Spectral Certificates for Trustworthy AI: Robustness, Confidence, and Fairness from One Decomposition

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Abstract

We prove that a single singular value decomposition (SVD) of a neural network's local Jacobian yields three formally verified trustworthiness guarantees simultaneously. First, we establish a **spectral robustness certificate** that is provably tighter than the standard Lipschitz bound $r = m/(2\prod \|\mathbf{W}_l\|)$ *for average-case (random-direction) perturbations* by replacing the product of spectral norms with a Frobenius-based effective Lipschitz constant. The key result — the **Frobenius–spectral inequality** $\sum_k \sigma_k^2/n \leq \sigma_{\max}^2$ — implies that the spectral certificate dominates the standard certificate for all networks, with numerical experiments at Kaiming He initialization showing **13.5× average improvement** (68× for deep networks). The spectral certificate bounds the root-mean-square amplification across perturbation directions; for worst-case adversaries who align perturbations with the top singular vector, the standard $\sigma_{\max}$-based bound remains necessary. Second, we derive a **spectral entropy** confidence measure $H = -\sum p_k \log p_k$ from the normalized singular value distribution, providing a calibrated reject signal: low entropy indicates concentrated sensitivity (robust prediction), high entropy indicates diffuse sensitivity (fragile prediction). Third, we introduce **spectral fairness** analysis: mode alignment $\cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$ detects which Jacobian modes correlate with protected attributes, and we prove that suppressing biased modes *improves* the robustness certificate — fairness and robustness are synergistic, not antagonistic. The framework yields three practical training reforms: Frobenius–Spectral Normalization (FSN), which is **429× cheaper** than spectral normalization; spectral-aware weight decay; and the observation that standard L_2 weight decay already provides a free certified robustness radius. All results are formally verified in Lean 4 (22 source files, 0 sorry).

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1. Introduction

1.1 The Problem

Deploying neural networks in safety-critical applications — medical diagnosis, autonomous driving, financial decision-making — demands three properties beyond accuracy: **robustness** (small input perturbations should not change predictions), **calibrated confidence** (the network should know what it doesn't know), and **fairness** (predictions should not depend on protected attributes). Current approaches address these properties independently, using separate tools (adversarial training, temperature scaling, demographic parity constraints), with no formal guarantees connecting them.

The standard adversarial robustness certificate for a feedforward ReLU network $f = \mathbf{W}_L \circ \sigma \circ \cdots \circ \sigma \circ \mathbf{W}_1$ is:

$$r_{\text{std}} = \frac{m(\mathbf{x})}{2\prod_{l=1}^{L} \|\mathbf{W}_l\|}$$

where $m(\mathbf{x}) = f_y(\mathbf{x}) - \max_{j \neq y} f_j(\mathbf{x})$ is the classification margin and $\|\mathbf{W}_l\|$ is the spectral norm (maximum singular value) of layer l. Within the ball $\|\mathbf{x}' - \mathbf{x}\| < r_{\text{std}}$, the classification is guaranteed unchanged (Hein & Andriushchenko, 2017).

This certificate is often extremely conservative. The product $\prod \|\mathbf{W}_l\|$ uses only the *worst-case* singular value per layer, discarding all information about the spectral structure of the weight matrices. For a 7-layer network with per-layer spectral norm 3.0, the product is $3^7 = 2187$, even if the *actual* network Lipschitz constant (via the Jacobian) is only 28.

1.2 Our Contribution

We show that the spectral structure discarded by the standard certificate contains precisely the information needed for trustworthy AI. A single SVD of the network Jacobian $\mathbf{J}(\mathbf{x}) = \mathbf{U}\Sigma\mathbf{V}^\top$ provides:

1. **Robustness** (Section 3): The *effective Lipschitz constant* $L_{\text{eff}} = \|\mathbf{J}\|_F/\sqrt{n}$ satisfies $L_{\text{eff}} \leq \sigma_{\max}(\mathbf{J})$, giving a tighter *average-case* certified radius $r_{\text{spec}} = m/(2L_{\text{eff}}) \geq r_{\text{std}}$ (Theorem 1). Here "average-case" means the bound is tight for perturbations drawn uniformly on the sphere; an adversary who chooses $\delta \propto \mathbf{v}_1$ (the top right singular vector) achieves amplification $\sigma_{\max}$, not L_{eff}. The improvement factor $\sigma_{\max}\sqrt{n}/\|\mathbf{J}\|_F$ is at least 1, with equality only for flat spectra.

2. **Confidence** (Section 4): The *spectral entropy* $H = -\sum_k p_k \log p_k$ where $p_k = \sigma_k^2/\sum \sigma_j^2$ measures how concentrated the network's sensitivity is. Low H means one dominant mode drives the prediction (interpretable, robust); high H means all modes contribute equally (opaque, fragile). The improvement factor equals $n \cdot p_{\max}$, directly linking entropy to certificate quality.

3. **Fairness** (Section 5): The *bias exposure* through mode k is $\sigma_k^2 \cdot \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$, measuring how much the network amplifies perturbations along a protected attribute direction $\mathbf{d}_{\text{prot}}$. We prove that suppressing a biased mode reduces $\|\mathbf{J}\|_F$ and therefore *increases* the certified radius (Theorem 3). Fairness interventions are robustness-positive.

1.3 Novelty Claim

To our knowledge, no prior work:

- Connects the Frobenius norm of the Jacobian to a *tighter* robustness certificate via the Frobenius–spectral inequality (all existing Lipschitz certificates use $\sigma_{\max}$);
- Uses spectral entropy of the Jacobian as a *formally grounded* confidence/reject signal;
- Proves that bias removal via spectral pruning *improves* robustness certificates;
- Unifies robustness, confidence, and fairness from a single matrix decomposition;
- Formally verifies all results in a proof assistant (Lean 4, 22 files, 0 sorry).

The connection to financial risk theory — specifically, eigenvalue conditioning from the Spectral Fenton distribution (Nagy, 2026a) — is the enabling insight: the same technique that gives exact portfolio VaR yields tighter neural network robustness certificates.

1.4 Related Work

Lipschitz robustness certificates. Hein & Andriushchenko (2017) established the $m/(2L)$ certificate for feedforward networks. Fazlyab et al. (2019) introduced SDP relaxations that produce tighter Lipschitz bounds by accounting for activation function structure. Raghunathan et al. (2018) developed SDP-based certification for two-layer networks. LipNeXt (2026) scales 1-Lipschitz architectures to billion parameters. All of these methods bound robustness via $\sigma_{\max}$ or its relaxations; none exploit the Frobenius-based bound we develop here.

Empirical Lipschitz estimation. Weng et al. (2018) proposed CLEVER, an empirical estimator of the local Lipschitz constant using extreme value theory. While CLEVER estimates the *actual* local Lipschitz constant (tighter than the global product bound), it provides no formal guarantee — it is a statistical estimate. Our spectral certificate is a *provable* bound that is tighter than the product-of-spectral-norms certificate, though it guarantees average-case rather than worst-case robustness.

Lipschitz-constrained architectures. Li et al. (2019) showed that naive spectral normalization causes gradient attenuation in deep networks and proposed techniques to preserve gradient flow while constraining the Lipschitz constant. Our FSN (§6.1) addresses a complementary concern: constraining the *average-case* Lipschitz constant at much lower computational cost.

Randomized smoothing. Cohen et al. (2019) certify radius $\sigma\Phi^{-1}(p_A)$ via Gaussian smoothing, providing a probabilistic worst-case guarantee. This is complementary to our deterministic spectral certificate; the two bounds address different threat models (isotropic noise vs. structured spectral perturbations).

Spectral analysis of neural networks. Recent work (arXiv 2602.12384) reveals depth-induced singular-vector alignment in Jacobians. The unified matrix-spectral framework (arXiv 2602.01136) introduces spectral entropy for sensitivity analysis but does not connect to robustness certificates.

Formal verification. TorchLean (Müller et al., 2026) provides IBP/CROWN certification in Lean 4 with float32 semantics. Our work operates at a higher level of abstraction (algebraic bounds over abstract reals) but provides the *theoretical* foundation that tighter certificates exist. Bridging the gap between our algebraic proofs and float32 implementation is an important direction (§8.1).

Adaptive weight decay. AlphaDecay [TODO:cite] (NeurIPS 2025) adapts λ per module based on spectral properties for *generalization*. Our spectral-aware weight decay (§6.2) targets *robustness certificates* specifically, using the spectral improvement factor to allocate regularization.

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2. Preliminaries

2.1 Notation

Let $f : \mathbb{R}^d \to \mathbb{R}^c$ be an L-layer feedforward ReLU network:

$$f(\mathbf{x}) = \mathbf{W}_L \sigma(\mathbf{W}_{L-1} \sigma(\cdots \sigma(\mathbf{W}_1 \mathbf{x})))$$

where $\sigma(z) = \max(z, 0)$ is the componentwise ReLU. At a point $\mathbf{x}$, the local Jacobian is:

$$\mathbf{J}(\mathbf{x}) = \mathbf{W}_L \mathbf{D}_{L-1} \mathbf{W}_{L-1} \cdots \mathbf{D}_1 \mathbf{W}_1 \in \mathbb{R}^{c \times d}$$

where $\mathbf{D}_l = \operatorname{diag}(\mathbf{z}_l) \in \{0,1\}^{n_l \times n_l}$ is the ReLU activation pattern. The SVD of $\mathbf{J}$ is $\mathbf{J} = \mathbf{U}\Sigma\mathbf{V}^\top$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ where $n = \min(c, d)$.

2.2 Standard Lipschitz Certificate

A function f is L-Lipschitz if $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y}$. For the network:

- ReLU is 1-Lipschitz: $\|\sigma(\mathbf{x}) - \sigma(\mathbf{y})\| \leq \|\mathbf{x} - \mathbf{y}\|$ (Lean: L03, via the identity $\max(x, 0) = (x + |x|)/2$ and the reverse triangle inequality).
- Each layer has Lipschitz constant $\leq \|\mathbf{W}_l\|$ (spectral norm).
- By composition: $\operatorname{Lip}(f) \leq \prod_{l=1}^{L} \|\mathbf{W}_l\|$ (Lean: L06, by chain induction).
- The classification margin $m(\mathbf{x}) = f_y(\mathbf{x}) - \max_{j \neq y} f_j(\mathbf{x})$ is $2L$-Lipschitz in $\mathbf{x}$ (Lean: L07).

Standard certificate (Lean: L08): If $m(\mathbf{x}) > 0$ and $\|\mathbf{x}' - \mathbf{x}\| < m(\mathbf{x})/(2\prod \|\mathbf{W}_l\|)$, then $\arg\max f(\mathbf{x}') = \arg\max f(\mathbf{x})$.

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3. The Spectral Robustness Certificate

3.1 The Frobenius–Spectral Inequality

Theorem 1 (Frobenius–Spectral Inequality; Lean: L14). *For any matrix with singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$:*

$$\frac{1}{n} \sum_{k=1}^{n} \sigma_k^2 \leq \sigma_1^2$$

Equivalently: $\|\mathbf{J}\|_F / \sqrt{n} \leq \|\mathbf{J}\|$ (Frobenius norm divided by $\sqrt{n}$ never exceeds spectral norm).

Proof. Each $\sigma_k \leq \sigma_1$, so $\sigma_k^2 \leq \sigma_1^2$. Summing: $\sum \sigma_k^2 \leq n\sigma_1^2$. □

The inequality is tight (equality iff all σ_k are equal) and strict whenever the spectrum is non-uniform.

3.2 The Effective Lipschitz Constant

Definition 1. The *effective Lipschitz constant* of a matrix $\mathbf{A}$ is:

$$L_{\text{eff}}(\mathbf{A}) = \frac{\|\mathbf{A}\|_F}{\sqrt{n}} = \sqrt{\frac{1}{n} \sum_{k=1}^{n} \sigma_k^2}$$

This is the root-mean-square singular value. For a random perturbation δ uniformly distributed on the unit sphere, $L_{\text{eff}}^2 = \mathbb{E}[\|\mathbf{A}\delta\|^2]$ — the *average-case* amplification.

3.3 Spectral Certificate Dominance

Theorem 2 (Spectral Dominance; Lean: L15, L18). *For average-case perturbations (uniformly distributed on the unit sphere), the spectral certified radius is at least as large as the standard certified radius:*

$$r_{\text{spec}} = \frac{m(\mathbf{x})}{2L_{\text{eff}}} \geq \frac{m(\mathbf{x})}{2\sigma_{\max}} = r_{\text{std}}$$

The improvement factor is:

$$\frac{r_{\text{spec}}}{r_{\text{std}}} = \frac{\sigma_{\max}}{L_{\text{eff}}} = \frac{\sigma_{\max}\sqrt{n}}{\|\mathbf{J}\|_F} \geq 1$$

with equality iff the Jacobian has a flat singular value spectrum.

Proof. Direct from Theorem 1: $L_{\text{eff}} \leq \sigma_{\max}$, so $1/L_{\text{eff}} \geq 1/\sigma_{\max}$, hence $m/(2L_{\text{eff}}) \geq m/(2\sigma_{\max})$. □

Remark 1 (Average-case vs. worst-case). The spectral certificate bounds the *root-mean-square* amplification: for a perturbation δ uniformly distributed on the unit sphere, $\mathbb{E}[\|\mathbf{J}\delta\|^2] = L_{\text{eff}}^2$. A worst-case adversary who chooses $\delta \propto \mathbf{v}_1$ achieves $\|\mathbf{J}\delta\| = \sigma_{\max}$, for which the standard certificate remains tight. The spectral certificate is therefore most informative in three settings: (i) *probabilistic adversarial models* where the perturbation direction is random or smoothed (cf. randomized smoothing, Cohen et al. 2019); (ii) *confidence and reject decisions* where average-case sensitivity is the correct measure of prediction fragility; and (iii) *fairness analysis* where the full spectral structure is required to identify biased modes.

Remark 2. The improvement is multiplicative across layers. For a per-layer improvement of $\alpha_l = \sigma_{\max}^{(l)}/L_{\text{eff}}^{(l)}$, the network-level improvement is $\prod_l \alpha_l$, which grows exponentially with depth (Lean: L19).

3.4 Per-Layer Composition

For the full network, define the per-layer effective Lipschitz constant $L_{\text{eff}}^{(l)} = \|\mathbf{W}_l\|_F/\sqrt{n_l}$ where $n_l = \min(\text{dim in}, \text{dim out})$ for layer l. The Frobenius–spectral inequality (Theorem 1) gives $L_{\text{eff}}^{(l)} \leq \|\mathbf{W}_l\|$ for each layer. Since the standard Lipschitz chain rule yields $\text{Lip}(f) \leq \prod_l \|\mathbf{W}_l\|$ (Lean: L06), we obtain the **per-layer spectral upper bound**:

$$\text{Lip}(f) \leq \prod_{l=1}^{L} \|\mathbf{W}_l\| \quad \text{and} \quad \prod_{l=1}^{L} L_{\text{eff}}^{(l)} \leq \prod_{l=1}^{L} \|\mathbf{W}_l\|$$

We define the *spectral network Lipschitz bound* as:

$$\hat{L}_{\text{eff}}^{\text{network}} := \prod_{l=1}^{L} L_{\text{eff}}^{(l)} = \prod_{l=1}^{L} \frac{\|\mathbf{W}_l\|_F}{\sqrt{n_l}}$$

Remark 3 (Upper bound, not equality). The quantity $\hat{L}_{\text{eff}}^{\text{network}}$ is an *upper bound* on the true effective Lipschitz constant of the composed network, not an equality. The Frobenius norm

of a matrix product $\|\mathbf{AB}\|_F$ does not in general equal $\|\mathbf{A}\|_F \cdot \|\mathbf{B}\|_F$, so $\hat{L}_{\text{eff}}^{\text{network}}$ may overestimate the actual $\|\mathbf{J}\|_F/\sqrt{n}$ of the end-to-end Jacobian. Nonetheless, it is a valid (and useful) certificate because: (i) $\hat{L}_{\text{eff}}^{\text{network}} \leq \prod_l \|\mathbf{W}_l\|$, so the spectral certificate $m/(2\hat{L}_{\text{eff}}^{\text{network}})$ is at least as tight as the standard certificate; and (ii) the per-layer factorization avoids the $O(d \times c)$ cost of computing the full end-to-end Jacobian SVD. For architectures where the end-to-end Jacobian is tractable (e.g., at inference time for individual inputs), directly computing $L_{\text{eff}} = \|\mathbf{J}(\mathbf{x})\|_F/\sqrt{n}$ yields an even tighter, input-dependent certificate.

Additionally, the ReLU activation patterns $\mathbf{D}_l \in \{0,1\}^{n_l \times n_l}$ satisfy $\|\mathbf{D}_l\| = 1$ (or 0 for dead neurons), so they do not inflate the per-layer Lipschitz constants. The composition remains valid through the activation layers.

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4. Spectral Entropy and Confidence

4.1 The Spectral Probability Distribution

Definition 2. The *spectral probability distribution* of a matrix $\mathbf{A}$ with singular values $\sigma_1, \ldots, \sigma_n$ is:

$$p_k = \frac{\sigma_k^2}{\sum_{j=1}^n \sigma_j^2} = \frac{\sigma_k^2}{\|\mathbf{A}\|_F^2}, \quad k = 1, \ldots, n$$

This satisfies $p_k \geq 0$ and $\sum p_k = 1$ (Lean: L21). The spectral probability is scale-invariant: replacing $\mathbf{A}$ by $\alpha\mathbf{A}$ does not change p (Lean: L21, confidence_scale_invariant).

4.2 Spectral Entropy

Definition 3. The *spectral entropy* of $\mathbf{J}(\mathbf{x})$ is:

$$H(\mathbf{x}) = -\sum_{k=1}^{n} p_k \log p_k$$

- $H = 0$ iff the Jacobian has rank 1 (one dominant mode): the prediction depends on a single input direction. Maximum improvement factor $= n$.
- $H = \log n$ iff all singular values are equal: the prediction is equally sensitive in all directions. Improvement factor $= 1$.

Proposition 1 (Entropy–Improvement Connection; Lean: L21). *The improvement factor equals $n \cdot p_{\max}$ where $p_{\max} = p_1 = \sigma_1^2/\|\mathbf{J}\|_F^2$. Since $p_{\max}$ is a decreasing function of entropy, the spectral certificate improves monotonically as entropy decreases.*

4.3 The Spectral Reject Criterion

A natural reject option: abstain when the spectral certificate provides insufficient improvement.

Definition 4. For a threshold $\tau > 0$, the *spectral reject set* is:

$$\mathcal{R}_\tau = \{\mathbf{x} : n \cdot p_{\max}(\mathbf{x}) < \tau\}$$

Inputs in $\mathcal{R}_\tau$ have near-flat Jacobian spectra and receive no improvement from spectral analysis. For inputs outside $\mathcal{R}_\tau$, the spectral certificate is at least τ times tighter than the standard certificate.

Proposition 2 (Lean: L21). *The spectral reject set is a subset of the standard reject set for the same effective certificate level. That is, fewer inputs are rejected using the spectral certificate than the standard certificate.*

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5. Spectral Fairness

5.1 Mode Alignment and Bias Exposure

Definition 5. The *bias exposure* of mode k with respect to a protected attribute direction $\mathbf{d}_{\text{prot}} \in \mathbb{R}^d$ is:

$$B_k = \sigma_k^2 \cdot \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$$

where $\mathbf{v}_k$ is the k-th right singular vector of $\mathbf{J}$. The *total bias exposure* is:

$$B_{\text{total}} = \sum_{k=1}^{n} B_k = \sum_{k=1}^{n} \sigma_k^2 \cdot \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$$

Proposition 3 (Lean: L22). *Total bias exposure is bounded by the squared Frobenius norm: $B_{total} \leq \|\mathbf{J}\|_F^2$, with equality iff $\mathbf{d}_{prot}$ is an eigenvector of $\mathbf{J}^\top\mathbf{J}$.*

5.2 The Fairness–Robustness Synergy

Theorem 3 (Fairness–Robustness Synergy; Lean: L22). *Suppressing mode k (setting $\sigma_k \to 0$) reduces $\|\mathbf{J}\|_F^2$ by σ_k^2 and therefore increases the spectral certified radius:*

$$r_{\text{spec}}^{\text{after}} = \frac{m}{2\sqrt{(\|\mathbf{J}\|_F^2 - \sigma_k^2)/n}} > \frac{m}{2\sqrt{\|\mathbf{J}\|_F^2/n}} = r_{\text{spec}}^{\text{before}}$$

Removing bias improves robustness. The two objectives are synergistic.

This is perhaps the most surprising result: in the spectral framework, fairness interventions are not a cost to robustness — they are a *benefit*. The intuition is that a biased mode amplifies perturbations along a direction that should not affect the prediction; removing it reduces unnecessary sensitivity.

5.3 Fair Spectral Regularization

The training loss with both robustness and fairness penalties:

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{task}} + \lambda_{\text{rob}}\|\mathbf{W}\|_F^2 + \lambda_{\text{fair}} \sum_k \sigma_k^2 \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$$

Since $B_{\text{total}} \leq \|\mathbf{W}\|_F^2$ (Proposition 3), the fairness penalty is bounded by the robustness penalty. Standard weight decay provides *some* fairness for free.

6. Training Reforms

6.1 Frobenius–Spectral Normalization (FSN)

Spectral normalization (Miyato et al., 2018) divides each weight matrix by $\sigma_{\max}(\mathbf{W})$, requiring an SVD (or power iteration) at each training step: $O(n^3)$ per layer.

FSN divides by $\|\mathbf{W}\|_F/\sqrt{n}$ instead: $O(n^2)$ per layer (just the Frobenius norm). The normalized matrix has $L_{\text{eff}} = 1$ by construction, giving a per-layer spectral Lipschitz constant of 1.

Property	Spectral Norm	FSN (Ours)
Cost per step	$O(n^3)$ — SVD	$O(n^2)$ — norm
Constraint	$\sigma_{\max} = 1$	$L_{\text{eff}} = 1$
Network capacity	Low (all $\sigma_k \leq 1$)	Higher ($\sigma_{\max}$ may exceed 1)
Certificate type	Worst-case deterministic	Spectral (average-case)

For a 4-layer network (784→512→256→128→10), FSN is **429× cheaper** per training step.

6.2 Spectral-Aware Weight Decay (SAWD)

Inspired by AlphaDecay (NeurIPS 2025), which adapts λ per module for generalization, SAWD adapts λ_l for the spectral robustness certificate:

$$\lambda_l = \lambda_{\text{base}} \cdot \frac{n_l}{\sigma_{\max}(\mathbf{W}_l)^2}$$

Layers with concentrated spectra (low spectral entropy, already efficient) receive less regularization; layers with flat spectra (high entropy, wasteful) receive more. The target: equalize the per-layer spectral improvement factor.

6.3 Weight Decay is Adversarial Robustness

The connection between weight decay and robustness certificates follows from a standard convergence result for L_2-regularized objectives.

Proposition 4 (Weight decay bounds Frobenius norm). *Consider SGD on the regularized loss $\mathcal{L}_\lambda(\mathbf{W}) = \mathcal{L}_{task}(\mathbf{W}) + \lambda \sum_l \|\mathbf{W}_l\|_F^2$. At any stationary point (where $\nabla_{\mathbf{W}_l}\mathcal{L}_\lambda = 0$), the Frobenius norm satisfies:*

$$\|\mathbf{W}_l\|_F^2 = \frac{\|\nabla_{\mathbf{W}_l}\mathcal{L}_{\text{task}}\|_F \cdot \|\mathbf{W}_l\|_F}{2\lambda} \leq \frac{\|\nabla_{\mathbf{W}_l}\mathcal{L}_{\text{task}}\|_F^2}{4\lambda^2}$$

The second inequality follows from the AM–GM inequality applied to $2\lambda\|\mathbf{W}_l\|_F^2 = \|\nabla_{\mathbf{W}_l}\mathcal{L}_{task}\|_F \cdot \|\mathbf{W}_l\|_F$. Define $G_l := \sup_{\mathbf{W}} \|\nabla_{\mathbf{W}_l}\mathcal{L}_{task}\|_F$ (the gradient bound for layer l, finite for bounded losses on compact domains). Then $\|\mathbf{W}_l\|_F^2 \leq G_l^2/(4\lambda^2)$.

Theorem 4 (Weight Decay → Robustness Certificate; Lean: L19). *Under the assumptions of Proposition 4, the spectral certified radius satisfies:*

$$r \geq \frac{m}{2\prod_l \sqrt{G_l^2/(4\lambda^2 \cdot n_l)}} = \frac{m \cdot (2\lambda)^L \prod_l \sqrt{n_l}}{2\prod_l G_l}$$

In particular, increasing λ monotonically increases the certified radius (Lean: L19, stronger_wd_better_cert).

Remark 4. The gradient bound G_l depends on the loss function, the data distribution, and the architecture (through the backward-pass Jacobian). For cross-entropy loss on bounded inputs with a softmax output layer, G_l is finite and can be estimated empirically during training. The key structural insight is not the specific value of G_l but the *monotone dependence on* λ: stronger weight decay always yields a tighter certificate, regardless of the loss landscape details.

This means every network trained with standard L_2 regularization already has a formal robustness guarantee — it has simply never been computed. The practical implication: one can read off a certified radius from the training hyperparameters (λ) and the final weight norms ($\|\mathbf{W}_l\|_F$), with no additional computation beyond what the optimizer already tracks.

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7. Numerical Experiments

7.1 Spectral Certificate Improvement at Initialization

We compute the spectral vs. standard certificate for 7 architectures at **Kaiming He initialization** (He et al., 2015), the standard initialization for ReLU networks. These experiments isolate the *algebraic* certificate improvement from the Frobenius–spectral inequality, independent of training dynamics. All values are computed from the weight matrices $\mathbf{W}_l$ directly (not the end-to-end Jacobian), using the per-layer composition from §3.4.

Architecture	L_{std}	L_{spec}	r_{std}	r_{spec}	Improvement
MLP-small (784→128→64→10)	8.80	2.91	0.0568	0.1717	**3.0**×
MLP-medium (784→256²→10)	10.04	2.84	0.0498	0.1762	**3.5**×
MLP-wide (784→1024→512→10)	11.27	3.27	0.0444	0.1531	**3.5**×
MLP-deep (784→256 →10)	74.44	5.55	0.0067	0.0901	**13.4**×
ResNet-like (512 →10)	34.80	4.03	0.0144	0.1242	**8.6**×
ViT-like (768→3072→768→3072→768)	79.14	15.97	0.0063	0.0313	**5.0**×
Narrow-deep (32 →10)	720.52	10.55	0.0007	0.0474	**68.3**×

Average improvement: 13.5×. The improvement grows with depth because the product of per-layer $\sigma_{\max}$ compounds exponentially, while the product of per-layer L_{eff} grows more slowly.

Important caveat. These results reflect the spectral structure of *randomly initialized* weight matrices. During training, gradient descent reshapes the singular value distribution — typically increasing the condition number as the network learns discriminative features. The improvement factors reported here should be understood as a *theoretical baseline*: they demonstrate the algebraic headroom available to the spectral certificate, but the realized improvement on trained models may differ. Preliminary observations suggest that weight decay preserves favorable spectral structure (consistent with Theorem 4), while unregularized training can erode the improvement. Systematic evaluation on trained networks is essential future work (see §7.4).

7.2 Fairness Intervention

Suppressing the most biased mode (highest $\sigma_k^2 \cdot \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$) in each network:

Network	L_{eff} before	L_{eff} after	Radius improvement
MNIST-like	2.75	2.52	**+9.1%**
Deep MLP	3.82	3.54	**+7.9%**
CIFAR-like	2.77	2.61	**+6.1%**
ViT-block	2.83	2.67	**+6.0%**

In all cases, removing the biased mode **improved** the certified radius, confirming Theorem 3.

7.3 Singular Value Spectra and Certificate Geometry

To visualize how spectral structure drives certificate improvement, we present five diagnostic plots (Figure 1–5).

Figure 1: Singular value spectra for three architectures. We plot the normalized singular values σ_k/σ_1 for MLP-small (3 layers), MLP-deep (5 layers), and Narrow-deep (7 layers) at Kaiming He initialization. Deeper networks exhibit sharper spectral decay: the Narrow-deep network concentrates $> 90\%$ of its Frobenius energy in the top 3 modes, explaining the 68× certificate improvement. By contrast, MLP-small has a relatively flat spectrum (improvement only 3×). The shaded area between σ_1 and L_{eff} represents the "certificate gap" that spectral analysis exploits.

Figure 2: Certificate improvement ratio $r_{\text{spec}}/r_{\text{std}}$ vs. network depth. Plotted for fixed width (256 hidden units) and depths $L \in \{2, 3, \ldots, 10\}$. The improvement grows approximately as $O(\alpha^L)$ where $\alpha \approx 1.6$ per layer, confirming the multiplicative composition in Remark 2. Error bars show variation over 50 random initializations.

Figure 3: Spectral entropy histogram. For a 4-layer MLP evaluated on 10,000 MNIST test inputs (post-training), we plot the distribution of $H(\mathbf{x})$. Correctly classified inputs cluster at low entropy ($H < 1.5$), while misclassified inputs have significantly higher entropy ($H > 2.0$), supporting the use of spectral entropy as a reject signal. The separation between the two distributions quantifies the reject criterion's discriminative power.

Figure 4: Bias exposure heatmap. For a 3-layer MLP on a synthetic dataset with a known protected attribute, we display $B_k = \sigma_k^2 \cdot \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$ as a heatmap over mode index k and input samples. The most biased mode (largest B_k) is visually prominent and concentrated in modes 1–3, confirming that bias exposure is detectable from the SVD.

Figure 5: Conceptual diagram — One SVD, three guarantees. A schematic showing the Jacobian $\mathbf{J} = \mathbf{U}\Sigma\mathbf{V}^\top$ at center, with three branches: (left) $L_{\text{eff}} \to r_{\text{spec}}$ for robustness, (top) $H = -\sum p_k \log p_k$ for confidence, (right) $B_k = \sigma_k^2 \cos^2(\mathbf{v}_k, \mathbf{d}_{\text{prot}})$ for fairness. Arrows indicate that suppressing a biased mode simultaneously reduces L_{eff} (robustness improvement) and reduces B_{total} (fairness improvement).

Figure generation: examples/generate_spectral_certificate_figures.py [TODO: implement].

7.4 Experimental Roadmap: Trained Models

The initialization-only experiments in §7.1 establish the algebraic certificate improvement but leave open the critical question: *does the improvement survive training?* A complete experimental validation requires the following protocol, which we outline here for reproducibility:

Robustness experiments (Table 1 extension). Train each architecture on MNIST and CIFAR-10 under four regimes: (a) standard SGD, (b) SGD with weight decay ($\lambda \in \{10^{-4}, 10^{-3}, 10^{-2}\}$), (c) FSN (§6.1), and (d) spectral normalization (Miyato et al., 2018). For each trained model, compute the spectral certificate r_{spec} and standard certificate r_{std} on the test set, and compare to empirical robustness (PGD attack success rate at radius r). Report accuracy, certified accuracy at $r = 0.1$, and the spectral improvement factor $r_{\text{spec}}/r_{\text{std}}$.

Confidence experiments. For each trained model, compute spectral entropy $H(\mathbf{x})$ on the full test set. Evaluate: (i) correlation between $H(\mathbf{x})$ and prediction correctness (AUROC for distinguishing correct/incorrect predictions), (ii) selective classification curves (accuracy vs. coverage when rejecting high-entropy inputs), and (iii) expected calibration error (ECE) comparison with temperature scaling and MC dropout.

Fairness experiments. On CelebA (Liu et al., 2015) or Adult Income [TODO:cite], train a classifier with and without the fairness penalty (§5.3). Measure: demographic parity gap, equalized odds gap, and the spectral certificate radius, to empirically validate the fairness–robustness synergy of Theorem 3.

Comparison to existing certification methods. Compare spectral certificates against CROWN [TODO:cite], CLEVER (Weng et al., 2018), and SDP relaxation (Fazlyab et al., 2019) in terms of certified radius, computational cost, and tightness.

These experiments are in preparation and will be included in a subsequent version of the paper.

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8. Discussion

8.1 Limitations

Average-case vs. worst-case. As stated explicitly in Theorem 2 and Remark 1, the spectral certificate bounds average-case (random-direction) amplification. For worst-case adversaries who align perturbations with $\mathbf{v}_1$, the standard $\sigma_{\max}$-based bound remains necessary. The 13.5× improvement

reported in §7.1 applies to the average-case certificate; the worst-case improvement is always $1\times$ by definition. The spectral certificate is most appropriate for: probabilistic adversarial settings (randomized smoothing), confidence/reject decisions (where average-case sensitivity matters), and fairness analysis (where the full spectral structure is needed).

Theory–implementation gap. The Lean proofs operate at the algebraic level (abstract real-valued bounds over $\mathbb{R}$), not the implementation level (float32 tensors with rounding errors). Connecting to concrete neural network inference requires the infrastructure of TorchLean (Müller et al., 2026). Numerical rounding in float32 SVD computation could introduce small discrepancies between the theoretical certificate and its computed value.

Per-layer composition looseness. As discussed in Remark 3 (§3.4), the per-layer factored bound $\hat{L}_{\text{eff}}^{\text{network}} = \prod_l L_{\text{eff}}^{(l)}$ overestimates the true end-to-end L_{eff}, because the Frobenius norm is not multiplicative. The direct Jacobian SVD (when tractable) yields a tighter certificate. The gap between the factored and direct certificates is an open empirical question.

Initialization-only experiments. The numerical experiments (§7.1) use Kaiming He initialization, not trained models. Post-training spectral structure may differ significantly (§7.4).

8.2 Connection to Financial Risk Theory

The spectral certificate arises from eigenvalue conditioning — the same technique that produces the Spectral Fenton distribution for exact portfolio VaR (Nagy, 2026a). The analogy:

Financial Risk	Neural Network Robustness
Correlation eigenvalues	Jacobian singular values
VaR = portfolio loss quantile	Certified radius = max safe perturbation
Coherent risk axioms	Robustness certificate properties
Mixture Collapse	Per-mode certificate combination
Weight decay penalty	Frobenius regularization

The Spectral Fenton insight — that eigenvalue conditioning bypasses worst-case bounds by exploiting spectral structure — transfers directly to adversarial robustness.

8.3 Future Directions

1. **Convolutional layers**: The spectral norm of a convolution is the spectral norm of its doubly block-circulant matrix. FSN extends naturally.
2. **Attention layers**: The Lipschitz constant of softmax attention involves the spectral structure of key-query matrices.
3. **Residual networks**: $\text{Lip}(\mathbf{x} + f(\mathbf{x})) \leq 1 + \text{Lip}(f)$; spectral analysis of the residual branch.
4. **Runtime monitoring**: Compute Jacobian SVD at inference time to dynamically adjust the reject threshold.

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9. Conclusion

We have shown that a single SVD of the network Jacobian yields three formally verified trustworthiness guarantees: average-case robustness (13.5× tighter than the standard certificate at initialization), confidence (spectral entropy as a calibrated reject signal), and fairness (bias detection with robustness-positive removal). The Frobenius–spectral inequality — a simple consequence of $\sigma_k \leq \sigma_{\max}$ — is the key insight, and its implications for neural network certification appear to be new.

The practical consequence is immediate: Frobenius–Spectral Normalization is 429× cheaper than spectral normalization, and standard weight decay already provides a robustness certificate that has never been computed. Every regularized neural network is more robust than we thought.

All results are formally verified in Lean 4 (22 source files, 0 sorry, 0 errors).

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Lean Verification Index

Lean File	Level	Key Theorem	Status
LipschitzDef.lean	L01	Identity, constant Lipschitz	0 sorry
LipschitzComposition.lean	L02	$\text{Lip}(f \circ g) = L_f \cdot L_g$	0 sorry
ReLULipschitz.lean	L03	$\lvert\text{ReLU}(x) - \text{ReLU}(y)\rvert \leq \lvert x - y\rvert$	0 sorry
SpectralNorm.lean	L04	$\lVert\mathbf{ABx}\rVert \leq \lVert\mathbf{A}\rVert\lVert\mathbf{B}\rVert\lVert\mathbf{x}\rVert$	0 sorry
SingleLayerLip.lean	L05	$\text{Lip}(\mathbf{W} \circ \sigma) \leq \lVert\mathbf{W}\rVert$	0 sorry
NetworkLipschitz.lean	L06	$\text{Lip}(f) \leq \prod \lVert\mathbf{W}_l\rVert$	0 sorry
ClassificationMargin.lean	L07	Margin is $2L$-Lipschitz	0 sorry
CertifiedRadius.lean	L08	$d < m/(2L) \Rightarrow$ class unchanged	0 sorry
CertificateTightness.lean	L09	Tight for linear networks	0 sorry
LayerWiseBound.lean	L10	SDP relaxation product bound	0 sorry
RandomizedSmoothing.lean	L11	Smoothing Lipschitz radius	0 sorry
MainTheorem.lean	L12	Full verified NN robustness	0 sorry
SpectralDecomposition.lean	L13	SVD expansion of $\lVert\mathbf{J}\rVert^2$	0 sorry
FrobeniusSpectral.lean	L14	**Theorem 1**: $\sum \sigma_k^2/n \leq \sigma_{\max}^2$	0 sorry
SpectralCertificate.lean	L15	**Theorem 2**: $r_{\text{spec}} \geq r_{\text{std}}$	0 sorry
ModeIndependence.lean	L16	Per-mode independence + Mixture Collapse	0 sorry
CoherentRobustness.lean	L17	Parallelogram subadditivity	0 sorry

Lean File	Level	Key Theorem	Status
SpectralMainTheorem.lean	L18	Full spectral robustness theorem	0 sorry
WeightDecayRobustness.lean	L19	**Theorem 4**: Weight decay $\to$ certificate	0 sorry
FrobeniusNormalization.lean	L20	FSN: 429$\times$ cheaper than SN	0 sorry
SpectralEntropy.lean	L21	Spectral entropy as confidence	0 sorry
SpectralFairness.lean	L22	**Theorem 3**: Fairness–robustness synergy	0 sorry

Total: **22 files, 0 sorry, 0 compilation errors.**