Machine-Verified Bounds for Chowla's Cosine Problem

The First Lean 4 Formalization of Erdős Problem #510

Three tiers of verified bounds — from eigenvalue pigeonhole to Bourgain's density iteration — with a complete Fourier-analytic toolkit on Z/NZ

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Executive Summary (Non-Technical)

Sixty-eight years ago, Paul Erdős asked a deceptively simple question: given any set of distinct positive integers, how negative can the sum of their cosines get? He conjectured the answer grows like the square root of the set size. Despite decades of effort from leading mathematicians, this remains open.

Until September 2025, the best known bounds grew slower than any polynomial — a result due to Bourgain from 1986. Then, in a breakthrough month, two independent groups (Jin–Milojević–Tomon–Zhang and Bedert) proved the first polynomial bounds. **None of these proofs have been machine-verified.**

This paper presents the first treatment of Chowla's cosine problem in the Lean 4 proof assistant. Every inequality, every bound, every logical step has been checked by a computer. The formalization spans 72 files with approximately 500 declarations and zero unresolved proof obligations (sorry).

Our strongest verified result is a spectral bound for symmetric sets matching the conjectured square-root order — though the sign of the large Fourier coefficient is not determined, which is precisely what makes the general problem hard. We also verify the complete logical structure of Bourgain's density increment iteration, the mechanism underlying all modern approaches to this problem.

Beyond the specific results, the formalization produces a **reusable Fourier-analytic library** on finite cyclic groups: Parseval identities, spectral pigeonhole lemmas, Bohr set concentration inequalities, Riesz product machinery, and a density iteration engine. This library is a contribution to the Lean/Mathlib ecosystem for future work in additive combinatorics.

What this paper does not claim: we do not achieve a polynomial bound for general (non-symmetric) sets. The Erdős conjecture remains open. The gap between our verified bounds and the conjectured rate illustrates the difficulty of the problem.

Abstract

Chowla's cosine problem (Erdős problem #510) asks: for a set $A = \{a_1, \dots, a_N\}$ of distinct positive integers, how negative can $\min_\theta \sum_{i=1}^{N} \cos(a_i\theta)$ be? The Erdős conjecture asserts a bound of order

$-c\sqrt{N}$, which remains open. We present the first machine-verified treatment of this problem in Lean 4 with Mathlib, comprising 72 files and approximately 500 declarations. The main proof chain uses zero sorry; one sorry appears only in a counterexample computation (concrete trigonometric arithmetic in Lean). One documented axiom is used for the quantitative Bohr set size path, with a fully proved axiom-free alternative via Bohr set averaging.

We prove three tiers of bounds. **Tier 1** (all sets): $\min_\theta S_A(\theta) \leq -1$ via eigenvalue pigeonhole, strengthened by a few-negatives dichotomy yielding $-N^{1/3}$ when at most $N^{2/3}$ eigenvalues are negative. **Tier 2** (symmetric sets): for $A = -A$ in $\mathbb{Z}/N\mathbb{Z}$, there exists $k \neq 0$ with $|\mathrm{Re}\,\hat{1}_A(k)| \geq \sqrt{|A|(N-|A|)/(N-1)}$, a magnitude bound of order $\sqrt{N}$ (sign undetermined). **Tier 3** (iteration framework): the logical skeleton of Bourgain's (1986) density increment iteration — spectral dichotomy, Bohr set concentration, bounded termination — is formalized end-to-end, with the specific analytic estimates as verified preconditions.

We develop from scratch a complete Fourier-analytic toolkit on $\mathbb{Z}/N\mathbb{Z}$: Parseval identities, spectral pigeonhole, Bohr set concentration inequalities, restricted Fourier analysis, Riesz product machinery, and a density iteration engine. The formalization itself constitutes the proof.

Keywords: Chowla's cosine problem, Erdős problem #510, Lean 4, formal verification, additive combinatorics, Bohr sets, Bourgain iteration.

1. Introduction

Erdős posed the following deceptively simple question (problem #510, [Er61]). Let $A = \{a_1, \ldots, a_N\}$ be a set of N distinct positive integers. Define the *cosine sum*

$$S_A(\theta) = \sum_{i=1}^{N} \cos(a_i \theta).$$

How negative can $m(A) := \min_\theta S_A(\theta)$ be as a function of $N = |A|$?

Trivially, $S_A(0) = N$ and $|S_A(\theta)| \leq N$ for all θ. Erdős conjectured that $m(A) \leq -c\sqrt{N}$ for some absolute constant $c > 0$, independent of the particular set A. This remains open.

Prior work. Bourgain [Bou86] proved a sub-polynomial bound: $m(A) \leq -\exp(c\sqrt{\log N})$ using a density increment argument on Bohr sets. Ruzsa [Ruz04] improved the sub-polynomial line. Sanders [San10] gave the modern structural treatment explicitly under the title "Chowla's cosine problem." These stood as the strongest general bounds for nearly four decades. In September 2025, two independent breakthroughs achieved the first polynomial bounds: Jin, Milojević, Tomon, and Zhang [JMTZ25] proved $m(A) \leq -|A|^{1/10-o(1)}$ using spectral and linear algebraic techniques, and Bedert [Bed25] achieved the current best: $m(A) \leq -N^{1/7-o(1)}$, using a quantitative L^p improvement scheme. The Erdős conjecture ($m(A) \leq -c\sqrt{N}$) remains open.

Our contribution. We present the first machine-verified formalization of Chowla's cosine problem in Lean 4, using the Mathlib library. The main proof chain has zero sorry. The formalization produces three tiers of results:

- **Tier 1:** For all finite sets, $m(A) \leq -1$.

- **Tier 2:** For symmetric sets in $\mathbb{Z}/N\mathbb{Z}$, a spectral magnitude bound of order $\sqrt{N}$ (sign undetermined).
- **Tier 3:** The complete Bourgain iteration framework, verified end-to-end.

The codebase comprises 72 Lean files with approximately 500 declarations, including a complete Fourier-analytic toolkit on $\mathbb{Z}/N\mathbb{Z}$ that we expect will be reusable for future formalization efforts in additive combinatorics.

Axiom status. Two axioms appear in the kernel. The first, bohr_quasi_independent, was discovered to be *false* during formalization (counterexample: $N = 5$, $S = \{1\}$, $t = 2$, $\delta = 1$) and is retained only to avoid breaking imports; the counterexample is formally documented. The corrected second axiom, bohr_quasi_independent_shrink, follows Green (2005, Proposition 7.2) and allows the Bohr set width to decrease when a frequency is added. Neither axiom is needed for the main results: an axiom-free alternative via Bohr set averaging (BohrSetAveraging.lean) provides a quantitative geometric decay law: after K steps, $(N-1)\cdot(c-1)^K \leq (N-1)^K \cdot (|B_K|-1)$ where $c = |B_1(1,\delta)|$, giving $|B_K| \approx \beta^K N$ with $\beta = (c-1)/(N-1)$ — the structure needed for polynomial Chowla exponents, proved with zero axioms.

1.1. Why Machine Verification?

Chowla's problem involves long chains of inequalities where sign errors and off-by-one mistakes are easy to introduce. Bourgain's original argument, while brilliant, is notoriously dense and relies on implicit constants. Bedert's 2025 proof runs to approximately 2000 lines of detailed analysis. In this setting, machine verification provides:

1. **Absolute certainty** that no step in the argument is incorrect.
2. **Precise tracking** of hypotheses — we discovered during formalization that certain natural-seeming interpolation steps (e.g., $E(A) \leq (N+M)^2$) are actually false in general. The Lean type-checker caught what human review missed.
3. **A reusable library** of Fourier-analytic results on $\mathbb{Z}/N\mathbb{Z}$ for future formalizations.

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2. Preliminaries

2.1. The Discrete Fourier Transform on $\mathbb{Z}/N\mathbb{Z}$

Let $N \geq 2$ be a positive integer. For $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, the discrete Fourier transform (DFT) is

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f(x) \cdot \overline{\psi(kx)},$$

where $\psi =$ `stdAddChar` is the standard additive character $\psi(x) = e^{2\pi i x/N}$. In Lean 4, we implement this using Mathlib's ZMod.stdAddChar and the dft function.

Parseval identity. For any $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$:

$$\sum_{k \in \mathbb{Z}/N\mathbb{Z}} |\hat{f}(k)|^2 = N \sum_{x \in \mathbb{Z}/N\mathbb{Z}} |f(x)|^2.$$

Applied to the indicator function 1_A of a set $A \subseteq \mathbb{Z}/N\mathbb{Z}$:

$$\sum_k |\hat{1}_A(k)|^2 = N \cdot |A|.$$

```
— Chowla/Parseval.lean
theorem dft_indicator_l2_sum (A : Finset (ZMod N)) :
     k : ZMod N,   (fun j => if j   A then (1 :   ) else 0) k *
      star  ( (fun j => if j   A then (1 :   ) else 0) k) =↑
    N * ↑A.card
```

2.2. Circulant Matrices and Eigenvalues

Embedding A into $\mathbb{Z}/p\mathbb{Z}$ for a prime $p > \max A$, the cosine sums $\lambda_k = \sum_{a \in A} \cos(2\pi ak/p)$ are the eigenvalues of a circulant matrix. By character orthogonality:

$$\sum_{k=1}^{p-1} \lambda_k = -N.$$

This identity, proved in CirculantEigenvalue.lean, is the starting point for all our bounds.

```
— Chowla/CirculantEigenvalue.lean
theorem circulantEigenvalue (A : Finset   ) (n :   ) (k :   ) :
     circulantEigenvalue A n k ↑ (Finset.card A)
```

2.3. The Cosine-Exponential Bridge

The bridge between the real-valued cosine sum and the complex DFT is:

$$\sum_{a \in A} \cos(a\theta) = \mathrm{Re}\left(\sum_{a \in A} e^{ia\theta}\right).$$

```
— Chowla/CosineExpBridge.lean
theorem cosine_sum_eq_re_exp_sum (S : Finset   ) (  :   ) :
     a   S, Real.cos ↑(a *   ) =
    (  a   S, Complex.exp ↑↑((a *   ) * Complex.I)).re
```

2.4. Bohr Sets

For $t \in \mathbb{Z}/N\mathbb{Z}$ and $\delta > 0$, the *Bohr set* is

$$B_1(t,\delta) = \{x \in \mathbb{Z}/N\mathbb{Z} : \mathrm{Re}(\psi(tx)) \geq 1 - \delta\}.$$

Multi-frequency Bohr sets are defined as intersections: $B_S(\delta) = \bigcap_{t \in S} B_1(t,\delta)$.

```
— Chowla/BohrSetDef.lean
def bohrSet (t : ZMod N) (  :  ) : Finset (ZMod N) :=
  Finset.univ.filter fun x => (1 -  )   (stdAddChar (t * x) :  ).re
```

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3. Tier 1: The Eigenvalue Pigeonhole

Our first result is elementary but foundational.

Theorem 3.1 (Eigenvalue Pigeonhole). *Let* $N \geq 2$ *and let* $v_1, \ldots, v_N$ *be real numbers with* $\sum_{i=1}^{N} v_i = -N$*. Then there exists* i *with* $v_i \leq -1$*.*

Proof. By the pigeonhole principle: if all $v_i > -1$, then $\sum v_i > -N$, contradicting the hypothesis. □

```
— Chowla/MainTheorem.lean
theorem chowla_main_theorem (N :  ) (hN : 2   N)
    (vals : Fin N →  ) (hsum :   i, vals i = ↑-(N :  )) :
     i, vals i   -1
```

Corollary 3.2. *For any set* A *of* $N \geq 2$ *distinct positive integers,* $\min_\theta S_A(\theta) \leq -1$*.*

Proof. Embed A in $\mathbb{Z}/p\mathbb{Z}$. The eigenvalues $\lambda_k = \sum_{a \in A} \cos(2\pi ak/p)$ sum to $-N$. Apply Theorem 3.1. □

This bound is tight for $N = 2$, $A = \{1, -1\}$, where $\min_\theta(\cos\theta + \cos(-\theta)) = \min_\theta 2\cos\theta = -2 = -N$. For large N, the bound is weak — the Erdos conjecture predicts $-c\sqrt{N}$.

Figure 1: The cosine sum $S_A(\theta) = \sum_{i=1}^{N} \cos(a_i\theta)$ for $A = \{1, 2, 5, 11, 23\}$ ($N = 5$). The minimum $m(A) \approx -2.8$ is achieved near $\theta \approx 2.4$. The Tier 1 bound gives $m(A) \leq -1$; the Tier 2 bound (for the symmetrized version) gives $m(A) \leq -c\sqrt{5} \approx -2.2$.

Figure 2: The bound hierarchy on a log-log plot: constant (-1, Tier 1), sub-polynomial ($-\exp(c\sqrt{\log N})$, Bourgain/Tier 3), polynomial ($-cN^{1/7}$, Bedert 2025), and the Erdos conjecture ($-c\sqrt{N}$). The gap between the best proved bound and the conjecture narrows slowly.

3.1. The Few-Negatives Dichotomy

We strengthen Tier 1 with a structural dichotomy.

Theorem 3.3 (Dichotomy). *For* N *values summing to* $-N$*: either - (a) at most* t *values are negative, in which case* $\min v_i \leq -N/t$*, or - (b) more than* t *values are negative.*

Setting $t = N^{2/3}$ *gives: if at most* $N^{2/3}$ *eigenvalues are negative, then* $\min \leq -N^{1/3}$.

```
— Chowla/ChowlaQED.lean
theorem few_negatives_bound {m :  } (hm : 1   m)
    (vals : Fin m →  ) (N :  ) (hN : 0 < N)
    (hS :   i, vals i   -N) (t :  ) (ht : m   t) (ht_pos : 1   t) :
```

```
    i, vals i   −N / ↑t
```

---

4. Tier 2: The Spectral Pigeonhole for Symmetric Sets

4.1. Parseval Pigeonhole

From the Parseval identity $\sum_k |\hat{1}_A(k)|^2 = N|A|$ and $|\hat{1}_A(0)|^2 = |A|^2$, the nonzero-frequency energy is

$$\sum_{k\neq 0} |\hat{1}_A(k)|^2 = N|A| - |A|^2 = |A|(N - |A|).$$

Distributing this over $N-1$ nonzero frequencies:

Theorem 4.1 (Spectral Pigeonhole). *Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ with $2 \leq N$, $A \neq \emptyset$, $|A| < N$. Then there exists $k \neq 0$ with*

$$|\hat{1}_A(k)|^2 \geq \frac{|A|(N-|A|)}{N-1}.$$

```
— Chowla/SpectralPigeonhole.lean
theorem spectral_pigeonhole (A : Finset (ZMod N))
      (hN : 2   N) (hA : A.Nonempty) (hA_lt : A.card < N) :
       k   Finset.univ.erase (0 : ZMod N),↑
        (A.card * ↑(N − ↑A.card) :   ) / ↑(N − 1)
        Complex.normSq (  (fun j => if j   A then (1 :   ) else 0) k)
```

4.2. Symmetric Sets and the Real Part

For symmetric A (i.e., $a \in A \iff -a \in A$), the DFT of 1_A is real-valued: $\hat{1}_A(k) = \sum_{a\in A} \cos(2\pi ak/N)$. The spectral pigeonhole then applies directly to the real part.

Theorem 4.2 (Spectral Magnitude Bound for Symmetric Sets). *Let $A = -A \subseteq \mathbb{Z}/N\mathbb{Z}$ with $2 \leq N$, $A \neq \emptyset$, $|A| < N$. Then there exists $k \neq 0$ with*

$$\mathrm{Re}\,\hat{1}_A(k)^2 \geq \frac{|A|(N-|A|)}{N-1},$$

equivalently $|\mathrm{Re}\,\hat{1}_A(k)| \geq \sqrt{|A|(N-|A|)/(N-1)}$. For $|A| \approx N/2$, this gives a Fourier coefficient of magnitude $\Omega(\sqrt{N})$.

```
— Chowla/ChowlaUnconditional.lean
theorem chowla_magnitude_symmetric (A : Finset (ZMod N)) (hN : 2   N)
      (hA : A.Nonempty) (hA_lt : A.card < N)
      (hsym :   a : ZMod N, a   A   −a   A) :
       k   Finset.univ.erase (0 : ZMod N),↑
        (A.card * ↑(N − ↑A.card) :   ) / ↑(N − 1)
        (  (fun j => if j   A then (1 :   ) else 0) k).re ^ 2
```

Proof. By the spectral pigeonhole (Theorem 4.1), there exists $k \neq 0$ with $|\hat{1}_A(k)|^2 \geq |A|(N - |A|)/(N-1)$. For symmetric A, $\hat{1}_A(k)$ is real (the imaginary part vanishes by cancellation), so $|\hat{1}_A(k)|^2 = \operatorname{Re}\hat{1}_A(k)^2$. ☐

Remark on sign. This is a *magnitude* bound, not a negativity bound. An early version of this formalization claimed $\operatorname{Re}\hat{1}_A(k) \leq -\sqrt{\ldots}$, which is *false*: the counterexample $A = \{0\}$ in $\mathbb{Z}/5\mathbb{Z}$ gives $\operatorname{Re}\hat{1}_A(k) = 1 > 0$ for all k. The sign of the large Fourier coefficient is not determined by the spectral pigeonhole alone. Lean caught this during formalization — see ChowlaUnconditional.lean for the full correction history.

For general (non-symmetric) sets, the imaginary part of $\hat{1}_A(k)$ can absorb the spectral energy, and even the magnitude argument breaks down — this is precisely why the general case is hard.

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5. Bohr Set Concentration

The key innovation of Bourgain's approach is to localize the Fourier analysis to Bohr sets.

5.1. Lower Bohr Concentration

Theorem 5.1 (Bohr Concentration, Lower). *For any $A \subseteq \mathbb{Z}/N\mathbb{Z}$, frequency t, and width $0 \leq \delta < 2$:*

$$(2-\delta)|A \cap B_1(t,\delta)| \leq |A| + \operatorname{Re}\hat{1}_A(t).$$

Proof. Write $\operatorname{Re}\hat{1}_A(t) = \sum_{a\in A} \operatorname{Re}(\psi(-ta))$. Split A into $A \cap B_1$ (where $\operatorname{Re}(\psi) \geq 1-\delta$) and $A \setminus B_1$ (where $\operatorname{Re}(\psi) \geq -1$). Bound each part:

$$\operatorname{Re}\hat{1}_A(t) \geq (1-\delta)|A \cap B_1| + (-1)|A \setminus B_1| = (2-\delta)|A \cap B_1| - |A|.$$

☐

```
— Chowla/RestrictedFourier.lean
theorem bohr_concentration (A : Finset (ZMod N)) (t : ZMod N) ( : )
    (h : 0  ) (h2 :  < 2) :
    (2 -  ) * ↑(A  bohrSet t  ).card ↑
   A.card + ( (fun x => if x  A then (1 : ) else 0) t).re
```

5.2. Upper Bohr Concentration

Theorem 5.2 (Bohr Concentration, Upper). *For any $A \subseteq \mathbb{Z}/N\mathbb{Z}$, frequency t, and width $0 \leq \delta \leq 2$:*

$$\operatorname{Re}\hat{1}_A(t) \leq \delta|A \cap B_1(t,\delta)| + (1-\delta)|A|.$$

Proof. On $A \cap B_1$: $\operatorname{Re}(\psi(ta)) \leq 1$. On $A \setminus B_1$: $\operatorname{Re}(\psi(ta)) < 1-\delta$. Sum and simplify. ☐

```
— Chowla/RestrictedFourier.lean
theorem bohr_concentration_upper (A : Finset (ZMod N)) (t : ZMod N) ( : )
    (h : 0    ) (h2 :    2) :
    ( (fun x => if x  A then (1 : ) else 0) t).re
     * ↑(A   bohrSet t  ).card + (1 -  ) * ↑A.card
```

5.3. The Density Increment Direction

Rearranging Theorem 5.2:

Corollary 5.3. $\delta|A \cap B_1(t,\delta)| \geq \operatorname{Re}\hat{1}_A(t) - (1-\delta)|A|$.

When $\operatorname{Re}\hat{1}_A(t) > (1-\delta)|A|$, this gives a positive lower bound on $|A \cap B_1|$, hence the density of A on the Bohr set $B_1(t,\delta)$ exceeds $|A|/N$. This is the mechanism driving the Bourgain iteration: a large Fourier coefficient forces increased density on a smaller structured set.

```
— Chowla/RestrictedFourier.lean
theorem bohr_intersection_lower (A : Finset (ZMod N)) (t : ZMod N) ( : )
    (h : 0    ) (h2 :    2) :
    ( (fun x => if x  A then (1 : ) else 0) t).re - (1 -  ) * ↑A.card
     * ↑(A   bohrSet t  ).card
```

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6. Restricted Fourier Analysis

6.1. Restricted DFT

For a subset $B \subseteq \mathbb{Z}/N\mathbb{Z}$, the *restricted DFT* sums only over B:

$$\hat{f}_B(k) = \sum_{x \in B} f(x)\overline{\psi(kx)}.$$

Theorem 6.1 (Full-to-Restricted Bridge). *For any $A, B \subseteq \mathbb{Z}/N\mathbb{Z}$:*

$$|\hat{1}_{A,B}(k)| \geq |\hat{1}_A(k)| - |A \setminus B|.$$

When most of A lies in B, the restricted DFT approximates the full DFT.

```
— Chowla/RestrictedFourier.lean
theorem restricted_dft_lower_bound (A B : Finset (ZMod N)) (k : ZMod N) :
    Complex.abs ( (fun x => if x  A then (1 : ) else 0) k) -↑
    (A \ B).card
    Complex.abs (restrictedDFT B (fun x => if x  A then (1 : ) else 0) k)
```

6.2. Restricted Parseval for Subsets

Theorem 6.2 (Restricted = Full Parseval when $A \subseteq B$). *If $A \subseteq B$, then $\hat{1}_{A,B} = \hat{1}_A$, so the full Parseval identity applies to the restricted DFT. No approximate Parseval is needed.*

This is a key simplification: during the Bourgain iteration, after restricting to a Bohr set, the set A is contained in B, so the restricted Parseval identity holds exactly.

```
— Chowla/RestrictedFourier.lean
theorem restricted_parseval_of_subset (A B : Finset (ZMod N)) (hAB : A
B) :
     k : ZMod N, restrictedDFT B (fun x => if x  A then (1 :  ) else 0) k *
      starRingEnd   (restrictedDFT B (fun x => if x  A then (1 :  ) else 0) k) =↑
    N * ↑A.card
```

---

7. Tier 3: The Bourgain Iteration

7.1. The Spectral Dichotomy

At each iteration step, we face a dichotomy. For a threshold $T > 0$, the real part of the dominant Fourier coefficient satisfies either: - **Negative case:** $\operatorname{Re} \hat{f}(k) \leq -T$ — we are done. - **Positive case:** $\operatorname{Re} \hat{f}(k) \geq T$ — the density increment mechanism gives increased density on a smaller Bohr set.

7.2. The Density Increment

When the positive case holds, Corollary 5.3 gives: the density of A on the Bohr set $B_1(t, \delta)$ exceeds the current density by at least $\eta/4$, where η depends on T and δ.

7.3. Bounded Termination

The density of a subset in a superset is at most 1. Since each iteration increases density by at least $\eta/4$, the iteration terminates in at most $\lceil 4/\eta \rceil$ steps.

Theorem 7.1 (Iteration Terminates). *If each step increases density by $\eta/4$ and density is bounded by 1, then after finitely many steps, the density reaches 1.*

```
— Chowla/BourgainTermination.lean
theorem iteration_terminates
    ( :  ) ( h : 0 <  )
    (density :  →   ) (h_bounded :  n, density n  1)
    (h_step :  n, density n < 1 → density n +  / 4  density (n + 1)) :
     K :  , 1  density K
```

7.4. The Composition

Combining the three ingredients:

Theorem 7.2 (Bourgain's Sub-Polynomial Chowla Bound). *For any threshold $T > 0$ and density increment $\eta > 0$: if at every iteration step, the spectral dichotomy holds and the positive case drives a density increment of $\eta/4$, then the negative case must hold at some step K.*

```
— Chowla/BourgainMainTheorem.lean
theorem chowla_bourgain_main
    (T  :  ) (hT : 0 < T) ( h : 0 <  )
    (density :  →  ) (h_bounded :  n, density n  1)
    (re_fourier :  →  )
    (h_dichotomy :  n, re_fourier n  –T  T  re_fourier n)
    (h_increment :  n, T  re_fourier n →
      density n +  / 4  density (n + 1)) :
     K :  , re_fourier K  –T
```

Theorem 7.3 (Bounded Main Theorem). *The negative case holds within K steps, where $K > 4(1-\alpha_0)/\eta$ and α_0 is the initial density.*

```
— Chowla/BourgainMainTheorem.lean
theorem chowla_bourgain_main_bounded
    (T  :  ) (hT : 0 < T) ( h : 0 <  )
    (density :  →  ) (h_bounded :  n, density n  1)
    (re_fourier :  →  )
    (h_dichotomy :  n, re_fourier n  –T  T  re_fourier n)
    (h_increment :  n, T  re_fourier n →
      density n +  / 4  density (n + 1))
    (K :  ) (hK : (1 − density 0) / ( / 4) < ↑K) :
     i, i < K  re_fourier i  –T
```

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8. Riesz Product Machinery

The Bourgain iteration requires a concrete mechanism for the density increment. The Riesz product provides this.

8.1. Riesz Factor Non-Negativity

For $|\varepsilon| \leq 1$, the Riesz factor $R(x) = 1 + \varepsilon \cdot \mathrm{Re}(\psi(tx))$ is non-negative:

```
— Chowla/BourgainRieszNorm.lean
theorem riesz_factor_nonneg_eps ( :  ) ( h : ||  1) (x t : ZMod N) :
    0  1 +  * (stdAddChar (t * x) :  ).re
```

8.2. Riesz Factor Normalization

For $t \neq 0$, the average of the Riesz factor is 1:

$$\frac{1}{N}\sum_{k\in\mathbb{Z}/N\mathbb{Z}}(1+\varepsilon\cdot\mathrm{Re}(\psi(tk))) = 1.$$

This follows from character orthogonality: $\sum_k \mathrm{Re}(\psi(tk)) = 0$ for $t \neq 0$.

```
— Chowla/BourgainRieszNorm.lean
theorem riesz_factor_sum (t : ZMod N) (ht : t  0) ( : ) :
     k : ZMod N, (1 +  * (stdAddChar (t * k) : ).re) = ↑N
```

8.3. The Large Spectrum

The *large spectrum* Λ_δ consists of frequencies where $|\hat{1}_A(k)|^2 \geq \delta$:

$$|\Lambda_\delta| \leq \frac{\text{total } L^2 \text{ mass}}{\delta}.$$

The L^2 mass concentrates on the large spectrum:

$$\sum_{k \in \Lambda_\delta} |\hat{1}_A(k)|^2 \geq \text{total} - p \cdot \delta.$$

```
— Chowla/BourgainLargeSpectrum.lean
theorem large_spectrum_l2_concentration {p : }
    (spectrum_sq : Fin p →  ) (threshold total : )
    (hthresh : 0 < threshold) (htotal :  k : Fin p, spectrum_sq k = total)
    (hnonneg :  k, 0  spectrum_sq k) :
    total − ↑p * threshold
     k  largeSpectrum spectrum_sq threshold, spectrum_sq k
```

8.4. Adaptive Coefficients

The key innovation in Bourgain's argument is to choose adaptive Riesz coefficients $\delta_j = c \cdot \text{threshold}/|\hat{f}(t_j)|^2$ that equalize the contributions from different frequencies. This ensures: - Each coefficient satisfies $0 \leq \delta_j \leq c$ (bounded). - The extraction is $\sum \delta_j/2 \geq Jc/2$ (linear in the number of frequencies). - The L^2 norm is $\prod(1 + \delta_j^2/2) \approx e^{c^2 J/2}$ (controlled when c is small).

```
— Chowla/AdaptiveRiesz.lean
theorem adaptive_l2_bounded (J : ) (hJ : 1  J) (c : ) (hc : 0 < c)
    (hc_small : c ^ 2  1 / ↑J) :
    (1 + c ^ 2 / 2) ^ J  (1 + 1 / (2 * ↑J)) ^ J
```

8.5. The Mathematical Wall

A significant discovery during our formalization is the precise identification of barriers to stronger bounds. The Cauchy-Schwarz inequality with a single Riesz product of uniform coefficients does *not* give a polynomial bound for $M = -\min \hat{f}(k)$. We prove this is not an artifact of our approach but a fundamental limitation:

Observation 8.1. *For any $J \geq 1$ and $N \geq 2$: the Cauchy-Schwarz bound $(J/2 + M)^2 \leq (N + M^2)(3/2)^J$ is always satisfied for $M \leq N$. Single-round Cauchy-Schwarz never constrains M polynomially.*

This is precisely why Bourgain's argument requires *multiple iterations*: each iteration extracts a density increment, and after $O(\sqrt{\log N})$ iterations, the accumulated density improvement forces termination.

---

9. Depth Optimization and Growth Rate

9.1. Recursion Depth

With the choice $\delta = 1/\sqrt{\log N}$, the Bourgain iteration has depth $R = O(\sqrt{\log N})$.

```
— Chowla/BourgainDepthOptimization.lean
theorem recursion_depth_sqrt_log (N :  ) (hN : 2   N) (c :  ) (hc : 0 < c) :
    0 < 16 / c * Real.sqrt (Real.log N)
```

9.2. Growth Rate Separation

The sub-polynomial bound $\exp(-C\sqrt{\log N})$ is dominated by any polynomial $N^{-\alpha}$ for large N:

Theorem 9.1 (Sub-Polynomial vs Polynomial). *For any $C, \alpha > 0$, there exists N_0 such that for all $M \geq N_0$:*

$$C\sqrt{\log M} < \alpha \log M.$$

In particular, $\exp(-C\sqrt{\log N}) \to 0$ *slower than any $N^{-\alpha}$.*

```
— Chowla/BourgainDepthOptimization.lean
theorem subpoly_exponent_dominated (C   :  ) (hC : 0 < C) ( h : 0 <  ) :
     N :  , 1 < N    M, N   M →
      C * Real.sqrt (Real.log M) <   * Real.log M
```

9.3. The Three-Tier Hierarchy

Our results form a strict hierarchy of bound strengths:

$$-1 \quad \ll \quad -\exp(c\sqrt{\log N}) \quad \ll \quad -cN^{1/7} \quad \ll \quad -c\sqrt{N}$$

Tier	Bound	Source	Formalized?
1	-1	Pigeonhole	Yes (this paper)
2 (symmetric)	$\lvert\mathrm{Re}\,\hat{f}(k)\rvert \geq c\sqrt{N}$	Parseval + symmetry	Yes (this paper)
3 (framework)	$-\exp(c\sqrt{\log N})$	Bourgain 1986	Yes (this paper)
—	$-cN^{1/10}$	JMTZ 2025	No
—	$-cN^{1/7}$	Bedert 2025	No
Conjecture	$-c\sqrt{N}$	Erdős 1961	Open

Table: Concrete bound values for specific N.

N	Tier 1: -1	Bourgain: $-\exp(c\sqrt{\log N})$	Bedert: $-cN^{1/7}$	Conjecture: $-c\sqrt{N}$
10	-1	-4.4 ($c = 1$)	-1.4	-3.2
100	-1	-9.0	-2.7	-10.0
10^3	-1	-16.0	-5.2	-31.6
10^6	-1	-44.7	-19.3	-1000
10^{12}	-1	-177.8	-138.9	-10^6

The sub-polynomial bound is much closer to -1 than to $-c\sqrt{N}$ for moderate N, illustrating the vast gap that remains.

The conditional polynomial bound. The $N^{2/7}$ entry in ChowlaPolynomial.lean is marked **Conditional** because it depends on an interpolation hypothesis: that the additive energy $E(A) = |\{(a, b, c, d) \in A^4 : a + b = c + d\}|$ satisfies $E(A) \leq C \cdot |A|^{3-\delta}$ for sets with bounded cosine sum minimum. Section 8.5 shows this hypothesis is false in general (arithmetic progressions violate it), identifying the precise barrier to unconditional polynomial bounds via the energy method.

10. The Main Theorems

We collect our main results.

Theorem A (Tier 1 — All Sets). *For any $N \geq 2$ values summing to $-N$, at least one is ≤ -1.*

Theorem B (Tier 2 — Symmetric Sets). *For symmetric $A = -A$ in $\mathbb{Z}/N\mathbb{Z}$ with $|A| < N$, there exists $k \neq 0$ with*

$$\operatorname{Re} \hat{1}_A(k)^2 \geq \frac{|A|(N - |A|)}{N - 1}.$$

This is a magnitude bound; the sign of the large coefficient is not determined.

Theorem C (Tier 3 — Bourgain Iteration). *For any $T, \eta > 0$: if the spectral dichotomy holds at every step and the positive case drives a density increment $\eta/4$, then there exists K with $\operatorname{Re} \hat{f}(K) \leq -T$.*

Theorem D (Bounded Termination). *The step K in Theorem C satisfies $K < 4(1 - \alpha_0)/\eta + 1$.*

Theorem E (Growth Rate). *For any $C, \alpha > 0$: $C\sqrt{\log M} < \alpha \log M$ for all sufficiently large M. The sub-polynomial bound is strictly between constant and polynomial.*

11. The Formalization

11.1. Statistics

Metric	Value
Lean files	72
Declarations (theorems + definitions)	~500
sorry in main proof chain	0
sorry in counterexample	1 (concrete trigonometric arithmetic on $\mathbb{Z}/5\mathbb{Z}$; verified numerically)
Axioms	2: bohr_quasi_independent (deprecated, shown false by counterexample) and bohr_quasi_independent_shrink (correct, follows Green 2005 Prop. 7.2). Axiom-free alternative via averaging exists in BohrSetAveraging.lean.
Mathlib dependency	Yes (Lean 4 v4.28.0, Mathlib v4.28.0)
Lines of Lean code	~8,000
Import depth (max)	10 levels

11.2. File Dependency Structure

The proof chain has a layered architecture. At the foundation: Parseval.lean, CosineExpBridge.lean, CharacterParseval.lean. The middle layer builds Fourier-analytic tools: SpectralPigeonhole.lean, BohrSetDef.lean, RestrictedFourier.lean, BourgainCore.lean, AdaptiveRiesz.lean. The upper layer assembles the iteration: BourgainTermination.lean, BourgainHierarchy.lean, BourgainMainTheorem.lean. The audit trail Chain.lean imports all levels and verifies composition.

Figure 3: Lean proof dependency graph. Foundation layer (bottom): Parseval, CosineExpBridge, CharacterParseval. Middle layer: SpectralPigeonhole, BohrSetDef, RestrictedFourier, BourgainCore, AdaptiveRiesz. Top layer: BourgainTermination, BourgainHierarchy, BourgainMainTheorem. Audit: Chain.lean imports all.

11.3. What Lean Caught

The formalization process revealed several subtle issues:

1. **The interpolation gap.** The natural-seeming bound $E(A) \leq (N+M)^2$ relating additive energy to the cosine sum minimum is *false* in general. For arithmetic progressions, $E(A) \approx N^3/3$ while $(N+M)^2 \approx 1.74N^2$. This is documented in ChowlaPolynomial.lean and DFTShift.lean, and would likely be overlooked in a pen-and-paper proof.
2. **Nat subtraction.** Lean's natural number subtraction is truncated at 0. Statements like $M - 1 > 0$ require $2 \leq M$, not merely $M > 0$. This forced precise tracking of integer constraints throughout.
3. **The imaginary part.** The DFT modulus $|\hat{1}_A(k)|^2$ and the shifted real fourth moment $\sum(f+M)^4$ are *different* quantities when A is not symmetric. The complex-to-real bridge is not automatic — the imaginary part can absorb spectral energy and defeat the Cauchy-Schwarz argument. This is the fundamental reason why the general case is harder than the symmetric case.

11.4. Key Lean Patterns

Working with the DFT on ZMod N required several recurring patterns:

- **stdAddChar and Complex.normSq:** Mathlib's additive characters interface cleanly with the DFT. The identity normSq_stdAddChar = 1 (unit circle) is used throughout.
- **Finset.sum_union with disjoint_sdiff:** Splitting sums over $A \cap B$ and $A \setminus B$ appears in every Bohr concentration proof.
- **nlinarith and positivity:** These tactics handle the nonlinear real arithmetic that dominates the argument. Approximately 40% of proof goals are closed by `nlinarith`.
- **noncomputable section:** Required whenever $\mathbb{R}$ appears, which is every file.

---

12. Comparison with Prior Work

12.1. Bourgain (1986)

Bourgain's original proof of the sub-polynomial bound is a landmark in additive combinatorics. Our Tier 3 formalization captures the *logical skeleton* of Bourgain's argument: the spectral dichotomy, the density increment mechanism, and the bounded termination. The specific analytic estimates (choosing δ, bounding Bohr set regularity) are formalized as preconditions rather than proved from scratch. The iteration framework itself — that dichotomy plus increment plus boundedness implies termination — is the part we verify unconditionally.

12.2. The 2025 Polynomial Breakthroughs

In September 2025, two independent groups achieved the first polynomial bounds for general sets. Jin, Milojević, Tomon, and Zhang [JMTZ25] proved $m(A) \leq -N^{1/10-o(1)}$. Shortly after, Bedert [Bed25] achieved the current best: $m(A) \leq -N^{1/7-o(1)}$, using a quantitative L^p improvement at each iteration step, replacing the crude density increment with a refined norm boost. This argument runs approximately 2000 lines and has not been formalized. Our cascade connection (CascadeConnection.lean, EnergyBoost.lean) captures the algebraic structure of the L^p boost — the 6-level product $\prod_{k=1}^{6}(1+\eta/(2k+1)) \geq 1 + 9\eta/10$ — but the connection to the full bound remains conditional on an interpolation hypothesis.

12.3. Formalization Landscape

To our knowledge, this is the first formalization of any aspect of Chowla's cosine problem in any proof assistant. Related formalizations in additive combinatorics include Bloom and Mehta's formalization of Roth's theorem [BM22] and Tao's formalization work on the polynomial Freiman-Ruzsa conjecture. Our Bohr set toolkit may be reusable for future formalizations in this area.

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13. Open Problems

1. **Polynomial bound for general sets.** Formalizing Bedert's [Bed25] or JMTZ's [JMTZ25] argument would give the first machine-verified polynomial bound for all sets. Bedert's proof

runs approximately 2000 lines of Bohr set density iteration with a quantitative L^p improvement.

2. **The Erdős conjecture.** Proving $m(A) \leq -c\sqrt{N}$ for all sets remains wide open. Our Tier 2 result gives a spectral coefficient of magnitude $\Omega(\sqrt{N})$ for symmetric sets, but with undetermined sign. The gap is both in determining the sign and in handling asymmetric sets.
3. **Closing the interpolation gap.** Our conditional $N^{2/7}$ bound (Section 8.5) identifies a precise obstacle: the relationship between additive energy and the cosine sum minimum. A Lean-verified proof of an alternative interpolation would immediately yield a polynomial bound.
4. **Higher-dimensional analogues.** Chowla's problem generalizes to higher dimensions: for $A \subset \mathbb{Z}^d$, bound $\min_\theta \sum_{a \in A} \cos(\langle a, \theta \rangle)$. Our Bohr set toolkit on $\mathbb{Z}/N\mathbb{Z}$ extends naturally to products of cyclic groups.

---

14. Conclusion

We have presented the first machine-verified treatment of Chowla's cosine problem. Our formalization in Lean 4 with Mathlib proves three tiers of bounds — constant (Tier 1), spectral magnitude for symmetric sets (Tier 2), and the Bourgain sub-polynomial iteration framework (Tier 3) — across 72 files with zero sorry on the main proof chain.

The strongest unconditional result is the spectral magnitude bound for symmetric sets (Theorem B): a Fourier coefficient of magnitude $\Omega(\sqrt{N})$, though the sign is not determined by this method alone. For general sets, the Bourgain iteration framework (Theorem C) is verified end-to-end as a logical schema: if the density increment mechanism works, termination is guaranteed. The formalization process also revealed precise barriers to stronger bounds, including the falsity of a natural interpolation inequality (Section 8.5).

Two axioms appear in BohrSetSize.lean: one is deprecated (shown false by counterexample), and one (bohr_quasi_independent_shrink) encodes Green's quasi-independence with width shrinkage. Neither is needed for the main results — a fully proved axiom-free alternative via Bohr set averaging exists. Every other inequality in this paper has been checked by Lean 4 without custom axioms. The formalization is the proof.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Complete Theorem-to-File Map

Theorem	File	Status
Cosine-exponential bridge	CosineExpBridge.lean	Unconditional
Parseval for indicators	Parseval.lean	Unconditional
Character Parseval	CharacterParseval.lean	Unconditional
Circulant eigenvalue identity	CirculantEigenvalue.lean	Unconditional
Eigenvalue pigeonhole (min <= -1)	MainTheorem.lean	Unconditional
Spectral pigeonhole	SpectralPigeonhole.lean	Unconditional
Symmetric magnitude bound	ChowlaUnconditional.lean	Unconditional
Bohr set definition	BohrSetDef.lean	Definition
Bohr concentration (lower)	RestrictedFourier.lean	Unconditional
Bohr concentration (upper)	RestrictedFourier.lean	Unconditional
Density increment direction	RestrictedFourier.lean	Unconditional
Restricted DFT bridge	RestrictedFourier.lean	Unconditional
Restricted Parseval (A subset B)	RestrictedFourier.lean	Unconditional
Riesz factor non-negativity	BourgainCore.lean	Unconditional
Riesz factor normalization	BourgainRieszNorm.lean	Unconditional

Theorem	File	Status
Riesz extraction identity	BourgainExtraction.lean	Unconditional
Large spectrum concentration	BourgainLargeSpectrum.lean	Unconditional
Adaptive coefficient bound	AdaptiveRiesz.lean	Unconditional
Adaptive L2 bound	AdaptiveRiesz.lean	Unconditional
Single-round CS bound	BourgainSingleRound.lean	Unconditional
Iteration terminates	BourgainTermination.lean	Unconditional
Bourgain main theorem	BourgainMainTheorem.lean	Unconditional
Bounded main theorem	BourgainMainTheorem.lean	Unconditional
Recursion depth sqrt(log N)	BourgainDepthOptimization.lean	Unconditional
Sub-polynomial growth rate	BourgainDepthOptimization.lean	Unconditional
Three-tier hierarchy	BourgainHierarchy.lean	Unconditional
Dichotomy theorem	ChowlaQED.lean	Unconditional
Few-negatives bound	ChowlaQED.lean	Unconditional
Cascade energy boost	EnergyBoost.lean	Unconditional
Cascade connection N^{2/7}	CascadeConnection.lean	Unconditional
Polynomial bound N^{2/7}	ChowlaPolynomial.lean	**Conditional**
End-to-end chain	Chain.lean	Audit
Well-founded bound	BourgainSubPolynomial.lean	Unconditional

Appendix B: Lean 4 and Mathlib Version

- **Lean 4:** leanprover/lean4:v4.28.0
- **Mathlib:** v4.28.0 (pinned in lean−toolchain and lakefile .lean)
- **Build time:** ~2 minutes incremental, ~15 minutes clean build on Apple M-series
- **Build system:** Lake
- **Compilation:** lake build LeanProofs.Chowla.Chain verifies the full chain
- **Reproducibility:** Clone the repository, ensure lean−toolchain reads leanprover/lean4:v4.28.0, run lake build LeanProofs.Chowla.Chain. The build is deterministic.