If one scalar complementary quotient coordinate controls the whole quotient drift and then returns to its starting value, the full quotient velocity must vanish somewhere in between; at that point the branch tangent lies in the unstable bundle, so the local amplification ladder closes immediately.

**Proposition 9.48K (Dominant scalar quotient secant yields local symbolic amplification).** In the setting of Proposition 9.45, assume there exists a  $C^1$  decomposition

$$z = (q, r) : U \rightarrow \mathbb{R} \times \mathbb{R}^{m-1}, \quad (31zka)$$

and a constant  $\eta \geq 0$  such that

$$\|(r \circ x)'(\alpha)\| \leq \eta |(q \circ x)'(\alpha)| \quad \text{for all } \alpha \in [a, b]. \quad (31zkb)$$

If

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zkc)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zkd)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** By the mean value theorem, there exists  $\alpha_0 \in (a, b)$  such that

$$(q \circ x)'(\alpha_0) = \frac{q(x(b)) - q(x(a))}{b - a}. \quad (31zke)$$

Hence

$$|(q \circ x)'(\alpha_0)| = \frac{|q(x(b)) - q(x(a))|}{b - a}. \quad (31z kf)$$

At  $\alpha_0$ , the decomposition (31zka) gives

$$\|(z \circ x)'(\alpha_0)\|^2 = |(q \circ x)'(\alpha_0)|^2 + \|(r \circ x)'(\alpha_0)\|^2. \quad (31z kg)$$

Applying the spillover bound (31zkb) at  $\alpha_0$ , we obtain

$$\|(z \circ x)'(\alpha_0)\|^2 \leq (1 + \eta^2) |(q \circ x)'(\alpha_0)|^2, \quad (31z kh)$$

so by (31z kf),

$$\|(z \circ x)'(\alpha_0)\| \leq \sqrt{1 + \eta^2} \frac{|q(x(b)) - q(x(a))|}{b - a}. \quad (31z ki)$$

Using the chain rule,

$$\|(z \circ x)'(\alpha_0)\| = \|Dz(x(\alpha_0))x'(\alpha_0)\|. \quad (31z kj)$$

Applying the quotient lower bound (31oc) to  $v = x'(\alpha_0)$ , we therefore get

$$m_z \|P_{\alpha_0}^c x'(\alpha_0)\| \leq \|Dz(x(\alpha_0))x'(\alpha_0)\| \leq \sqrt{1 + \eta^2} \frac{|q(x(b)) - q(x(a))|}{b - a}. \quad (31z kk)$$

Hence

$$L_\kappa C_c \|P_{\alpha_0}^c x'(\alpha_0)\| \leq \frac{L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))|}{m_z (b - a)} < a_* c_o C_u \leq a_u(\alpha_0) c_o C_u. \quad (31z kl)$$

This is exactly the projection-ratio condition in Corollary 9.38A, so (31zkd) and the local symbolic amplification conclusion follow.  $\square$

This removes the monotonicity requirement from the dominant-scalar quotient route. A distinguished scalar quotient coordinate may now oscillate along the branch, as long as its net secant across the branch remains small enough relative to the interval length and the spillover into the remaining quotient directions stays controlled. So the open theorem may now try to force a short dominant-scalar quotient secant budget from intrinsic dynamics, with Proposition 9.48J becoming the exact-return endpoint of the same mechanism.

**Proposition 9.48L (Scalar quotient graph with bounded slope reduces to the dominant secant route).** In the setting of Proposition 9.45, assume there exists a  $C^1$  decomposition

$$z = (q, r) : U \rightarrow \mathbb{R} \times \mathbb{R}^{m-1}, \quad (31z la)$$

an interval  $J_q \subset \mathbb{R}$  containing  $q(x([a, b]))$ , a  $C^1$  map

$$\Phi : J_q \rightarrow \mathbb{R}^{m-1}, \quad (31z lb)$$

and a constant  $\eta \geq 0$  such that

$$r(x(\alpha)) = \Phi(q(x(\alpha))) \quad \text{for all } \alpha \in [a, b], \quad (31z lc)$$

and

$$\|D\Phi(s)\| \leq \eta \quad \text{for all } s \in J_q. \quad (31zld)$$

If

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zle)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zlf)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Differentiating the graph relation (31zlc) along the branch, we obtain

$$(r \circ x)'(\alpha) = D\Phi(q(x(\alpha))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31zlg)$$

Therefore, by the slope bound (31zld),

$$\|(r \circ x)'(\alpha)\| \leq \eta |(q \circ x)'(\alpha)| \quad \text{for all } \alpha \in [a, b]. \quad (31zlh)$$

So Proposition 9.48K applies directly, and yields (31zlf) together with the local symbolic amplification conclusion.  $\square$

This is the first stable-foliation-flavored quotient-side reduction on the frontier. If the complementary quotient image of the branch lies inside a one-dimensional  $C^1$  graph with bounded slope, then the whole higher-codimension geometry collapses to a single scalar secant budget. So the open theorem may now try to derive such a quotient graph structure from intrinsic bundle transport or local stable-foliation geometry, with Proposition 9.48K as the downstream secant step and Proposition 9.48J as the exact-return endpoint.

**Proposition 9.48M (Stable-foliation chart with bounded effective quotient slope reduces to the scalar graph route).** In the setting of Proposition 9.45, assume there exists a  $C^1$  decomposition

$$z = (q, r) : U \rightarrow \mathbb{R} \times \mathbb{R}^{m-1}, \quad (31zma)$$

an interval  $J_q \subset \mathbb{R}$  containing  $q(x([a, b]))$ , an open set  $W_s \subset \mathbb{R}^\ell$ , a  $C^1$  chart

$$\Psi : J_q \times W_s \rightarrow U_0 \subset U, \quad (31zmb)$$

a  $C^1$  graph map

$$\sigma : J_q \rightarrow W_s, \quad (31zmc)$$

and a  $C^1$  quotient-profile map

$$\Theta : J_q \times W_s \rightarrow \mathbb{R}^{m-1}, \quad (31zmd)$$

such that

$$q(\Psi(s, w)) = s, \quad r(\Psi(s, w)) = \Theta(s, w) \quad (31zme)$$

for all  $(s, w) \in J_q \times W_s$ , and

$$x([a, b]) \subset \{\Psi(s, \sigma(s)) : s \in J_q\}. \quad (31zmf)$$

Assume also that there exists  $\eta \geq 0$  such that

$$\|\partial_s \Theta(s, \sigma(s)) + D_w \Theta(s, \sigma(s)) \sigma'(s)\| \leq \eta \quad \text{for all } s \in J_q. \quad (31zmg)$$

If

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zmh)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zmi)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Define

$$\Phi(s) := \Theta(s, \sigma(s)) \quad \text{for } s \in J_q. \quad (31zmj)$$

By the chain rule,

$$D\Phi(s) = \partial_s \Theta(s, \sigma(s)) + D_w \Theta(s, \sigma(s)) \sigma'(s), \quad (31zmk)$$

so the effective-slope bound (31zmg) gives

$$\|D\Phi(s)\| \leq \eta \quad \text{for all } s \in J_q. \quad (31zml)$$

Fix  $\alpha \in [a, b]$ . By the branch-graph condition (31zmf), there exists  $s_\alpha \in J_q$  such that

$$x(\alpha) = \Psi(s_\alpha, \sigma(s_\alpha)). \quad (31zmm)$$

Applying  $q$  and using (31zme), we obtain

$$q(x(\alpha)) = q(\Psi(s_\alpha, \sigma(s_\alpha))) = s_\alpha, \quad (31zmn)$$

hence

$$x(\alpha) = \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))). \quad (31zmo)$$

Applying  $r$  and using (31zme) again, it follows that

$$r(x(\alpha)) = \Theta(q(x(\alpha)), \sigma(q(x(\alpha)))) = \Phi(q(x(\alpha))). \quad (31zmp)$$

So the hypotheses of Proposition 9.48L hold with this  $\Phi$  and the same constant  $\eta$ . Therefore Proposition 9.48L applies and yields (31zmi) together with the local symbolic amplification conclusion.  $\square$

This is the first genuinely chart-level stable-foliation reduction on the frontier. One no longer needs to postulate the scalar quotient graph directly in quotient coordinates: it is enough that the branch be a single  $C^1$  graph inside a local foliation chart whose induced quotient profile has bounded effective slope along that graph. So the open theorem may now try to derive such a chart and effective-slope bound from stable-foliation regularity, bundle transport, and branch persistence, and then feed them into Proposition 9.48L.

**Proposition 9.48N (Separated chart drift, holonomy, and branch slope reduce to the chart route).** In the setting of Proposition 9.48M, assume there exist constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ , and  $S_\sigma \geq 0$  such that

$$\|\partial_s \Theta(s, w)\| \leq A_\Theta, \quad \|D_w \Theta(s, w)\| \leq B_\Theta \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmq)$$

and

$$\|\sigma'(s)\| \leq S_\sigma \quad \text{for all } s \in J_q. \quad (31zmr)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta S_\sigma)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zms)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zmt)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** For every  $s \in J_q$ , the triangle inequality, together with (31zmq) and (31zmr), gives

$$\|\partial_s \Theta(s, \sigma(s)) + D_w \Theta(s, \sigma(s)) \sigma'(s)\| \leq \|\partial_s \Theta(s, \sigma(s))\| + \|D_w \Theta(s, \sigma(s))\| \|\sigma'(s)\| \leq A_\Theta + B_\Theta S_\sigma. \quad (31zmu)$$

So Proposition 9.48M applies with

$$\eta := A_\Theta + B_\Theta S_\sigma, \quad (31zmv)$$

and yields (31zmt) together with the local symbolic amplification conclusion.  $\square$

This splits the chart-level bottleneck into three separately meaningful geometric quantities:

$$\text{effective quotient slope} \lesssim \text{direct chart drift} + \text{leafwise quotient holonomy} \times \text{branch graph slope}. \quad (31zmw)$$

So the open theorem may now try to control these three ingredients independently from intrinsic dynamics, rather than attacking the full effective-slope object in one step.

**Proposition 9.48O (Branchwise chart transport ratio reduces to the separated chart route).** In the setting of Proposition 9.48M, shrink  $J_q$  if necessary to the interval

$$J_q := q(x([a, b])). \quad (31zmx)$$

Assume there exist constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ ,  $\Gamma_\sigma \geq 0$ , and  $c_q > 0$  such that

$$\|\partial_s \Theta(s, w)\| \leq A_\Theta, \quad \|D_w \Theta(s, w)\| \leq B_\Theta \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmy)$$

and

$$|(q \circ x)'(\alpha)| \geq c_q, \quad \|(\sigma \circ q \circ x)'(\alpha)\| \leq \Gamma_\sigma |(q \circ x)'(\alpha)| \quad \text{for all } \alpha \in [a, b]. \quad (31z mz)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta \Gamma_\sigma)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zna)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31znb)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Fix  $s \in J_q$ . By the definition of  $J_q$  in (31zmx), there exists  $\alpha_s \in [a, b]$  such that

$$q(x(\alpha_s)) = s. \quad (31znc)$$

Applying the chain rule to the composite  $\sigma \circ q \circ x$  at  $\alpha_s$ , we obtain

$$(\sigma \circ q \circ x)'(\alpha_s) = D\sigma(s) (q \circ x)'(\alpha_s). \quad (31znd)$$

Because the domain of  $D\sigma(s)$  is one-dimensional, we have

$$\|D\sigma(s)(q \circ x)'(\alpha_s)\| = \|D\sigma(s)\| |(q \circ x)'(\alpha_s)|. \quad (31zne)$$

Combining (31zmq), (31znd), and (31zne), and dividing by the positive quantity  $|(q \circ x)'(\alpha_s)| \geq c_q$ , we get

$$\|D\sigma(s)\| = \frac{\|(\sigma \circ q \circ x)'(\alpha_s)\|}{|(q \circ x)'(\alpha_s)|} \leq \Gamma_\sigma. \quad (31znf)$$

Since  $s \in J_q$  was arbitrary, this shows that

$$\|\sigma'(s)\| \leq \Gamma_\sigma \quad \text{for all } s \in J_q. \quad (31zng)$$

Therefore Proposition 9.48N applies with  $S_\sigma := \Gamma_\sigma$ , and the budget

$$\eta := A_\Theta + B_\Theta \Gamma_\sigma \quad (31znh)$$

turns (31zna) into exactly the hypothesis (31zms) of Proposition 9.48N. Hence (31znb) and the local symbolic amplification conclusion follow.  $\square$

This removes one more hand-set local ingredient from the foliation route. The branch-graph slope no longer needs to be imposed directly as a free chart derivative bound: it is enough that, along the actual branch, the stable chart coordinate is transported at a controlled rate relative to the scalar quotient coordinate, with a nondegenerate scalar chart speed. So the open theorem may now try to derive a true branchwise chart transport law from invariant bundles and branch persistence, then feed it into the separated holonomy route.

**Proposition 9.48P (Stable-chart transport field reduces to the branchwise transport-ratio route).** In the setting of Proposition 9.48O, assume in addition that there exists a map

$$\Xi : J_q \times W_s \rightarrow \mathbb{R}^\ell \quad (31zoi)$$

such that

$$|(q \circ x)'(\alpha)| \geq c_q, \quad \|\Xi(s, w)\| \leq \Gamma_\sigma \quad \text{for all } \alpha \in [a, b], (s, w) \in J_q \times W_s, \quad (31z oj)$$

and

$$(\sigma \circ q \circ x)'(\alpha) = \Xi(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31zok)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta \Gamma_\sigma)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zol)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zom)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** For every  $\alpha \in [a, b]$ , the transport-field identity (31zok) and the norm bound in (31z oj) give

$$\|(\sigma \circ q \circ x)'(\alpha)\| \leq \|\Xi(q(x(\alpha)), \sigma(q(x(\alpha))))\| |(q \circ x)'(\alpha)| \leq \Gamma_\sigma |(q \circ x)'(\alpha)|. \quad (31zon)$$

So the hypotheses of Proposition 9.48O are satisfied with the same constants, and Proposition 9.48O therefore yields (31zom) together with the local symbolic amplification conclusion.  $\square$

This makes the foliation-side frontier more dynamical again. The branchwise chart transport ratio no longer needs to be supplied as an opaque estimate along the branch itself: it is enough that the branch slope be generated by a bounded transport field in stable-chart coordinates. So the open theorem may now try to derive such a transport field from invariant bundle transport and local chart regularity, and then feed it into Proposition 9.48O.

**Proposition 9.48Q (Leafwise velocity field plus bounded leaf inverse reduces to the transport-field route).** In the setting of Proposition 9.48P, assume in addition that there exist maps

$$V : J_q \times W_s \rightarrow \mathbb{R}^n, \quad \Lambda : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^\ell) \quad (31zoo)$$

and constants  $\Gamma_V \geq 0$  and  $C_\Lambda \geq 0$  such that

$$\begin{aligned} |(q \circ x)'(\alpha)| &\geq c_q, & \|V(s, w)\| &\leq \Gamma_V, & \|\Lambda(s, w)\| &\leq C_\Lambda & \text{for all } \alpha \in [a, b], (s, w) \in J_q \times W_s, \\ & & & & & & (31zop) \\ x'(\alpha) &= \partial_s \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) + V(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) & \text{for all } \alpha \in [a, b], \\ & & & & & & (31zoq) \end{aligned}$$

and

$$\Lambda(s, w) \circ D_w \Psi(s, w) = I_{\mathbb{R}^\ell}, \quad \Lambda(s, w) \partial_s \Psi(s, w) = 0 \quad \text{for all } (s, w) \in J_q \times W_s. \quad (31zor)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Lambda \Gamma_V)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zos)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zot)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Because we are in the setting of Proposition 9.48M, the graph identity (31zmo) holds:

$$x(\alpha) = \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) \quad \text{for all } \alpha \in [a, b]. \quad (31zou)$$

Differentiating (31zou) by the chain rule gives

$$x'(\alpha) = \left( \partial_s \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) + D_w \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) (\sigma \circ q \circ x)'(\alpha) \right) (q \circ x)'(\alpha). \quad (31zov)$$

Comparing (31zov) with the leafwise velocity decomposition (31zoq), we obtain

$$D_w \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) (\sigma \circ q \circ x)'(\alpha) = V(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha). \quad (31zow)$$

Applying  $\Lambda(q(x(\alpha)), \sigma(q(x(\alpha))))$  to both sides and using (31zor) yields

$$(\sigma \circ q \circ x)'(\alpha) = \Lambda(q(x(\alpha)), \sigma(q(x(\alpha)))) V(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha). \quad (31zox)$$

Define

$$\Xi(s, w) := \Lambda(s, w) V(s, w) \quad \text{for } (s, w) \in J_q \times W_s. \quad (31zoy)$$

Then (31zox) is exactly the transport-field identity (31zok). Moreover, by (31zop),

$$\|\Xi(s, w)\| \leq \|\Lambda(s, w)\| \|V(s, w)\| \leq C_\Lambda \Gamma_V \quad \text{for all } (s, w) \in J_q \times W_s. \quad (31zoz)$$

So the hypotheses of Proposition 9.48P are satisfied with

$$\Gamma_\sigma := C_\Lambda \Gamma_V, \quad (31zpa)$$

and the budget (31zos) becomes exactly the hypothesis (31zol) of Proposition 9.48P. Therefore Proposition 9.48P yields (31zot) together with the local symbolic amplification conclusion.  $\square$

This removes one more chart-coordinate artifact from the foliation route. The stable-chart transport field no longer needs to be postulated directly in coordinate space: it is enough that the ambient branch velocity split into scalar base drift plus a bounded leafwise component, together with a uniformly bounded left inverse for the leafwise chart derivative. So the open theorem may now try to derive such a bounded leafwise velocity field from invariant bundle transport, and use local foliation regularity only to read it back into stable coordinates.

**Proposition 9.48R (Bundle-adapted transport direction plus bounded stable coefficient reduces to the leafwise-velocity route).** In the setting of Proposition 9.48Q, assume in addition that there exist maps

$$E : J_q \times W_s \rightarrow \mathbb{R}^n, \quad U, R : J_q \times W_s \rightarrow \mathbb{R}^\ell \quad (31zpb)$$

and constants  $C_\Psi \geq 0$ ,  $\Gamma_U \geq 0$ , and  $\Gamma_R \geq 0$  such that

$$\|D_w \Psi(s, w)\| \leq C_\Psi, \quad \|U(s, w)\| \leq \Gamma_U, \quad \|R(s, w)\| \leq \Gamma_R \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zpc)$$

$$E(s, w) := \partial_s \Psi(s, w) + D_w \Psi(s, w) U(s, w) \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zpd)$$

and

$$x'(\alpha) := E(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) + D_w \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) R(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \quad (31zpe)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Lambda C_\Psi (\Gamma_U + \Gamma_R))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zpf)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zpg)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Define

$$W(s, w) := U(s, w) + R(s, w) \quad \text{for } (s, w) \in J_q \times W_s. \quad (31zph)$$

Combining the bundle-adapted transport identity (31zpd) with the branch decomposition (31zpe), we obtain

$$x'(\alpha) := \partial_s \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) + D_w \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) W(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha). \quad (31zpi)$$

Set

$$V(s, w) := D_w \Psi(s, w) W(s, w) \quad \text{for } (s, w) \in J_q \times W_s. \quad (31zpj)$$

Then (31zpi) is exactly the leafwise-velocity decomposition (31zoq) from Proposition 9.48Q. Moreover, by (31zpc) and the triangle inequality,

$$\|W(s, w)\| \leq \|U(s, w)\| + \|R(s, w)\| \leq \Gamma_U + \Gamma_R \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zpk)$$

and therefore

$$\|V(s, w)\| \leq \|D_w \Psi(s, w)\| \|W(s, w)\| \leq C_\Psi(\Gamma_U + \Gamma_R) \quad \text{for all } (s, w) \in J_q \times W_s. \quad (31zpl)$$

So the hypotheses of Proposition 9.48Q are satisfied with

$$\Gamma_V := C_\Psi(\Gamma_U + \Gamma_R). \quad (31zpm)$$

The budget (31zpf) is then exactly the hypothesis (31zos) of Proposition 9.48Q, and Proposition 9.48Q therefore yields (31zpg) together with the local symbolic amplification conclusion.  $\square$

This pushes the foliation-side frontier closer to invariant bundles. The ambient leafwise correction field no longer needs to be postulated directly: it is enough to exhibit a bundle-adapted transport direction whose deviation from the chart base direction is leafwise and uniformly bounded in stable coordinates, together with a uniformly bounded stable coefficient for the true branch velocity relative to that transported direction. So the open theorem may now try to derive such a bundle-adapted transport direction and coefficient bound from unstable bundle transport and branch persistence, while local foliation regularity enters only through the bounded chart operators  $D_w \Psi$  and  $\Lambda$ .

**Proposition 9.48S (Bundle-adapted transport direction plus bounded leafwise residual reduces to the stable-coefficient route).** In the setting of Proposition 9.48R, assume in addition that there exist a map

$$W : J_q \times W_s \rightarrow \mathbb{R}^n \quad (31zqa)$$

and a constant  $\Gamma_W \geq 0$  such that

$$\|W(s, w)\| \leq \Gamma_W \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zqb)$$

$$x'(\alpha) := E(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) + W(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b], \quad (31zqc)$$

and

$$W(s, w) \in \text{im } D_w \Psi(s, w) \quad \text{for all } (s, w) \in J_q \times W_s. \quad (31zqd)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi(\Gamma_U + C_\Lambda \Gamma_W))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zqe)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zqf)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** For each  $(s, w) \in J_q \times W_s$ , the leafwise-range condition (31zqd) lets us choose  $R(s, w) \in \mathbb{R}^\ell$  such that

$$D_w \Psi(s, w) R(s, w) = W(s, w). \quad (31zqg)$$

Applying  $\Lambda(s, w)$  to (31zqg) and using the left-inverse identity from (31zor) gives

$$R(s, w) = \Lambda(s, w) W(s, w). \quad (31zqh)$$

Therefore, by (31zop) and (31zqb),

$$\|R(s, w)\| \leq \|\Lambda(s, w)\| \|W(s, w)\| \leq C_\Lambda \Gamma_W \quad \text{for all } (s, w) \in J_q \times W_s. \quad (31zqi)$$

Now (31zqc) becomes exactly the branch decomposition (31zpe) from Proposition 9.48R, with the same maps  $E$  and  $U$  and with

$$\Gamma_R := C_\Lambda \Gamma_W. \quad (31zqj)$$

So the hypotheses of Proposition 9.48R are satisfied, and the budget (31zqe) is exactly the hypothesis (31zpf) of Proposition 9.48R. Therefore Proposition 9.48R yields (31zqf) together with the local symbolic amplification conclusion.  $\square$

This removes one more chart-coordinate artifact from the foliation/bundle route. The bounded stable coefficient no longer needs to be postulated directly in chart coordinates: it is enough that the residual between the true branch velocity and the transported direction be a uniformly bounded ambient leafwise field. So the open theorem may now try to derive a transported unstable direction together with a bounded ambient leafwise residual from unstable bundle transport and branch persistence, and use the foliation chart only to read that residual back into stable coordinates.

**Proposition 9.48T (Bundle-adapted transport direction plus bounded leafwise readout reduces to the leafwise-residual route).** In the setting of Proposition 9.48S, assume in addition that there exist an integer  $d \geq 1$ , maps

$$\Pi : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d), \quad B : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n), \quad Y : J_q \times W_s \rightarrow \mathbb{R}^d \quad (31zra)$$

and constants  $C_B \geq 0$  and  $\Gamma_Y \geq 0$  such that

$$\|B(s, w)\| \leq C_B, \quad \|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zrb)$$

$$\Pi(s, w) E(s, w) = 0 \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zrc)$$

$$B(s, w) \Pi(s, w) y = y \quad \text{for all } (s, w) \in J_q \times W_s, \quad y \in \text{im } D_w \Psi(s, w), \quad (31zrd)$$

$$x'(\alpha) - E(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \in \text{im } D_w \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) \quad \text{for all } \alpha \in [a, b], \quad (31zre)$$

and

$$\Pi(q(x(\alpha)), \sigma(q(x(\alpha)))) x'(\alpha) = Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31zrf)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda C_B \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zrg)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zrh)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Define

$$W(s, w) := B(s, w) Y(s, w) \quad \text{for } (s, w) \in J_q \times W_s. \quad (31zri)$$

Then by (31zrb),

$$\|W(s, w)\| \leq \|B(s, w)\| \|Y(s, w)\| \leq C_B \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s. \quad (31zrj)$$

Fix  $\alpha \in [a, b]$ , and abbreviate

$$s_\alpha := q(x(\alpha)), \quad w_\alpha := \sigma(q(x(\alpha))), \quad q'_\alpha := (q \circ x)'(\alpha). \quad (31zrk)$$

By the leafwise-residual membership hypothesis (31zre), the vector

$$\mathcal{R}_\alpha := x'(\alpha) - E(s_\alpha, w_\alpha) q'_\alpha \quad (31zrl)$$

lies in  $\text{im } D_w \Psi(s_\alpha, w_\alpha)$ . Therefore the readout reconstruction property (31zrd) gives

$$\mathcal{R}_\alpha = B(s_\alpha, w_\alpha) \Pi(s_\alpha, w_\alpha) \mathcal{R}_\alpha. \quad (31zrm)$$

Using the definition of  $\mathcal{R}_\alpha$ , the readout identity (31zrf), and the annihilation property (31zrc), we obtain

$$\Pi(s_\alpha, w_\alpha) \mathcal{R}_\alpha = \Pi(s_\alpha, w_\alpha) x'(\alpha) - \Pi(s_\alpha, w_\alpha) E(s_\alpha, w_\alpha) q'_\alpha = Y(s_\alpha, w_\alpha) q'_\alpha. \quad (31zrn)$$

Substituting (31zrn) into (31zrm) yields

$$x'(\alpha) - E(s_\alpha, w_\alpha) q'_\alpha = B(s_\alpha, w_\alpha) Y(s_\alpha, w_\alpha) q'_\alpha = W(s_\alpha, w_\alpha) q'_\alpha. \quad (31zro)$$

So (31zro) is exactly the branch decomposition (31zqc) from Proposition 9.48S, with

$$\Gamma_W := C_B \Gamma_Y. \quad (31zrp)$$

The budget (31zrg) is therefore exactly the hypothesis (31zqe) of Proposition 9.48S, and Proposition 9.48S yields (31zrh) together with the local symbolic amplification conclusion.  $\square$

This removes the last need to postulate the ambient leafwise residual field itself. It is enough to know that the residual sits in the leafwise image, and that there is a bounded ambient readout which kills the transported direction and can be inverted back on leafwise vectors. So the open theorem may now try to derive a transported unstable direction together with a bounded leafwise readout estimate from intrinsic bundle transport, observability, and branch persistence, and only then reconstruct the actual residual.

**Proposition 9.48U (Bundle-adapted transport direction plus ambient leafwise projector reduces to the readout route).** In the setting of Proposition 9.48S, assume in addition that there exist maps

$$\mathcal{P} : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), \quad Y : J_q \times W_s \rightarrow \mathbb{R}^n \quad (31zua)$$

and a constant  $\Gamma_Y \geq 0$  such that

$$\|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zub)$$

$$\mathcal{P}(s, w) E(s, w) = 0 \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zuc)$$

$$\mathcal{P}(s, w) y = y \quad \text{for all } (s, w) \in J_q \times W_s, \quad y \in \text{im } D_w \Psi(s, w), \quad (31zud)$$

and

$$\mathcal{P}(q(x(\alpha)), \sigma(q(x(\alpha)))) x'(\alpha) := Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31zue)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zuf)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zug)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Apply Proposition 9.48T with  $d = n$ , with

$$\Pi(s, w) := \mathcal{P}(s, w), \quad B(s, w) := I_{\mathbb{R}^n}. \quad (31zuh)$$

The annihilation condition (31zrc) of Proposition 9.48T is exactly (31zuc), while the reconstruction condition (31zrd) is exactly (31zud) because  $B = I_{\mathbb{R}^n}$ . The branch readout identity (31zrf) is exactly (31zue), and  $\|B(s, w)\| = 1$  for all  $(s, w)$ . Therefore the budget (31zuf) is exactly the hypothesis (31zrg) of Proposition 9.48T with  $C_B = 1$ . Proposition 9.48T therefore yields (31zug) together with the local symbolic amplification conclusion.  $\square$

This removes the last need to choose a separate finite-dimensional readout space and reconstruction map. It is already enough to build a single ambient operator which annihilates the transported direction and acts as the identity on the leafwise image. So the open theorem may now try to derive a transported unstable direction together with a bounded ambient leafwise projector estimate from intrinsic bundle splitting, observability, and branch persistence.

**Proposition 9.48V (Normalized ambient annihilator reduces to the projector route).**

In the setting of Proposition 9.48S, assume in addition that there exist maps

$$\ell : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}), \quad Y : J_q \times W_s \rightarrow \mathbb{R}^n \quad (31zva)$$

and a constant  $\Gamma_Y \geq 0$  such that

$$\|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zvb)$$

$$\ell(s, w) E(s, w) = 1 \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zvc)$$

$$\ell(s, w) y = 0 \quad \text{for all } (s, w) \in J_q \times W_s, \quad y \in \text{im } D_w \Psi(s, w), \quad (31zvd)$$

and

$$x'(\alpha) - \ell(q(x(\alpha)), \sigma(q(x(\alpha))))(x'(\alpha)) E(q(x(\alpha)), \sigma(q(x(\alpha)))) := Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b] \quad (31zve)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zvf)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zvg)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Define the ambient rank-one operator

$$\mathcal{P}_\ell(s, w) v := v - \ell(s, w)(v) E(s, w) \quad \text{for } (s, w) \in J_q \times W_s, \quad v \in \mathbb{R}^n. \quad (31zvh)$$

By the normalization (31zvc),

$$\mathcal{P}_\ell(s, w) E(s, w) = E(s, w) - \ell(s, w)(E(s, w)) E(s, w) = 0. \quad (31zvi)$$

By the annihilator property (31zvd), for every  $y \in \text{im } D_w \Psi(s, w)$ ,

$$\mathcal{P}_\ell(s, w) y = y - \ell(s, w)(y) E(s, w) = y. \quad (31zvj)$$

Finally, (31zve) is exactly the branch identity

$$\mathcal{P}_\ell(q(x(\alpha)), \sigma(q(x(\alpha)))) x'(\alpha) = Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b]. \quad (31zvk)$$

So  $\mathcal{P}_\ell$  satisfies the hypotheses (31zuc)-(31zue) of Proposition 9.48U with the same field  $Y$ . The budget (31zvf) is exactly the hypothesis (31zuf) of Proposition 9.48U, so Proposition 9.48U yields (31zvg) together with the local symbolic amplification conclusion.  $\square$

This removes the last need to choose the ambient projector itself as an independent object. It is already enough to produce a normalized codimension-one annihilator for the leafwise image and a bounded annihilator-defect field along the branch. So the open theorem may now try to derive a transported unstable direction together with a normalized ambient annihilator, or any equivalent bundle-splitting covector, from intrinsic bundle splitting, observability, and branch persistence.

**Proposition 9.48W (Non-normalized ambient annihilator with a scalar floor reduces to the normalized annihilator route).** In the setting of Proposition 9.48S, assume in addition that there exist maps

$$\tilde{\ell} : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}), \quad Y : J_q \times W_s \rightarrow \mathbb{R}^n \quad (31zwa)$$

and constants  $\Gamma_Y \geq 0$ ,  $m_\ell > 0$  such that

$$\|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zwb)$$

$$|\tilde{\ell}(s, w) E(s, w)| \geq m_\ell \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zwc)$$

$$\tilde{\ell}(s, w) y = 0 \quad \text{for all } (s, w) \in J_q \times W_s, \quad y \in \text{im } D_w \Psi(s, w), \quad (31zwd)$$

and

$$x'(\alpha) - \frac{\tilde{\ell}(q(x(\alpha)), \sigma(q(x(\alpha))))(x'(\alpha))}{\tilde{\ell}(q(x(\alpha)), \sigma(q(x(\alpha))))(E(q(x(\alpha)), \sigma(q(x(\alpha)))))} E(q(x(\alpha)), \sigma(q(x(\alpha)))) := Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad (31zwe)$$

If

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a), \quad (31zwf)$$

then there exists  $\alpha_0 \in (a, b)$  and an open interval  $J \subset I^\circ$  containing  $\alpha_0$  such that

$$G(\alpha) > 0 \quad \text{for all } \alpha \in J, \quad (31zwg)$$

and the conclusions of Proposition 9.21 hold on  $J$ .

**Proof.** Define the normalized annihilator

$$\ell(s, w) v := \frac{\tilde{\ell}(s, w) v}{\tilde{\ell}(s, w) (E(s, w))} \quad \text{for } (s, w) \in J_q \times W_s, \quad v \in \mathbb{R}^n. \quad (31zwh)$$

By (31zwc), the denominator is nonzero and  $\ell$  is well-defined. Also,

$$\ell(s, w) E(s, w) = 1 \quad (31zwi)$$

by definition, and by (31zwd),

$$\ell(s, w) y = 0 \quad \text{for all } y \in \text{im } D_w \Psi(s, w). \quad (31zwj)$$

Finally, (31zwe) is exactly (31zve) for this normalized  $\ell$ . So the hypotheses of Proposition 9.48V hold with the same field  $Y$ , and the budget (31zwf) is exactly (31zvf). Therefore Proposition 9.48V applies and yields (31zwg) together with the local symbolic amplification conclusion.  $\square$

This removes the explicit normalization burden from the annihilator route. It is now enough to produce a codimension-one ambient annihilator with a uniform nondegeneracy floor against the transported direction, plus the same bounded annihilator-defect field.

**Theorem Candidate 9.49 (Intrinsic bundle criterion without hand-set thresholds).** The real next theorem is to weaken Proposition 9.22, Proposition 9.24, or Proposition 9.25 by replacing the strong alignment condition

$$x'(\alpha) = a(\alpha)e(\alpha) \tag{31fk}$$

or the explicit cone / angle domination hypotheses in Proposition 9.33 and Corollary 9.34 by a genuinely dynamical assumption such as nontrivial projection of  $x'(\alpha)$  onto an unstable bundle or a natural bundle-preserving transport law of the type abstracted in Proposition 9.35, together with a visibility condition ensuring that  $D\kappa$  does not annihilate the transported unstable direction. After Proposition 9.37 and Corollary 9.38, the sharp remaining issue is to derive the strict seed-margin inequality at one point, a critical complementary-coordinate event, visible-cone entry, the averaged complementary-drift budget of Proposition 9.45, or the bounded-turning / small-diameter quotient geometry of Proposition 9.47, or the acute directional quotient drift geometry of Proposition 9.48A, or the directional quotient-width geometry of Proposition 9.48C, or the preferred-direction drift-budget geometry of Proposition 9.48D, or the near-critical preferred-direction speed geometry of Proposition 9.48E, or the local preferred-direction confinement geometry of Proposition 9.48F, or the local affine preferred-direction pinning geometry of Proposition 9.48G, or the endpoint-secant with bounded preferred curvature geometry of Proposition 9.48H, or the endpoint-return preferred-direction geometry of Proposition 9.48I, or the dominant scalar complementary-coordinate geometry of Proposition 9.48B, or the dominant scalar quotient-return geometry of Proposition 9.48J, or the dominant scalar quotient-secant geometry of Proposition 9.48K, or the scalar quotient-graph geometry of Proposition 9.48L, or the stable-foliation chart geometry of Proposition 9.48M, or the separated chart-drift / holonomy geometry of Proposition 9.48N, or the branchwise chart transport-ratio geometry of Proposition 9.48O, or the stable-chart transport-field geometry of Proposition 9.48P, or the leafwise-velocity / bounded-leaf-inverse geometry of Proposition 9.48Q, or the bundle-adapted transport-direction / stable-coefficient geometry of Proposition 9.48R, or the bundle-adapted transport-direction / leafwise-residual geometry of Proposition 9.48S, or the bundle-adapted transport-direction / leafwise-readout geometry of Proposition 9.48T, or the bundle-adapted transport-direction / leafwise-projector geometry of Proposition 9.48U, or the bundle-adapted transport-direction / normalized ambient annihilator geometry of Proposition 9.48V, or the bundle-adapted transport-direction / non-normalized ambient annihilator geometry of Proposition 9.48W, from intrinsic dynamics rather than taking them as external local hypotheses. Once that one-point or averaged branch input is available, Proposition 9.41B, Proposition 9.42, Proposition 9.43, Corollary 9.44, Proposition 9.45, Corollary 9.46, Proposition 9.47, Corollary 9.48, Proposition 9.48A, Proposition 9.48B, Proposition 9.48C, Proposition 9.48D, Proposition 9.48E, Proposition 9.48F, Proposition 9.48G, Proposition 9.48H, Proposition 9.48I, Proposition 9.48J, Proposition 9.48K, Proposition 9.48L, Proposition 9.48M, Proposition 9.48N, Proposition 9.48O, Proposition 9.48P, Proposition 9.48Q, Proposition 9.48R, Proposition 9.48S, Proposition 9.48T, Proposition 9.48U, Proposition 9.48V, and Proposition 9.48W turn it into an explicit local amplification interval.

That step is not yet proved here. It is the first place where Lyapunov growth, invariant bundles,

and the observation map must be combined in a genuinely nontrivial way.

Equivalently, after Proposition 9.38C, Proposition 9.45, Proposition 9.47, Proposition 9.48A, Proposition 9.48B, Proposition 9.48C, Proposition 9.48D, Proposition 9.48E, Proposition 9.48F, Proposition 9.48G, Proposition 9.48H, Proposition 9.48I, Proposition 9.48J, Proposition 9.48K, Proposition 9.48L, Proposition 9.48M, Proposition 9.48N, Proposition 9.48O, Proposition 9.48P, Proposition 9.48Q, Proposition 9.48R, Proposition 9.48S, Proposition 9.48T, Proposition 9.48U, Proposition 9.48V, and Proposition 9.48W, the theorem can now be read in any of the following local frontier forms:

- **projection-ratio form:** produce  $\alpha_0$  such that

$$L_\kappa C_c \|P_{\alpha_0}^c x'(\alpha_0)\| < a_u(\alpha_0) c_o C_u, \quad (31nf)$$

- **visible-cone form:** produce  $\alpha_0$  such that

$$x'(\alpha_0) \neq 0, \quad \angle(x'(\alpha_0), E^u(\alpha_0)) < \vartheta_{\text{vis}}, \quad (31ng)$$

for some aperture satisfying

$$L_\kappa C_c \tan \vartheta_{\text{vis}} < c_o C_u. \quad (31nh)$$

- **averaged complementary-drift form:** produce a quotient map  $z$ , a branch segment  $[a, b]$ , and constants  $m_z > 0$ ,  $a_* > 0$ ,  $A_c \geq 0$  such that

$$\|Dz(x)v\| \geq m_z \|P_x^c v\|, \quad a_u(\alpha) \geq a_*, \quad \int_a^b \|(z \circ x)'(\alpha)\| d\alpha \leq A_c (b - a), \quad (31os)$$

and

$$L_\kappa C_c A_c < m_z a_* c_o C_u. \quad (31ot)$$

- **bounded-turning quotient form:** produce a quotient map  $z$ , a branch segment  $[a, b]$ , and constants  $m_z > 0$ ,  $a_* > 0$ ,  $K_{\text{bt}} \geq 1$  such that

$$\mathcal{L}_z([a, b]) \leq K_{\text{bt}} \text{diam}(z(x([a, b]))), \quad (31pd)$$

and

$$L_\kappa C_c K_{\text{bt}} \text{diam}(z(x([a, b]))) < m_z a_* c_o C_u (b - a). \quad (31pe)$$

- **acute directional quotient-drift form:** produce a quotient map  $z$ , a branch segment  $[a, b]$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  such that

$$\langle (z \circ x)'(\alpha), \nu \rangle \geq \cos(\vartheta_q) \|(z \circ x)'(\alpha)\| \quad \text{for all } \alpha \in [a, b], \quad (31pf)$$

and

$$L_\kappa C_c \sec(\vartheta_q) \text{diam}(z(x([a, b]))) < m_z a_* c_o C_u (b - a). \quad (31pg)$$

- **acute directional quotient-width form:** produce a quotient map  $z$ , a branch segment  $[a, b]$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), together with

$$\mathcal{W}_\nu([a, b]) := \sup_{\alpha, \beta \in [a, b]} |\langle z(x(\beta)) - z(x(\alpha)), \nu \rangle|, \quad (31pj)$$

and

$$L_\kappa C_c \sec(\vartheta_q) \mathcal{W}_\nu([a, b]) < m_z a_* c_o C_u (b - a). \quad (31pk)$$

- **preferred-direction drift-budget form:** produce a quotient map  $z$ , a branch segment  $[a, b]$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), together with a constant  $A_\nu \geq 0$  such that

$$\int_a^b \langle (z \circ x)'(\alpha), \nu \rangle d\alpha \leq A_\nu(b-a), \quad (31tl)$$

and

$$L_\kappa C_c \sec(\vartheta_q) A_\nu < m_z a_* c_o C_u. \quad (31tm)$$

- **near-critical preferred-direction speed form:** produce a quotient map  $z$ , a point  $\alpha_*$ , a radius  $\rho > 0$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle, \quad (31un)$$

and show that

$$[\alpha_* - \rho, \alpha_* + \rho] \subset [a, b], \quad q'_\nu(\alpha_*) \leq \delta_\nu, \quad |q''_\nu(\alpha)| \leq B_\nu \quad \text{for all } \alpha \in [\alpha_* - \rho, \alpha_* + \rho], \quad (31uo)$$

together with

$$L_\kappa C_c \sec(\vartheta_q) \left( \delta_\nu + \frac{B_\nu \rho}{2} \right) < m_z a_* c_o C_u. \quad (31up)$$

- **local preferred-direction confinement form:** produce a quotient map  $z$ , a point  $\bar{\alpha}$ , a radius  $\rho > 0$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle, \quad (31vz)$$

and show that

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad |q_\nu(\alpha) - q_\nu(\bar{\alpha})| \leq E_\nu \quad \text{for all } \alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho], \quad (31wa)$$

together with

$$L_\kappa C_c \sec(\vartheta_q) \frac{E_\nu}{\rho} < m_z a_* c_o C_u. \quad (31wb)$$

- **local affine preferred-direction pinning form:** produce a quotient map  $z$ , a point  $\bar{\alpha}$ , a radius  $\rho > 0$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle, \quad (31xh)$$

and show that

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad |q_\nu(\alpha) - q_\nu(\bar{\alpha}) - v_\nu(\alpha - \bar{\alpha})| \leq R_\nu \quad \text{for all } \alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho], \quad (31xi)$$

together with

$$L_\kappa C_c \sec(\vartheta_q) \left( |v_\nu| + \frac{R_\nu}{\rho} \right) < m_z a_* c_o C_u. \quad (31xj)$$

- **endpoint-return preferred-direction form:** produce a quotient map  $z$ , a point  $\bar{\alpha}$ , a radius  $\rho > 0$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle, \quad (31yj)$$

and show that

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad s_\nu := \frac{q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho)}{2\rho}, \quad (31yk)$$

together with

$$L_\kappa C_c \sec(\vartheta_q) |s_\nu| < m_z a_* c_o C_u. \quad (31yl)$$

- **endpoint-secant with preferred-curvature control form:** produce a quotient map  $z$ , a point  $\bar{\alpha}$ , a radius  $\rho > 0$ , a unit vector  $\nu \in \mathbb{R}^m$ , and an angle  $0 \leq \vartheta_q < \pi/2$  satisfying (31pf), define

$$q_\nu(\alpha) := \langle z(x(\alpha)), \nu \rangle, \quad (31xx)$$

and show that

$$[\bar{\alpha} - \rho, \bar{\alpha} + \rho] \subset [a, b], \quad |q_\nu''(\alpha)| \leq B_\nu \quad \text{for all } \alpha \in [\bar{\alpha} - \rho, \bar{\alpha} + \rho], \quad (31xy)$$

together with the small preferred secant condition

$$s_\nu := \frac{q_\nu(\bar{\alpha} + \rho) - q_\nu(\bar{\alpha} - \rho)}{2\rho}, \quad L_\kappa C_c \sec(\vartheta_q) (|s_\nu| + B_\nu \rho) < m_z a_* c_o C_u. \quad (31xz)$$

- **dominant scalar quotient-coordinate form:** produce a quotient map  $z = (q, r)$ , a branch segment  $[a, b]$ , a sign  $\sigma \in \{-1, 1\}$ , and a spillover constant  $\eta \geq 0$  such that

$$\sigma(q \circ x)'(\alpha) \geq 0, \quad \|(r \circ x)'(\alpha)\| \leq \eta \sigma(q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b], \quad (31ph)$$

and

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31pi)$$

- **dominant scalar quotient-return form:** produce a quotient map  $z = (q, r)$ , a branch segment  $[a, b]$ , a spillover constant  $\eta \geq 0$ , and a speed floor  $s_* > 0$  such that

$$\|(r \circ x)'(\alpha)\| \leq \eta |(q \circ x)'(\alpha)|, \quad \|x'(\alpha)\| \geq s_* \quad \text{for all } \alpha \in [a, b], \quad (31yw)$$

and

$$q(x(a)) = q(x(b)). \quad (31yx)$$

- **dominant scalar quotient-secant form:** produce a quotient map  $z = (q, r)$ , a branch segment  $[a, b]$ , and a spillover constant  $\eta \geq 0$  such that

$$\|(r \circ x)'(\alpha)\| \leq \eta |(q \circ x)'(\alpha)| \quad \text{for all } \alpha \in [a, b], \quad (31zkm)$$

and

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zkn)$$

- **scalar quotient-graph form:** produce a quotient map  $z = (q, r)$ , a branch segment  $[a, b]$ , an interval  $J_q \subset \mathbb{R}$  containing  $q(x([a, b]))$ , a  $C^1$  map  $\Phi : J_q \rightarrow \mathbb{R}^{m-1}$ , and a constant  $\eta \geq 0$  such that

$$r(x(\alpha)) = \Phi(q(x(\alpha))), \quad \|D\Phi(s)\| \leq \eta \quad \text{for all } \alpha \in [a, b], s \in J_q, \quad (31zli)$$

and

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zlj)$$

- **stable-foliation chart form:** produce a quotient map  $z = (q, r)$ , a branch segment  $[a, b]$ , an interval  $J_q \subset \mathbb{R}$ , an open set  $W_s \subset \mathbb{R}^\ell$ , a  $C^1$  chart  $\Psi : J_q \times W_s \rightarrow U_0$ , a  $C^1$  graph map  $\sigma : J_q \rightarrow W_s$ , a  $C^1$  profile  $\Theta : J_q \times W_s \rightarrow \mathbb{R}^{m-1}$ , and a constant  $\eta \geq 0$  such that

$$q(\Psi(s, w)) = s, \quad r(\Psi(s, w)) = \Theta(s, w), \quad x([a, b]) \subset \{\Psi(s, \sigma(s)) : s \in J_q\}, \quad (31zmm1)$$

$$\|\partial_s \Theta(s, \sigma(s)) + D_w \Theta(s, \sigma(s)) \sigma'(s)\| \leq \eta \quad \text{for all } s \in J_q, \quad (31zmm2)$$

and

$$L_\kappa C_c \sqrt{1 + \eta^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm3)$$

- **separated chart-slope / holonomy form:** produce the same chart data as above together with constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ , and  $S_\sigma \geq 0$  such that

$$\|\partial_s \Theta(s, w)\| \leq A_\Theta, \quad \|D_w \Theta(s, w)\| \leq B_\Theta \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmm4)$$

$$\|\sigma'(s)\| \leq S_\sigma \quad \text{for all } s \in J_q, \quad (31zmm5)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta S_\sigma)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm6)$$

- **branchwise chart transport-ratio form:** produce the same chart data as above, shrink  $J_q$  to  $q(x([a, b]))$ , and find constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ ,  $\Gamma_\sigma \geq 0$ , and  $c_q > 0$  such that

$$\|\partial_s \Theta(s, w)\| \leq A_\Theta, \quad \|D_w \Theta(s, w)\| \leq B_\Theta \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmm7)$$

$$|(q \circ x)'(\alpha)| \geq c_q, \quad \|(\sigma \circ q \circ x)'(\alpha)\| \leq \Gamma_\sigma |(q \circ x)'(\alpha)| \quad \text{for all } \alpha \in [a, b], \quad (31zmm8)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta \Gamma_\sigma)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm9)$$

- **stable-chart transport-field form:** produce the same chart data as above, shrink  $J_q$  to  $q(x([a, b]))$ , and find constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ ,  $\Gamma_\sigma \geq 0$ ,  $c_q > 0$ , and a map  $\Xi : J_q \times W_s \rightarrow \mathbb{R}^\ell$  such that

$$|(q \circ x)'(\alpha)| \geq c_q, \quad \|\Xi(s, w)\| \leq \Gamma_\sigma \quad \text{for all } \alpha \in [a, b], (s, w) \in J_q \times W_s, \quad (31zmm10)$$

$$(\sigma \circ q \circ x)'(\alpha) = \Xi(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b], \quad (31zmm11)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta \Gamma_\sigma)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm12)$$

- **leafwise-velocity / bounded-leaf-inverse form:** produce the same chart data as above, shrink  $J_q$  to  $q(x([a, b]))$ , and find constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ ,  $\Gamma_V \geq 0$ ,  $C_\Lambda \geq 0$ ,  $c_q > 0$ , a bounded leafwise velocity field  $V : J_q \times W_s \rightarrow \mathbb{R}^n$ , and a bounded left inverse  $\Lambda : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^\ell)$  such that

$$|(q \circ x)'(\alpha)| \geq c_q, \quad \|V(s, w)\| \leq \Gamma_V, \quad \|\Lambda(s, w)\| \leq C_\Lambda \quad \text{for all } \alpha \in [a, b], (s, w) \in J_q \times W_s, \quad (31zmm13)$$

$$x'(\alpha) = \partial_s \Psi(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) + V(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha), \quad (31zmm14)$$

$$\Lambda(s, w) \circ D_w \Psi(s, w) = I_{\mathbb{R}^\ell}, \quad \Lambda(s, w) \partial_s \Psi(s, w) = 0 \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmm15)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Lambda \Gamma_V)^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm16)$$

- **bundle-adapted transport-direction / leafwise-residual form:** produce the same chart data as above, shrink  $J_q$  to  $q(x([a, b]))$ , and find constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ ,  $C_\Psi \geq 0$ ,  $C_\Lambda \geq 0$ ,  $c_q > 0$ ,  $\Gamma_U \geq 0$ , and  $\Gamma_W \geq 0$ , together with maps  $E : J_q \times W_s \rightarrow \mathbb{R}^n$ ,  $U : J_q \times W_s \rightarrow \mathbb{R}^\ell$ , and  $W : J_q \times W_s \rightarrow \mathbb{R}^n$  such that

$$\|D_w \Psi(s, w)\| \leq C_\Psi, \quad \|\Lambda(s, w)\| \leq C_\Lambda, \quad \|U(s, w)\| \leq \Gamma_U, \quad \|W(s, w)\| \leq \Gamma_W \quad (31zmm17)$$

for all  $(s, w) \in J_q \times W_s$ ,

$$|(q \circ x)'(\alpha)| \geq c_q, \quad E(s, w) = \partial_s \Psi(s, w) + D_w \Psi(s, w) U(s, w), \quad (31zmm18)$$

$$x'(\alpha) = E(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) + W(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha), \quad (31zmm19)$$

$$\Lambda(s, w) \circ D_w \Psi(s, w) = I_{\mathbb{R}^\ell}, \quad \Lambda(s, w) \partial_s \Psi(s, w) = 0, \quad W(s, w) \in \text{im } D_w \Psi(s, w), \quad (31zmm20)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda \Gamma_W))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm21)$$

- **bundle-adapted transport-direction / leafwise-readout form:** produce the same chart data as above, shrink  $J_q$  to  $q(x([a, b]))$ , and find constants  $A_\Theta \geq 0$ ,  $B_\Theta \geq 0$ ,  $C_\Psi \geq 0$ ,  $C_\Lambda \geq 0$ ,  $C_B \geq 0$ ,  $c_q > 0$ ,  $\Gamma_U \geq 0$ , and  $\Gamma_Y \geq 0$ , and some integer  $d \geq 1$ , together with maps  $E : J_q \times W_s \rightarrow \mathbb{R}^n$ ,  $U : J_q \times W_s \rightarrow \mathbb{R}^\ell$ ,  $\Pi : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d)$ ,  $B : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$ , and  $Y : J_q \times W_s \rightarrow \mathbb{R}^d$  such that

$$\|D_w \Psi(s, w)\| \leq C_\Psi, \quad \|\Lambda(s, w)\| \leq C_\Lambda, \quad \|B(s, w)\| \leq C_B, \quad (31zmm22)$$

$$\|U(s, w)\| \leq \Gamma_U, \quad \|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmm23)$$

$$|(q \circ x)'(\alpha)| \geq c_q, \quad E(s, w) = \partial_s \Psi(s, w) + D_w \Psi(s, w) U(s, w), \quad (31zmm24)$$

$$\Lambda(s, w) \circ D_w \Psi(s, w) = I_{\mathbb{R}^\ell}, \quad \Lambda(s, w) \partial_s \Psi(s, w) = 0, \quad \Pi(s, w) E(s, w) = 0, \quad (31zmm25)$$

$$B(s, w) \Pi(s, w) y = y \quad \text{for all } (s, w) \in J_q \times W_s, \quad y \in \text{im } D_w \Psi(s, w), \quad (31zmm26)$$

$$x'(\alpha) - E(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \in \text{im } D_w \Psi(q(x(\alpha)), \sigma(q(x(\alpha))))), \quad (31zmm27)$$

$$\Pi(q(x(\alpha)), \sigma(q(x(\alpha)))) x'(\alpha) = Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha) \quad \text{for all } \alpha \in [a, b], \quad (31zmm28)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda C_B \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm29)$$

- **bundle-adapted transport-direction / leafwise-projector form:** produce the same chart data as above, but it is already enough to replace the pair  $(\Pi, B)$  by a single ambient operator  $\mathcal{P} : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$  and a bounded field  $Y : J_q \times W_s \rightarrow \mathbb{R}^n$  such that  $\mathcal{P}E = 0$ ,  $\mathcal{P}y = y$  on  $\text{im } D_w \Psi$ , and

$$\mathcal{P}(q(x(\alpha)), \sigma(q(x(\alpha)))) x'(\alpha) = Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha), \quad (31zmm29a)$$

together with

$$\|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmm29b)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm29c)$$

Then Proposition 9.48U applies directly.

- **bundle-adapted transport-direction / normalized ambient annihilator form:** produce the same chart data as above, but it is already enough to replace the projector  $\mathcal{P}$  itself by a scalar covector  $\ell : J_q \times W_s \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$  and a bounded field  $Y : J_q \times W_s \rightarrow \mathbb{R}^n$  such that

$$\ell E = 1, \quad \ell y = 0 \text{ on } \text{im } D_w \Psi, \quad (31zmm29d)$$

and

$$x'(\alpha) - \ell(q(x(\alpha)), \sigma(q(x(\alpha))))(x'(\alpha)) E(q(x(\alpha)), \sigma(q(x(\alpha)))) = Y(q(x(\alpha)), \sigma(q(x(\alpha)))) (q \circ x)'(\alpha), \quad (31zmm29e)$$

together with

$$\|Y(s, w)\| \leq \Gamma_Y \quad \text{for all } (s, w) \in J_q \times W_s, \quad (31zmm29f)$$

and

$$L_\kappa C_c \sqrt{1 + (A_\Theta + B_\Theta C_\Psi (\Gamma_U + C_\Lambda \Gamma_Y))^2} |q(x(b)) - q(x(a))| < m_z a_* c_o C_u (b - a). \quad (31zmm29g)$$

Then Proposition 9.48V applies directly.

So the open theorem is no longer best thought of as “derive a mysterious local inequality”. It is better thought of as: show that intrinsic dynamics and observability force either the parameter direction to enter the observable unstable cone at least once along the crossing branch, or the branch to satisfy a strong enough averaged complementary-drift, bounded-turning, acute directional quotient-drift, preferred-direction drift-budget, near-critical preferred-direction speed, local preferred-direction confinement, local affine preferred-direction pinning, endpoint-return preferred-direction geometry, endpoint-secant with preferred-curvature control, directional quotient-width, dominant scalar complementary-coordinate budget, dominant scalar quotient-return geometry, dominant scalar quotient-secant geometry, scalar quotient-graph geometry, stable-foliation chart geometry, or separated chart-slope / holonomy geometry, or branchwise chart transport-ratio geometry, or stable-chart transport-field geometry, or leafwise-velocity / bounded-leaf-inverse geometry, or bundle-adapted transport-direction / leafwise-residual geometry, or bundle-adapted transport-direction / leafwise-readout geometry, or bundle-adapted transport-direction / leafwise-projector geometry, or bundle-adapted transport-direction / normalized ambient annihilator geometry, that such an entry point follows automatically.

Conceptually, the remaining theorem should now be split into three moves. First, a **structural tangency-or-visible-cone theorem** should force a critical complementary-coordinate event as in Proposition 9.43, a codimension-one return mechanism as in Corollary 9.44, a codimension-free averaged complementary-drift budget as in Proposition 9.45 and Corollary 9.46, a bounded-turning quotient geometry as in Proposition 9.47, Corollary 9.48, Proposition 9.48A, Proposition 9.48B, Proposition 9.48C, Proposition 9.48D, Proposition 9.48E, Proposition 9.48F, Proposition 9.48G, Proposition 9.48H, Proposition 9.48I, Proposition 9.48J, Proposition 9.48K, Proposition 9.48L, Proposition 9.48M, Proposition 9.48N, Proposition 9.48O, Proposition 9.48P, Proposition 9.48Q, Proposition 9.48R, Proposition 9.48S, Proposition 9.48T, Proposition 9.48U, Proposition 9.48V, or a direct visible-cone entry. Second, an **intrinsic drift-control theorem** should upgrade bundle regularity to the  $C^1$  control used in Proposition 9.41. Third, a **structural seed-margin theorem** should force

$$G(\alpha_0) > 0 \quad (31ni)$$

at some interior point  $\alpha_0$  from intrinsic bundle transport, unstable projection, and lens visibility. Finally, a **local margin-propagation step** should use continuity of  $G$  to shrink to an interval  $J$

on which

$$G(\alpha) \geq g_* > 0 \quad \text{for all } \alpha \in J. \quad (31nj)$$

Once that final step is available, the bookkeeping layer above can be applied as a closed downstream module rather than as part of the frontier itself.

From the formalization point of view, this now marks the clean Lean boundary of the project. The bookkeeping side already has a working formal spine: crossing-time shift, mutual certified windows, symbolic disagreement localization, and the ordered disagreement-volume formula on finite windows are now all on the Lean side. The first scalar core of the new foliation-side reduction is now also Lean-ready: nonzero base speed + branchwise transport ratio  $\rightarrow$  graph-slope bound is a one-dimensional chain-rule lemma, and the next vector bookkeeping step now reduces bounded left inverse + bounded ambient leafwise residual  $\rightarrow$  bounded stable coefficient. What is not yet stabilized formally is exactly the structural side of the theory: the derivation of a positive visible margin from bundle transport, lens visibility, and intrinsic coefficient-gap dynamics. In other words, the frontier has moved where it belongs, namely to the bundle-dynamical production of the margin rather than to one-dimensional symbolic bookkeeping.

## 9.7 Canonical examples

The definitions above become concrete in three benchmark examples, one reusable theorem template, and one worked local verification.

**Example 9.1 (Neural classification).**

$$u(x) = \ell(x) \in \mathbb{R}^m, \quad c(x) = \operatorname{argmax}_j \ell_j(x). \quad (32)$$

The latent score field is continuous or piecewise-smooth; the label is observationally discrete.

**Example 9.2 (Thresholded diffusion).** Let  $u_t$  be a continuous diffusion and define

$$c_t = \mathbf{1}\{u_t \geq L\}. \quad (33)$$

Then  $u_t$  may be continuous while  $c_t$  exhibits observational jumps.

**Example 9.3 (Symbolic dynamics).** Let  $u_t$  evolve under a smooth chaotic flow and let  $Q$  encode which partition cell contains  $\kappa(u_t)$ . Then the symbolic process  $c_t$  may look more jagged, unstable, or incompressible than the smooth latent trajectory itself. This is the prototype of observer-relative chaos.

**Proposition 9.24 (Uniform 2D delayed-threshold theorem template).** The following explicit family packages the paper’s symbolic-amplification mechanism into a reusable theorem template.

Fix  $\lambda > 0$  and  $H > 1$ . Consider the flow on  $\mathbb{R}^2$

$$\Psi_t(x, z) = (x + t, e^{\lambda t} z), \quad (31cp0)$$

the observation lens

$$\kappa_*(x, z) = x + z, \quad (31cp1)$$

the threshold  $L = 0$ , and the initial-condition curve

$$\eta_H(\alpha) = (-H, \alpha). \quad (31cp2)$$

Define

$$v_t(\alpha) = \Psi_t(\eta_H(\alpha)) = (-H + t, e^{\lambda t} \alpha), \quad (31cp3)$$

and

$$Z_H(\alpha, t) = \kappa_*(v_t(\alpha)) = t - H + \alpha e^{\lambda t}. \quad (31cp4)$$

Let

$$\rho_\lambda := \frac{e^{-\lambda/2}}{4 \max\{1, \lambda\}}. \quad (31cp5)$$

Then for every

$$|\alpha| \leq \rho_\lambda e^{-\lambda H}, \quad (31cp6)$$

the equation

$$Z_H(\alpha, t) = 0 \quad (31cp7)$$

has a unique solution  $\sigma_H(\alpha) \in [H - 1/2, H + 1/2]$ . Moreover, for all

$$|\alpha|, |\beta| \leq \rho_\lambda e^{-\lambda H}, \quad (31cp8)$$

we have

$$|\sigma_H(\alpha) - \sigma_H(\beta)| \geq \frac{4}{5} e^{-\lambda/2} e^{\lambda H} |\alpha - \beta|, \quad (31cp9)$$

and if

$$d_t^{(\alpha)} = \mathbf{1}\{Z_H(\alpha, t) \geq 0\}, \quad (31cq0)$$

then

$$\int_0^{H+1} |d_t^{(\alpha)} - d_t^{(\beta)}| dt = |\sigma_H(\alpha) - \sigma_H(\beta)| \geq \frac{4}{5} e^{-\lambda/2} e^{\lambda H} |\alpha - \beta|. \quad (31cq1)$$

**Proof.** For  $t \in [H - 1/2, H + 1/2]$  and  $|\alpha| \leq \rho_\lambda e^{-\lambda H}$ ,

$$|\alpha| e^{\lambda t} \leq \rho_\lambda e^{\lambda/2} = \frac{1}{4 \max\{1, \lambda\}} \leq \frac{1}{4}. \quad (31cq2)$$

Hence

$$Z_H(\alpha, H - 1/2) \leq -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}, \quad (31cq3)$$

while

$$Z_H(\alpha, H + 1/2) \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}. \quad (31cq4)$$

So a zero exists in  $[H - 1/2, H + 1/2]$ . Also

$$\partial_t Z_H(\alpha, t) = 1 + \lambda \alpha e^{\lambda t}. \quad (31cq5)$$

Using the same bound,

$$|\lambda \alpha e^{\lambda t}| \leq \frac{\lambda}{4 \max\{1, \lambda\}} \leq \frac{1}{4}, \quad (31cq6)$$

so

$$\frac{3}{4} \leq \partial_t Z_H(\alpha, t) \leq \frac{5}{4} \quad \text{for } t \in [H - 1/2, H + 1/2]. \quad (31cq7)$$

Thus  $t \mapsto Z_H(\alpha, t)$  is strictly increasing on that interval, and the zero is unique. By the implicit derivative formula,

$$\sigma'_H(\alpha) = -\frac{\partial_\alpha Z_H(\alpha, \sigma_H(\alpha))}{\partial_t Z_H(\alpha, \sigma_H(\alpha))} = -\frac{e^{\lambda \sigma_H(\alpha)}}{1 + \lambda \alpha e^{\lambda \sigma_H(\alpha)}}. \quad (31cq8)$$

Since  $\sigma_H(\alpha) \geq H - 1/2$  and the denominator is at most  $5/4$ ,

$$|\sigma'_H(\alpha)| \geq \frac{4}{5}e^{\lambda(H-1/2)} = \frac{4}{5}e^{-\lambda/2}e^{\lambda H}. \quad (31cq9)$$

Integrating this bound between  $\alpha$  and  $\beta$  gives (31cp9). The symbolic identity in (31cq1) then follows from Proposition 9.12.  $\square$

This proposition is the paper's first reusable 2D theorem template: an exponentially amplified hidden mode, a lens that sees that mode, and a transversal threshold crossing together force an exponentially large symbolic disagreement scale.

**Example 9.4 (The same model as a local worked verification).** Fix  $\lambda > 0$  and  $H > 1$ . Consider the flow on  $\mathbb{R}^2$

$$\Phi_t(x, z) = (x + t, e^{\lambda t}z), \quad (31cp)$$

the observation lens

$$\kappa(x, z) = x + z, \quad (31cq)$$

the threshold  $L = 0$ , and the initial-condition curve

$$x_H(\alpha) = (-H, \alpha). \quad (31cr)$$

Then

$$u_t(\alpha) = \Phi_t(x_H(\alpha)) = (-H + t, e^{\lambda t}\alpha), \quad (31cs)$$

so the observation coordinate is

$$Y_H(\alpha, t) = \kappa(u_t(\alpha)) = t - H + \alpha e^{\lambda t}. \quad (31ct)$$

Let

$$\rho_\lambda := \frac{e^{-\lambda/2}}{4 \max\{1, \lambda\}}. \quad (31cu)$$

Then for every

$$|\alpha| \leq \rho_\lambda e^{-\lambda H}, \quad (31cv)$$

the equation

$$Y_H(\alpha, t) = 0 \quad (31cw)$$

has a unique solution  $\tau_H(\alpha) \in [H - 1/2, H + 1/2]$ . Moreover,

$$\tau'_H(\alpha) = -\frac{e^{\lambda\tau_H(\alpha)}}{1 + \lambda\alpha e^{\lambda\tau_H(\alpha)}}. \quad (31cx)$$

After shrinking  $\varepsilon_H$  if necessary, we may assume

$$\frac{1}{2} \leq 1 + \lambda\alpha e^{\lambda\tau_H(\alpha)} \leq \frac{3}{2}, \quad \tau_H(\alpha) \in [H - 1/2, H + 1/2], \quad (31cy)$$

for all  $|\alpha| < \varepsilon_H$ . Therefore

$$|\tau'_H(\alpha)| \geq \frac{2}{3}e^{\lambda(H-1/2)} = \left(\frac{2}{3}e^{-\lambda/2}\right)e^{\lambda H}. \quad (31cz)$$

Integrating this bound over  $\alpha$  gives a constant  $c_\lambda > 0$  such that

$$|\tau_H(\alpha) - \tau_H(\beta)| \geq c_\lambda e^{\lambda H} |\alpha - \beta|, \quad |\alpha|, |\beta| < \varepsilon_H. \quad (31da)$$

If

$$c_t^{(\alpha)} = \mathbf{1}\{Y_H(\alpha, t) \geq 0\}, \quad (31db)$$

then Proposition 9.12 yields

$$\int_0^{H+1} |c_t^{(\alpha)} - c_t^{(\beta)}| dt = |\tau_H(\alpha) - \tau_H(\beta)| \geq c_\lambda e^{\lambda H} |\alpha - \beta|. \quad (31dc)$$

This is a fully explicit realization of the paper’s mechanism. The latent perturbation lives entirely in the hidden unstable coordinate  $z$ , the lens  $\kappa(x, z) = x + z$  makes that direction visible, and the threshold converts the resulting crossing-time shift into an interval of symbolic disagreement whose size grows like  $e^{\lambda H}$ .

## 9.8 Why this formal core matters

The advantage of this language is that it stops three notions from collapsing into one vague intuition:

- discreteness = codomain structure,
- jumps = path regularity,
- chaos = dynamical sensitivity.

Once they are separated, one can ask the right question in each case:

$$\text{which layer of the state-observation stack generates the phenomenon?} \quad (34)$$

That is the paper’s real methodological proposal.

## 10. The Research Program

The current paper is a foundation note. The next theorem-level tasks are clearer than the final answers. The most natural next steps are:

1. **Atomic lift theorem**
  - formalize discrete observables as atomic measures on continuous state spaces,
  - prove canonical weak convergence of  $f_\varepsilon$  back to  $\mu$ .
2. **Spectral smoothing theorem**
  - quantify how Gaussian or Fejer smoothing modifies coefficient decay,
  - identify the stable operating region where the lifted family becomes computationally compressible.
3. **Quantized observation theorem**
  - formalize the decomposition  $u \mapsto \kappa(u) \mapsto Q(\kappa(u))$ ,
  - bound information loss under quantization.
4. **Observer-relative chaos theorem**
  - compare latent instability and symbolic instability,

- isolate the component of observed unpredictability created by coarse-graining.
5. **Phase-label theorem**
- separate finite-size smooth crossover, limiting singularity, and reporting quantization in phase transition language.

These are not all proved here. The point of the present paper is to define the object language in which they can be stated.

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## 11. Limitations

This paper has four explicit limitations.

First, it is mainly a conceptual and mathematical reframing. It does not yet provide a complete verified theorem chain for the proposed program.

Second, the smoothing step introduces a family of nearby objects, not the unique truth by fiat. The scientific burden remains to justify which observation scale  $\varepsilon$  matches the real system.

Third, the paper does not claim that all discrete observables come from smooth latent structure. Some systems may be fundamentally singular or may be best modeled at the discrete level.

Fourth, the paper does not dissolve classical chaos. It argues only that the observer's representation can intensify, sharpen, or partially create the *appearance* of discontinuity.

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## 12. Conclusion

The right opposition is not discrete versus continuous. It is **latent state versus recorded observable**.

A discrete recorded law can be embedded as an atomic measure on a continuous space. A smoothing kernel then generates a nearby family whose spectral behavior can be studied explicitly. An observation lens  $\kappa$  and a quantizer  $Q$  explain how a smooth latent state becomes a finite recorded output. This one architecture already covers categorical variables, classifier labels, threshold events, phase labels, and some coarse-grained manifestations of chaos.

The paper's main claim can therefore be stated in one sentence:

Many discrete outcomes are quantized projections of smoother latent structure.

The most interesting consequence is not philosophical but methodological. If the latent object is smoother than the record, then the right representational surface may also be more spectrally compressible. In that case, what looks like jump, collapse, or chaos at the observation layer may become mathematically clearer once the latent smooth state is written down at the right level.

This is the point where the paper stops. It does not say that the world is secretly continuous everywhere. It says something narrower and more useful: that our observables are often rougher than the states that generate them, and that spectral language is a natural way to make that mismatch precise.

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*During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.*

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