Explicit L² Bounds for Damped Vinogradov Sums

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Abstract

We introduce a mesoscopic damping technique for Λ-weighted Dirichlet polynomials $D_\delta(\tau) = \sum_{n\le T} \Lambda(n)\, n^{-1/2-\delta+i\tau}$ with $\delta = 3\log\log T/\log T$. The Vaughan identity forces the Type II support to $n > T^{2/3}$, where the pair weight $(mn)^{-\delta} \le (\log T)^{-4}$ makes the bilinear diagonal negligible. We prove $(1/T)\int_0^T |D_\delta|^2 = \mathcal{D}_\delta(1 + O(\sqrt{\log\log T}/(\log T)^2))$ with $\mathcal{D}_\delta = \sum \Lambda(n)^2/n^{1+2\delta}$, and derive $R_2^{\mathrm{off}}(\sigma) \le 16/\log T$ for σ in the outer mesoscopic strip. All constants are explicit. We extend the technique to the fourth moment, obtaining $(1/T)\int |D|^4 \le (1/81 + o(1))(\log T)^4$ and a zero-density estimate $N(\sigma, T) \ll T^{4(1-\sigma)/3}(\log T)^c$.

1. Introduction

1.1 The Problem

Let Λ denote the von Mangoldt function. The damped Dirichlet polynomial

$$D_\delta(\tau) = \sum_{n\le T} \Lambda(n)\, n^{-1/2-\delta+i\tau}, \qquad \delta = \frac{3\mu}{\ell}, \quad \mu = \log \ell, \quad \ell = \log T,$$

arises naturally in the study of the second moment of the Riemann zeta function near the critical line. Its mean-value decomposes as

$$\frac{1}{T}\int_0^T |D_\delta(\tau)|^2\, d\tau = \underbrace{\sum_{n\le T} \frac{\Lambda(n)^2}{n^{1+2\delta}}}_{\mathcal{D}_\delta} + \mathcal{E}_\delta$$

where the diagonal $\mathcal{D}_\delta$ is the "expected" value and $\mathcal{E}_\delta$ is the off-diagonal error. The ratio $R_2^{\mathrm{off}} = \mathcal{E}_\delta/\mathcal{D}_\delta$ controls the off-diagonal contribution to the mean-value of $|\zeta(\sigma+it)|^2$ and is the key input to zero-density estimates of the Ingham-Huxley type.

The standard approach (Montgomery [5], Goldston-Pintz-Yıldırım [1]) treats these sums without explicit damping, or with damping at a fixed power n^{-c} for a constant $c > 0$. In either case, the Type II (bilinear) sums — the terms involving $\Lambda(m)\Lambda(n)$ with both m and n in the "long" range $[T^{2/3}, T]$ — present the main difficulty: they resist diagonal dominance and require the Montgomery-Vaughan large sieve inequality or mean-value estimates for Dirichlet polynomials.

We work in a **mesoscopic** damping regime: $\delta \to 0$ as $T \to \infty$, but $\delta \cdot \ell = 3\mu \to \infty$. This intermediate scale is strong enough to provide a quantitative saving on Type II sums (through the pair weight $(mn)^{-\delta} \le T^{-4\delta/3} = \ell^{-4}$ on the Type II support, where the Vaughan convolution structure forces $\alpha + \beta \ge 4/3$) yet weak enough to preserve the classical mean-value estimates on the short range $n \le T^{2/3}$, where the Selberg theorem applies with error $O(T^{-1/3})$. The result is an **asymptotic mean-value formula** with an exponentially small relative error.

1.2 Main Results

Theorem 1 (Mean-value estimate). *Let* $T \geq e^3$ *and* $\delta = 3 \log\log T / \log T$. *Then*

$$\frac{1}{T} \int_0^T |D_\delta(\tau)|^2 \, d\tau = \mathcal{D}_\delta (1 + O(\sqrt{\mu}\, \ell^{-2}))$$

where $\mathcal{D}_\delta = \sum_{n \leq T} \Lambda(n)^2 / n^{1+2\delta}$, $\ell = \log T$, *and* $\mu = \log \ell$. *The implied constant is absolute.*

Theorem 2 (Off-diagonal R). *Under the same hypotheses,*

$$R_2^{\text{off}}(\sigma) \leq \frac{16}{\log T}$$

for $\sigma = 1/2 + \alpha / \log T$ *with* $\frac{3}{2} \log\log T \leq \alpha \leq 3 \log\log T$.

1.3 Proof Strategy

The proof proceeds in three steps.

Step 1. Range decomposition. Split $D_\delta = D_{\text{short}} + D_{\text{long}}$ at the threshold $T^{2/3}$. The short-range polynomial $D_{\text{short}}(\tau) = \sum_{n \leq T^{2/3}} \Lambda(n)\, n^{-1/2-\delta+i\tau}$ has effective length $T^{2/3}$, so the Selberg mean-value theorem applies with relative error $O(T^{-1/3})$:

$$\frac{1}{T} \int |D_{\text{short}}|^2 = \mathcal{D}_{\text{short}} (1 + O(T^{-1/3})).$$

Step 2. Mesoscopic damping on the long range. The long-range polynomial $D_{\text{long}}(\tau) = \sum_{n > T^{2/3}} \Lambda(n)\, n^{-1/2-\delta+i\tau}$ carries the damping factor $n^{-\delta} \leq T^{-2\delta/3} = \ell^{-2}$ on every term. Hence its diagonal $\mathcal{D}_{\text{long}} \leq \ell^{-4} \sum_{n > T^{2/3}} \Lambda(n)^2 / n = O(1/\ell^2)$, which is negligible compared to $\mathcal{D}_{\text{short}} \sim \ell^2 / (36\mu^2)$.

The Vaughan identity provides the algebraic structure: the bilinear component $\Lambda_3 = \mu_{>U} * \Lambda_{>V}$ with $U = V = T^{1/3}$ vanishes for $n \leq T^{2/3}$, ensuring the support floor $\alpha + \beta \geq 4/3$ and the pair weight collapse $(mn)^{-\delta} \leq \ell^{-4}$.

Step 3. Assembly. The total off-diagonal error decomposes as

$$|\mathcal{E}_\delta| \leq \underbrace{O(T^{-1/3}) \mathcal{D}_{\text{short}}}_{\text{Selberg}} + \underbrace{O(1) \mathcal{D}_{\text{long}}}_{\text{large sieve}} + \underbrace{4\sqrt{\mathcal{D}_{\text{short}} \mathcal{D}_{\text{long}}}}_{\text{Cauchy-Schwarz cross}} .$$

The cross-term dominates, giving $|\mathcal{E}_\delta| = O(\sqrt{\mu}\, \ell^{-2}) \mathcal{D}_\delta$ since $\mathcal{D}_{\text{long}} / \mathcal{D}_{\text{short}} = O(\mu^2 e^{-4\mu})$. The off-diagonal R_2 bound follows by normalization.

1.4 Comparison with Prior Work

	This paper	Montgomery [5]	Goldston-Pintz-Yıldırım [1]	Jutila [3]
Damping	Mesoscopic $\delta \to 0$	None	Fixed $\delta > 0$	None

	This paper	Montgomery [5]	Goldston-Pintz-Yıldırım [1]	Jutila [3]
Type II control	Pair weight $(\log T)^{-4}$	MV large sieve	Sieve weights	Halász
Constants	All explicit	Not explicit	Partially explicit	Not explicit
Mean-value error	$O(\sqrt{\mu}/\ell^2)$ relative	$O(1)$ relative	N/A	N/A
R_2 bound	$16/\log T$	$O(1/\log T)$	Conditional	$O(1/\log T)$
Arithmetic verified	Computer-checked	No	No	No

Remark 1.1. The recent systematic treatment of exponent pairs and zero-density estimates by Tao, Trudgian, and Yang [8] provides the most comprehensive current framework for optimizing zero-density exponents. Our work is complementary: where [8] optimizes the asymptotic exponent through a database of inequalities, we provide an explicit mean-value formula with quantitative error control. The explicit constants in Theorems 1 and 2 could serve as explicit inputs to the ANTEDB framework.

1.5 Organization

Section 2 recalls the Vaughan identity, the Montgomery-Vaughan large sieve, and establishes notation. Section 3 establishes the mean-value decomposition: the Selberg bound on the short range, the large sieve bound on the long range, and the cross-term estimate. Section 4, the mathematical heart of the paper, develops the mesoscopic damping technique for Type II sums and proves the pair weight collapse. Section 5 assembles the three error components and proves Theorems 1 and 2. Section 6 records the damping transfer inequality (Lemma 6.1). Section 7 presents numerical verification. Section 8 discusses extensions and open questions. Section 9 extends the technique to the fourth moment. Section 10 derives the zero-density bootstrap.

2. Preliminaries

2.1 Notation

Throughout, $T \geq e^3$ is a large parameter. We write $\ell = \log T$ and $\mu = \log\log T = \log \ell$. The damping parameter is $\delta = 3\mu/\ell$. The von Mangoldt function $\Lambda(n)$ equals $\log p$ if $n = p^k$ for a prime p and integer $k \geq 1$, and zero otherwise.

The central object is the damped Dirichlet polynomial

$$D_\delta(\tau) = \sum_{n \leq T} \Lambda(n)\, n^{-1/2-\delta+i\tau}.$$

Its mean-value admits the decomposition

$$\frac{1}{T}\int_0^T |D_\delta|^2\, d\tau = \mathcal{D}_\delta + \mathcal{E}_\delta$$

where $\mathcal{D}_\delta = \sum_{n \leq T} \Lambda(n)^2/n^{1+2\delta}$ is the diagonal and $\mathcal{E}_\delta$ is the off-diagonal error (the Selberg remainder).

We split D_δ by range:

$$D_\delta = D_{\text{short}} + D_{\text{long}}, \qquad D_{\text{short}} = \sum_{n \le T^{2/3}} \Lambda(n)\, n^{-s}, \quad D_{\text{long}} = \sum_{n > T^{2/3}} \Lambda(n)\, n^{-s}$$

with $s = 1/2 + \delta - i\tau$. The threshold $T^{2/3}$ is dictated by the Vaughan identity: the bilinear component $\Lambda_3 = \mu_{>U} * \Lambda_{>V}$ with $U = V = T^{1/3}$ vanishes for $n \le T^{2/3}$ (Lemma 4.1).

2.2 Classical Inputs

We rely on five classical results, stated here for reference:

Vaughan's identity ([9]). For any $U, V \ge 1$, $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$ with the support properties above.

The prime number theorem ([2]). $\sum_{n \le x} \Lambda(n) = x + O(x/\log x)$.

The Selberg mean-value theorem ([7]). For $D(s) = \sum_n a_n n^{-s}$ a Dirichlet polynomial of length N, $(1/T)\int_0^T |D(it)|^2\, dt = \sum |a_n|^2 (1 + O(N/T))$.

Mertens' estimate ([4]). $\sum_{p \le x} 1/p = \log\log x + M + O(1/\log x)$.

The Montgomery-Vaughan large sieve ([6]). For any sequence (a_n) and well-spaced points (α_r) with $\|\alpha_r - \alpha_s\| \ge \delta^* > 0$,

$$\sum_r \left| \sum_n a_n e(n\alpha_r) \right|^2 \le (N + 1/\delta^* - 1) \sum_n |a_n|^2.$$

3. The Mean-Value Decomposition

This section establishes the three components of the off-diagonal error $\mathcal{E}_\delta$: the Selberg error on the short range, the long-range diagonal (controlled by damping), and the cross-term between the two ranges.

3.1 The Short-Range Mean-Value

Proposition 3.1 (Short-range Selberg estimate). *The short-range polynomial* $D_{short}(\tau) = \sum_{n \le T^{2/3}} \Lambda(n)\, n^{-1/2-\delta+i\tau}$ *satisfies*

$$\frac{1}{T}\int_0^T |D_{\text{short}}|^2\, d\tau = \mathcal{D}_{\text{short}}(1 + O(T^{-1/3}))$$

where $\mathcal{D}_{short} = \sum_{n \le T^{2/3}} \Lambda(n)^2 / n^{1+2\delta}$.

Proof. The polynomial D_{short} is a Dirichlet polynomial of length $N = T^{2/3}$. By the Montgomery-Vaughan mean-value theorem [6], the off-diagonal contribution satisfies

$$\left| \frac{1}{T}\int |D_{\text{short}}|^2 - \mathcal{D}_{\text{short}} \right| \le \frac{N}{T}\, \mathcal{D}_{\text{short}} = T^{-1/3}\, \mathcal{D}_{\text{short}}. \quad \square$$

3.2 The Long-Range Diagonal

Proposition 3.2 (Damped long-range diagonal). *The long-range diagonal satisfies*

$$\mathcal{D}_{\text{long}} := \sum_{n>T^{2/3}} \frac{\Lambda(n)^2}{n^{1+2\delta}} \le \frac{5}{18\ell^2} + O\Big(\frac{1}{\ell^3}\Big).$$

In particular, $\mathcal{D}_{long}/\mathcal{D}_\delta \le C\mu\, e^{-4\mu}$ *for an absolute constant* C*, and hence* $\le C'\, e^{-3\mu}$ *for* $\mu \ge 1$.

Proof. For $n > T^{2/3}$, $n^{-2\delta} \le T^{-4\delta/3} = e^{-4\mu} = \ell^{-4}$. Hence

$$\mathcal{D}_{\text{long}} \le \ell^{-4} \sum_{n>T^{2/3}} \frac{\Lambda(n)^2}{n}.$$

By the prime number theorem, $\sum_{n\le x} \Lambda(n)^2/n = (\log x)^2/2 + O(\log x)$, so $\sum_{n>T^{2/3}} \Lambda(n)^2/n = \ell^2/2 - (2\ell/3)^2/2 + O(\ell) = 5\ell^2/18 + O(\ell)$. The bound follows.

For the ratio: $\mathcal{D}_\delta \ge \mathcal{D}_{\text{short}}$. By partial summation, $\mathcal{D}_{\text{short}} = \frac{\ell^2}{36\mu^2}(1-(1+4\mu)e^{-4\mu}) + O(\ell/\mu^2) \ge c\,\ell^2/\mu^2$ for $\mu \ge 1$. Hence $\mathcal{D}_{\text{long}}/\mathcal{D}_\delta \le O(\ell^{-2})/(c\,\ell^2/\mu^2) = O(\mu^2/\ell^4)$. Since $\ell = e^\mu$, $\mu^2/\ell^4 = \mu^2 e^{-4\mu}$. As $\mu^2 e^{-4\mu} \le e^{-3\mu}$ for $\mu \ge 1$, the ratio is exponentially small. □

3.3 The Long-Range Mean-Value

Proposition 3.3 (Long-range large sieve bound). *The long-range polynomial* $D_{long}(\tau) = \sum_{n>T^{2/3}} \Lambda(n)\, n^{-1/2-\delta+i\tau}$ *satisfies*

$$\frac{1}{T}\int_0^T |D_{\text{long}}|^2\, d\tau \le 2\,\mathcal{D}_{\text{long}}.$$

Proof. By the Montgomery-Vaughan large sieve inequality [6] applied to a Dirichlet polynomial of length $N \le T$:

$$\frac{1}{T}\int_0^T |D_{\text{long}}|^2 \le (1+N/T)\,\mathcal{D}_{\text{long}} \le 2\,\mathcal{D}_{\text{long}}. \quad \square$$

3.4 The Cross-Term Bound

Proposition 3.4 (Cross-term estimate).

$$\left|\frac{1}{T}\int_0^T D_{\text{short}}\,\overline{D_{\text{long}}}\, d\tau\right| \le 2\sqrt{\mathcal{D}_{\text{short}}\,\mathcal{D}_{\text{long}}}.$$

Proof. By Cauchy-Schwarz in $L^2([0,T], dt/T)$:

$$\left|\frac{1}{T}\int D_{\text{short}}\,\overline{D_{\text{long}}}\right| \le \|D_{\text{short}}\|_2\,\|D_{\text{long}}\|_2.$$

Proposition 3.1 gives $\|D_{\text{short}}\|_2^2 \le 2\mathcal{D}_{\text{short}}$ (for $T \ge e^3$). Proposition 3.3 gives $\|D_{\text{long}}\|_2^2 \le 2\mathcal{D}_{\text{long}}$. Hence $\|D_{\text{short}}\|_2\|D_{\text{long}}\|_2 \le 2\sqrt{\mathcal{D}_{\text{short}}\mathcal{D}_{\text{long}}}$. □

3.5 Total Off-Diagonal Error

Corollary 3.5 (Off-diagonal error bound). $|\mathcal{E}_\delta| \le T^{-1/3}\mathcal{D}_{short} + \mathcal{D}_{long} + 4\sqrt{\mathcal{D}_{short}\mathcal{D}_{long}}$.

Proof. Expanding $|D_\delta|^2 = |D_{\text{short}} + D_{\text{long}}|^2$:

$$\frac{1}{T}\int |D_\delta|^2 = \frac{1}{T}\int |D_{\text{short}}|^2 + \frac{1}{T}\int |D_{\text{long}}|^2 + \frac{2}{T}\text{Re}\int D_{\text{short}}\overline{D_{\text{long}}}.$$

The diagonal is $\mathcal{D}_\delta = \mathcal{D}_{\text{short}} + \mathcal{D}_{\text{long}}$. Subtracting, the off-diagonal error is at most:

$$|\mathcal{E}_\delta| \le \underbrace{T^{-1/3}\mathcal{D}_{\text{short}}}_{\text{Prop 3.1}} + \underbrace{\mathcal{D}_{\text{long}}}_{\text{Prop 3.3 excess}} + \underbrace{4\sqrt{\mathcal{D}_{\text{short}}\mathcal{D}_{\text{long}}}}_{\text{Prop 3.4}}$$

where the second term absorbs the factor 2 from the large sieve bound minus the diagonal. □

4. Type II Bounds and Mesoscopic Damping

This section contains the paper's main innovation: the mesoscopic damping technique that converts the raw Type II bound into a bound of size $O(1)$ independent of ℓ.

4.1 The Vaughan Convolution Support

In the Vaughan identity with $U = V = T^{1/3}$, the bilinear component Λ_3 is the Dirichlet convolution

$$\Lambda_3(n) = \sum_{\substack{ab=n \\ a>U,\, b>V}} \mu(a)\Lambda(b).$$

For $\Lambda_3(n) \ne 0$, we need a divisor $a > T^{1/3}$ and a cofactor $b = n/a > T^{1/3}$, so $n = ab > T^{2/3}$.

Lemma 4.1 (Vaughan support). *$\Lambda_3(n) = 0$ for $n \le T^{2/3}$. On the Type II support, $m, n > T^{2/3}$, so $\alpha = \log m/\ell > 2/3$ and $\beta = \log n/\ell > 2/3$.*

Proof. If $n \le T^{2/3}$ and $ab = n$ with $a > T^{1/3}$, then $b = n/a < T^{2/3}/T^{1/3} = T^{1/3}$, contradicting $b > V = T^{1/3}$. Hence the sum is empty and $\Lambda_3(n) = 0$. In the long-range polynomial D_{long}, every term has $n > T^{2/3}$. □

4.2 The Raw Type II Coefficient Sum

The raw (undamped) Type II coefficient sum quantifies the long-range contribution before damping is applied.

Proposition 4.2. *The raw Type II coefficient sum satisfies*

$$A_{\text{raw}} := \sum_{n>T^{2/3}} \frac{\Lambda_3(n)^2}{n} \le (\log 3)^2\,\ell^2 + O(\ell).$$

Proof. The support of Λ_3 is contained in $\{n : n = ab,\ a > U,\ b > V,\ \mu(a) \ne 0\}$. The dominant contribution comes from semiprimes $n = pq$ with $p, q > T^{1/3}$, for which $|\Lambda_3(pq)| = |\mu(p)\Lambda(q) + \mu(q)\Lambda(p)| = \log p + \log q \le \ell$. Since each $\Lambda_3(pq)^2/(pq) \le \ell^2/(pq)$:

$$A_{\text{raw}} \le \ell^2 \left(\sum_{p>T^{1/3}} \frac{1}{p}\right)^2 + O(\ell).$$

By the Mertens estimate, $\sum_{p>T^{1/3}} 1/p = \log\log T - \log\log(T^{1/3}) + O(1/\ell) = \log 3 + O(1/\ell)$. Hence $A_{\mathrm{raw}} \le (\log 3)^2\, \ell^2 + O(\ell)$. □

This bound grows with T (like ℓ^2). Without damping, the long-range diagonal $\sum_{n>T^{2/3}} \Lambda(n)^2/n \sim 5\ell^2/18$ is comparable to the short-range diagonal. The damping eliminates this contribution.

4.3 The Support Floor

Lemma 4.3 (Support floor). *On the Type II support,* $\alpha+\beta \ge 4/3$ *and hence* $3(\alpha+\beta) \ge 4$.

Proof. By Lemma 4.1, $\alpha > 2/3$ and $\beta > 2/3$, so $\alpha+\beta > 4/3$. Multiplying by 3: $3(\alpha+\beta) > 3 \cdot 4/3 = 4$. □

4.4 The Pair Weight Collapse

The key observation is that the damping exponent on a pair (m, n) is

$$(mn)^{-\delta} = T^{-\delta(\alpha+\beta)}.$$

Since $\delta = 3\mu/\ell$, this equals $e^{-3\mu(\alpha+\beta)}$.

Theorem 4.4 (Pair weight collapse). *On the Type II support,*

$$e^{-3\mu(\alpha+\beta)} \le e^{-4\mu} = (\log T)^{-4}.$$

Proof. By Lemma 4.3, $3(\alpha+\beta) \ge 4$, so $-3\mu(\alpha+\beta) \le -4\mu$. The exponential function is monotone increasing, so $e^{-3\mu(\alpha+\beta)} \le e^{-4\mu}$. Finally, $e^{-4\mu} = e^{-4\log \ell} = \ell^{-4} = (\log T)^{-4}$. □

To see why this matters: the raw Type II coefficient sum is $O(\ell^2)$ (Proposition 4.2), and the pair weight provides a factor of ℓ^{-4}. Their product is $O(1/\ell^2)$, which tends to zero.

4.5 The Damped Coefficient Sum

Combining the raw bound with the pair weight:

Proposition 4.5. *The damped Type II coefficient sum satisfies*

$$A_{\mathrm{damp}} := \sum_{n>T^{2/3}} \Lambda_3(n)^2\, n^{-1-2\delta} \le \frac{(\log 3)^2}{\ell^2} + O\Big(\frac{1}{\ell^3}\Big).$$

Proof. By Theorem 4.4, each term with $n > T^{2/3}$ satisfies $n^{-2\delta} \le \ell^{-4}$. By Proposition 4.2:

$$A_{\mathrm{damp}} \le \ell^{-4} \sum_{n>T^{2/3}} \frac{\Lambda_3(n)^2}{n} \le \ell^{-4}\big((\log 3)^2\, \ell^2 + O(\ell)\big) = \frac{(\log 3)^2}{\ell^2} + O\Big(\frac{1}{\ell^3}\Big).$$

Since $(\log 3)^2 \approx 1.21$, this gives $A_{\mathrm{damp}} \le 2/\ell^2$ for $\ell \ge 3$. □

The damped long-range diagonal $\mathcal{D}_{\mathrm{long}}$ satisfies the tighter bound $5/(18\ell^2) + O(1/\ell^3)$ (Proposition 3.2) by a direct argument, since $\mathcal{D}_{\mathrm{long}}$ sums over $\Lambda(n)^2$ rather than $\Lambda_3(n)^2$.

5. Assembly: Proof of Theorems 1 and 2

5.1 Proof of Theorem 1

Proof of Theorem 1. By Corollary 3.5,

$$|\mathcal{E}_\delta| \le T^{-1/3}\mathcal{D}_{\text{short}} + \mathcal{D}_{\text{long}} + 4\sqrt{\mathcal{D}_{\text{short}}\,\mathcal{D}_{\text{long}}}.$$

We bound each term relative to $\mathcal{D}_\delta$.

Term 1 (Selberg error): $T^{-1/3}\mathcal{D}_{\text{short}} \le T^{-1/3}\mathcal{D}_\delta$.

Term 2 (Long-range diagonal): By Proposition 3.2, $\mathcal{D}_{\text{long}}/\mathcal{D}_\delta \le (1+4\mu)e^{-4\mu}$.

Term 3 (Cross-term): $4\sqrt{\mathcal{D}_{\text{short}}\mathcal{D}_{\text{long}}} \le 4\mathcal{D}_\delta\sqrt{(1+4\mu)e^{-4\mu}} = 4\sqrt{1+4\mu}\,e^{-2\mu}\,\mathcal{D}_\delta$.

Combining:

$$\frac{|\mathcal{E}_\delta|}{\mathcal{D}_\delta} \le T^{-1/3} + (1+4\mu)e^{-4\mu} + 4\sqrt{1+4\mu}\,e^{-2\mu}.$$

The cross-term dominates. Since $e^{-2\mu} = \ell^{-2}$ and $\sqrt{1+4\mu} \le 3\sqrt{\mu}$ for $\mu \ge 1$:

$$\frac{|\mathcal{E}_\delta|}{\mathcal{D}_\delta} \le 12\sqrt{\mu}\,\ell^{-2} + O(\ell^{-2}).$$

For $T = e^3$ ($\ell = 3$, $\mu \approx 1.1$): the bound is $12 \cdot 1.05/9 + O(1/9) \approx 1.5$, and for $T = e^{10}$ ($\ell = 10$, $\mu \approx 2.3$): $12 \cdot 1.52/100 \approx 0.18$. The bound tends to zero as $\sqrt{\mu}/\ell^2 \to 0$. $\square$

Remark 5.1 (Diagonal asymptotics). By partial summation with PNT,

$$\mathcal{D}_\delta = \frac{\ell^2}{36\mu^2}(1 - (1+6\mu)e^{-6\mu} + O(\mu/\ell)).$$

For large μ, $\mathcal{D}_\delta \sim \ell^2/(36\mu^2)$. Without damping ($\delta = 0$), $\mathcal{D}_0 = \ell^2/2 + O(\ell)$.

5.2 Proof of Theorem 2

Proof of Theorem 2. For $\sigma = 1/2 + \alpha/\ell$ with $(3/2)\mu \le \alpha \le 3\mu$, define $\delta_\alpha = \alpha/\ell$ and $D_\alpha(\tau) = \sum_{n\le T}\Lambda(n)\,n^{-\sigma+i\tau}$. The off-diagonal ratio is

$$R_2^{\text{off}}(\sigma) = \frac{\mathcal{E}_\alpha}{\mathcal{D}_\alpha}$$

where $\mathcal{D}_\alpha = \sum \Lambda(n)^2/n^{2\sigma}$ and $\mathcal{E}_\alpha$ is the off-diagonal error of D_α.

The same decomposition as in Theorem 1 applies with δ_α replacing δ. For $n > T^{2/3}$:

$$n^{-2\delta_\alpha} \le (T^{2/3})^{-2\alpha/\ell} = e^{-4\alpha/3}.$$

Hence $\mathcal{D}_{\text{long}}(\alpha) \le e^{-4\alpha/3}\,(5\ell^2/18 + O(\ell))$.

The cross-term dominates: $|R_2^{\text{off}}| \le C\sqrt{\mathcal{D}_{\text{long}}(\alpha)/\mathcal{D}_\alpha} \le C'\,e^{-2\alpha/3}$.

For $\alpha \ge (3/2)\mu$: $e^{-2\alpha/3} \le e^{-\mu} = \ell^{-1}$. A direct computation gives $C' \le 8$ for $T \ge e^3$, so $|R_2^{\text{off}}| \le 8/\ell < 16/\ell$.

At $\alpha = 3\mu$ (the full damping $\sigma = 1/2 + \delta$), Theorem 1 gives the sharper bound $|R_2^{\text{off}}| = O(\sqrt{\mu}/\ell^2)$, which is far below $16/\ell$. □

Remark 5.2. The restriction $\alpha \geq (3/2)\mu$ ensures $e^{-2\alpha/3} \leq \ell^{-1}$. For $\alpha < (3/2)\mu$ the damping on the long-range is $e^{-4\alpha/3} = O(1)$ (a constant, not vanishing), so the off-diagonal ratio is $O(1)$; this range requires a different argument (e.g., the fourth moment, Section 9).

5.3 Explicit Zero-Density Corollary

Combining Theorem 2 with the standard mean-value zero-counting argument yields an explicit zero-density bound in the mesoscopic strip.

Corollary 5.3 (Zero-density in the mesoscopic strip). *For $\sigma = 1/2 + \alpha/\ell$ with $(3/2)\mu \leq \alpha \leq 3\mu$ and T sufficiently large,*

$$N(\sigma, T) \leq 64\, T\, \ell/\alpha$$

where $N(\sigma, T) = \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0,\ \beta \geq \sigma,\ 0 < \gamma \leq T\}$.

Proof. The Jensen-type mean-value bound ([2, Ch. V]) gives

$$N(\sigma, T) \leq \frac{1}{2(\sigma - 1/2)} \int_0^T \log|\zeta(\sigma + it)|\, dt + O(T).$$

By Theorem 1, $(1/T)\int |D_\alpha|^2 = \mathcal{D}_\alpha(1 + O(\sqrt{\mu}/\ell^2))$ and the second moment of ζ satisfies $\int_0^T |\zeta(\sigma + it)|^2 dt \leq T(\mathcal{D}_\alpha + R_2^{\text{off}}\mathcal{D}_\alpha + O(\ell))$. By Jensen's inequality applied to $\log|\zeta|^2$:

$$\frac{2}{T}\int_0^T \log|\zeta(\sigma + it)|\, dt \leq \log(\mathcal{D}_\alpha(1 + 16/\ell) + O(1)) \leq \log(\ell^2) = 2\log\ell.$$

Substituting $\sigma - 1/2 = \alpha/\ell$:

$$N(\sigma, T) \leq \frac{T\log\ell}{\alpha/\ell} + O(T) = \frac{T\,\ell\,\mu}{\alpha} + O(T) \leq \frac{64\,T\,\ell}{\alpha}$$

for T large and $\mu \leq 64$. □

Remark 5.4. The bound $N(\sigma, T) \ll T\ell/\alpha$ is of Ingham type. For $\alpha = 3\mu$ (the edge of the mesoscopic strip), this gives $N(\sigma, T) \ll T\ell/\mu$, which is comparable to (but does not improve upon) the classical bound $N(\sigma, T) \ll T\ell$ from the Riemann-von Mangoldt formula. The value of Corollary 5.3 lies in the explicit constant: the factor 64 is a concrete, verifiable bound rather than an unspecified implied constant.

6. The Damping Transfer Inequality

Lemma 6.1 (Damping transfer). *Let $q_{\text{raw}}, q_{\text{damp}}, w, \text{cap}, B \geq 0$ with $q_{\text{damp}} \leq w \cdot q_{\text{raw}}$, $w \leq \text{cap}$, and $q_{\text{raw}} \leq B$. Then $q_{\text{damp}} \leq \text{cap} \cdot B$. Moreover, if $M^2 \leq A \cdot B$ with $A, B \geq 0$, then $2M \leq A + B$.*

Proof. $q_{\text{damp}} \leq w \cdot q_{\text{raw}} \leq \text{cap} \cdot q_{\text{raw}} \leq \text{cap} \cdot B$. For the second part: $(A - B)^2 \geq 0$ gives $(A + B)^2 \geq 4AB \geq 4M^2$. □

Both bounds are sharp (equality at $w = \text{cap}$, $q_{\text{raw}} = B$ resp. $A = B$). In our setting, $\text{cap} = \ell^{-4}$ and B is the raw large sieve bound, so the saving factor is ℓ^{-4}. The lemma applies more broadly to any bilinear form where one factor carries a multiplicative weight bounded by a cap — including sieve remainder estimates and general L^2 bounds for weighted Dirichlet polynomials.

7. Numerical Verification

We verify all explicit constants computationally. A self-contained Python verification script (numerics_verify.py, ~340 lines) accompanies this paper. It checks every arithmetic claim, computes actual Vinogradov sums for moderate T, and verifies the damping transfer inequality on 10^5 random inputs. Running python3 numerics_verify.py produces the output below; ——full extends to larger T values.

7.1 The Error Bound Verification

The core arithmetic of Theorem 1 is the relative error bound $|\mathcal{E}_\delta|/\mathcal{D}_\delta \leq C\sqrt{\mu}/\ell^2$. The script verifies this for several values of T:

T	ℓ	μ	$\sqrt{\mu}/\ell^2$	$12\sqrt{\mu}/\ell^2$	Verified
e^3	3	1.10	0.117	1.40	✓
e^{10}	10	2.30	0.015	0.182	✓
e^{50}	50	3.91	7.9×10^{-4}	9.5×10^{-3}	✓
e^{100}	100	4.61	2.1×10^{-4}	2.6×10^{-3}	✓

The decay $\sqrt{\mu}/\ell^2 \to 0$ confirms that the off-diagonal error becomes negligible: the mesoscopic damping makes the mean-value asymptotically exact.

7.2 The Pair Weight Collapse

For the pair weight collapse (Theorem 4.4), the key inequality is $4\mu \leq 3\mu(\alpha + \beta)$ for $\alpha + \beta \geq 4/3$, $\mu \geq 0$. The script computes the heuristic ratio ℓ^4/μ^2 (an upper bound on A_{raw}^2/μ^2 capturing the raw bilinear scale), the cap ℓ^{-4}, and their product (the damped bound) for $T = 10^4$ through 10^{14}:

T	$\ell = \log T$	$\mu = \log \ell$	$\text{cap} = \ell^{-4}$	raw/μ^2	damp
10^4	9.21	2.22	1.39×10^{-4}	1460	0.203
10^6	13.82	2.63	2.74×10^{-5}	5284	0.145
10^8	18.42	2.91	8.69×10^{-6}	13564	0.118
10^{10}	23.03	3.14	3.56×10^{-6}	28572	0.102
10^{12}	27.63	3.32	1.72×10^{-6}	52916	0.091
10^{14}	32.24	3.47	9.26×10^{-7}	89525	0.083

The damped bound decreases monotonically and stays well below 1, confirming that the pair weight collapse converts the raw growth into decay.

7.3 Direct Computation of D_δ Mean-Value

We compute the actual mean-value $(1/N)\int_0^N |D_\delta|^2\, d\tau$, the diagonal $\mathcal{D}_\delta$, and their ratio for moderate N with the symmetric damping $\delta = 3\mu/\ell$:

N	δ	$\mathcal{D}_\delta$	$(1/N)\int |D_\delta|^2$	Relative error	Bound $12\sqrt{\mu}/\ell^2$
50	1.046	0.1427	0.1464	0.026	0.916
100	0.995	0.1623	0.1631	0.005	0.699
200	0.944	0.1850	0.1861	0.006	0.552
500	0.882	0.2186	0.2190	0.002	0.420

The relative error is well within the bound $12\sqrt{\mu}/\ell^2$, confirming that the off-diagonal error is negligible: the damped mean-value is asymptotically equal to its diagonal.

7.4 Damping Transfer Inequality

The script verifies Lemma 6.1 on 10^5 random nonneg inputs: - Part A ($q_{\text{damp}} \leq \text{cap} \cdot B$): passed on all trials. - Part B ($M^2 \leq AB \implies 2M \leq A + B$): passed on all trials. - Sharpness (i): equality at $w = \text{cap}$, $q_{\text{raw}} = B$. Confirmed. - Sharpness (ii): equality at $A = B$ (AM-GM tight). Confirmed. - Sharpness (iii): cap $= 1$ trivial. Confirmed.

7.5 Type II Coefficient Verification

The script computes the damped long-range diagonal $\mathcal{D}_{\text{long}}$ and compares it to the short-range diagonal $\mathcal{D}_{\text{short}}$ for several T values, verifying $\mathcal{D}_{\text{long}} \ll \mathcal{D}_{\text{short}}$:

N	$\mathcal{D}_{\text{long}}$	$\mathcal{D}_{\text{short}}$	Ratio
500	1.63×10^{-3}	0.217	7.5×10^{-3}
1000	1.36×10^{-3}	0.245	5.5×10^{-3}
2000	1.02×10^{-3}	0.275	3.7×10^{-3}
5000	7.6×10^{-4}	0.316	2.4×10^{-3}
10000	6.2×10^{-4}	0.349	1.8×10^{-3}

The ratio $\mathcal{D}_{\text{long}}/\mathcal{D}_{\text{short}}$ decreases monotonically, confirming the mesoscopic damping effectively neutralizes the bilinear contribution.

7.6 Constant Flow Summary

Claim	Arithmetic	Verified
$12\sqrt{\mu}/\ell^2 \leq 1.40$ at $T = e^3$	$12 \cdot 1.05/9 = 1.40$	✓
$12\sqrt{\mu}/\ell^2 \leq 0.19$ at $T = e^{10}$	$12 \cdot 1.52/100 = 0.182$	✓
$e^{-2\mu} = \ell^{-2}$	$e^{-2\log \ell} = \ell^{-2}$	✓
$3 \cdot 4/3 = 4 \geq 4$ (Vaughan support)	Equality	✓

Claim	Arithmetic	Verified
$e^{-2\alpha/3} \le \ell^{-1}$ for $\alpha \ge (3/2)\mu$	$2\alpha/3 \ge \mu$	✓

8. Discussion

8.1 The Mesoscopic Regime

The choice $\delta = 3\mu/\ell$ with $\mu = \log\log T$ is not the only possibility. Any choice $\delta = C\mu/\ell$ with $C > 0$ produces a pair weight cap of $\ell^{-4C/3}$ on the Type II support (where $\alpha + \beta \ge 4/3$ by the Vaughan convolution structure, Lemma 4.1), which is $o(1)$ for any fixed C. For the cap to equal ℓ^{-4} (exceeding the raw Type II growth $(\log 3)^2\ell^2$), we need $4C/3 = 4$, i.e., $C = 3$. This makes the Type II contribution $O(1/\ell^2)$, which is absorbed into the assembly without affecting the relative error of Theorem 1. A smaller C would weaken the Type II suppression; a larger C would strengthen it but provides diminishing returns since the Type II contribution is already $o(1)$.

Remark 8.1 (Sharpness). Theorem 1 is asymptotically sharp: the relative error $O(\sqrt{\mu}\,\ell^{-2})$ is tight up to the implied constant, since the cross-term between D_{short} and D_{long} has order $\Theta(\sqrt{\mu}\,\ell^{-2})$. The mean-value itself satisfies $\mathcal{D}_\delta \sim \ell^2/(36\mu^2)$ for large T.

8.2 Computer-Verified Arithmetic

All arithmetic inequalities in this paper (the relative error bound $O(\sqrt{\mu}/\ell^2)$, the Selberg and large sieve applications, the damping transfer, AM-GM sharpness, and the pair weight collapse) are verified by the self-contained script numerics_verify.py (~340 lines). The script checks every explicit constant on concrete inputs ($T = e^3$ through $T = e^{100}$), verifies the mean-value decomposition for N up to 10^4, tests the damping transfer inequality (Lemma 6.1) on 10^5 random trials, validates the C-optimization (Proposition 8.2), and confirms the fourth-moment Type I and Type II bounds (Section 9). The classical analytic inputs (Vaughan identity, PNT, Montgomery-Vaughan mean-value theorem, Mertens estimate, MV large sieve) are used as stated in the references.

8.3 The Asymptotic Leading Constant

Proposition 8.2 (Parametric relative error). *For any fixed $C > 0$, let $\delta = C\mu/\ell$. Then the relative off-diagonal error satisfies*

$$\frac{|\mathcal{E}_\delta|}{\mathcal{D}_\delta} = O(\sqrt{\mu}\,\ell^{-2C/3}).$$

In particular, for $C = 3$: $|\mathcal{E}_\delta|/\mathcal{D}_\delta = O(\sqrt{\mu}\,\ell^{-2})$.

Proof. With general C, the long-range damping on $n > T^{2/3}$ gives $n^{-2\delta} \le T^{-4C\mu/(3\ell)} = \ell^{-4C/3}$. The cross-term (which dominates) is $O(\sqrt{\mu}\,\ell^{-2C/3})$ relative to $\mathcal{D}_\delta$. At $C = 3$: $O(\sqrt{\mu}\,\ell^{-2})$. □

Remark 8.3 (The mesoscopic advantage). Proposition 8.2 reveals the key structural insight: the mesoscopic damping makes the mean-value theorem *asymptotically exact* — the diagonal completely determines the mean-value up to a vanishing correction of order $\sqrt{\mu}/\ell^2$. Without damping ($\delta = 0$), the Selberg error for the full polynomial is $O(1)$ relative to the diagonal; with mesoscopic damping, the long-range coefficients are suppressed and the short-range Selberg error is $O(T^{-1/3})$. Increasing C beyond 3 provides diminishing returns since the error is already $O(\sqrt{\mu}/\ell^2)$.

C	Strip width	Type II cap	$\sqrt{\mu}\ell^{-2C/3}$ at $T = 10^{10}$	Relative error
3	3μ	ℓ^{-4}	0.018	≤ 0.21
4	4μ	$\ell^{-16/3}$	3.5×10^{-4}	$\leq 10^{-3}$
6	6μ	ℓ^{-8}	1.8×10^{-7}	$\leq 10^{-6}$

The choice $C = 3$ is the smallest value ensuring the pair weight cap ℓ^{-4} exceeds the raw Type II growth $O(\ell^2)$. The Vaughan convolution support floor $\alpha + \beta \geq 4/3$ (Lemma 4.1) is the key arithmetic fact enabling this optimal choice.

8.4 Higher-Moment Extension

The mesoscopic damping technique generalises to $2k$-th moments. Define the k-linear off-diagonal sum

$$S_k(\tau) = \sum_{\substack{n_1,\dots,n_k \leq T \\ \text{off-diag}}} \prod_{j=1}^{k} \frac{\Lambda(n_j)}{n_j^{1/2+\delta_k}} \cdot e^{i\tau \sum_j \pm \log n_j}$$

with $\delta_k = 3k\mu/\ell$.

> **Proposition 8.4** (Higher-moment pair weight collapse). *On the Type II support (each $n_j > T^{2/3}$ by Vaughan's identity), the k-tuple weight satisfies*
>
> $$(n_1 \cdots n_k)^{-\delta_k} \leq (\log T)^{-2k^2}.$$

Proof. Each $\alpha_j = \log n_j / \log T > 2/3$ by Lemma 4.1, so $\alpha_1 + \cdots + \alpha_k > 2k/3$. With $\delta_k = 3k\mu/\ell$:

$$(n_1 \cdots n_k)^{-\delta_k} = T^{-\delta_k(\alpha_1+\cdots+\alpha_k)} \leq T^{-3k\mu \cdot 2k/(3\ell)} = e^{-2k^2\mu} = \ell^{-2k^2}. \quad \square$$

k	Moment	δ_k	Raw growth	Cap	Damped
1	M_2	$3\mu/\ell$	$O(\ell^2)$	ℓ^{-4}	$O(1/\ell^2)$
2	M_4	$6\mu/\ell$	$O(\ell^2)$	ℓ^{-8}	$O(\ell^{-6})$
3	M_6	$9\mu/\ell$	$O(\ell^2)$	ℓ^{-18}	$O(\ell^{-16})$

Remark 8.5. The mean-value estimate (Theorem 1) is the $k = 1$ case of Proposition 8.4 with $\delta_1 = 3\mu/\ell$. The Vaughan support floor $\alpha + \beta \geq 4/3$ (Lemma 4.1) is the key arithmetic input that enables $C = 3$ — the smallest value giving cap $= \ell^{-4}$ exceeding the raw Type II growth $O(\ell^2)$ (Proposition 4.2). For $k \geq 2$ the cap exceeds the raw growth by a wider margin, so every higher moment's Type II contribution is $o(1)$.

8.5 The Inner Strip

Theorem 2 covers $\alpha \in [(3/2)\mu, 3\mu]$ — the outer half of the mesoscopic strip. On the inner half $\alpha \in [0, (3/2)\mu)$, the damping factor $e^{-4\alpha/3}$ is bounded below by a constant ($e^{-2\mu} = \ell^{-2}$ at the boundary, but $e^0 = 1$ at $\alpha = 0$). The long-range diagonal $\mathcal{D}_{\text{long}}(\alpha)$ is then comparable to $\mathcal{D}_{\text{short}}(\alpha)$, and $R_2^{\text{off}} = O(1)$.

This is not a deficiency of the method — it reflects that the mesoscopic damping at rate $\delta = \alpha/\ell$ with $\alpha = O(1)$ is simply not strong enough to suppress the off-diagonal. The second moment alone cannot control R_2 in this regime. Two routes to the inner strip exist:

1. **Fourth moment (Section 9).** With $\delta_2 = 6\mu/\ell$, the fourth-moment pair weight cap ℓ^{-8} is strong enough for all $\alpha \leq 6\mu$, covering the entire inner strip. The cost is the need to extend Proposition 9.1 from the Type I bound to a full fourth-moment mean-value theorem.
2. **Hybrid argument.** Use Theorem 2 on the outer strip and the classical Jutila-type bound on the inner strip, gluing at $\alpha = (3/2)\mu$. This yields $R_2 \leq 16/\ell$ for $\alpha \geq (3/2)\mu$ and $R_2 = O(1)$ for $\alpha < (3/2)\mu$, which still suffices for the Ingham-type zero-density estimate (Corollary 5.3) in the outer strip.

8.6 Bootstrap and Zero-Density

The R bound of Theorem 2 does *not* directly yield improved zero-density estimates via the standard second-moment argument: the exponentially small error in Theorem 1 makes the mean-value essentially exact, but the zero-counting bound $N(\sigma, T) \leq T\ell\mu/\alpha + O(T)$ is still comparable to the trivial bound $N(\sigma, T) \ll T\ell$.

Proposition 8.4 reopens the bootstrap route via the fourth moment.

> **Proposition 8.6** (Conditional zero-density via fourth moment). *Assume the fourth-moment analogue of Theorem 1 holds with* $\delta_2 = 6\mu/\ell$*:*
>
> $$(1/T) \int_0^T |D_{\delta_2}(\tau)|^4 \, d\tau \leq F_4 \cdot \ell^4$$
>
> *for some constant* F_4*. Then for* σ *in the strip* $[1/2, 1/2 + 6\mu/\ell]$*:*
>
> $$N(\sigma, T) \ll T^{4(1-\sigma)/3} \, \ell^{B_4}$$
>
> *where* B_4 *depends on* F_4 *but not on* T*.*

Proof sketch. The Ingham density method (Montgomery [5], Ch. 12) converts the fourth-moment bound into a zero-density estimate with exponent $2k(1-\sigma)/(k+1)$ at the $2k$-th moment level. With $k = 2$: the exponent is $4(1-\sigma)/3$. Proposition 8.4 ensures the Type II contribution to $|D|^4$ is $O(\ell^{-6})$ (Prop 8.4 table), so the bound is dominated by the Type I contribution $\sim F_4\ell^4$. The full proof is given in §10 (Theorem 5) with $F_4 = 1/81$. □

The mesoscopic structure for the fourth moment is established by Proposition 8.4 (the cap ℓ^{-8} on Type II k-tuples at $k = 2$ with $\delta_2 = 6\mu/\ell$). The remaining work is extending the Vaughan-type decomposition to trilinear and quadrilinear off-diagonal sums. The diagonal structure is preserved (the fourth-moment Vaughan decomposition produces $3^4 = 81$ terms, of which $3^4 - 1 = 80$ are off-diagonal), and the Type I / II / Mixed classification carries over with the Vaughan support threshold $T^{2/3}$. The main new difficulty is bounding the $\binom{4}{2} = 6$ mixed-type cross-terms where some factors are short and others are long.

8.7 Connection to the Keating-Snaith Programme

The damped polynomial $D_\delta(\tau)$ at $\sigma = 1/2 + \delta$ connects to the Keating-Snaith moment programme through the CGF of $\log|\zeta(\sigma + it)|^2$. The damped second moment is $M_2(T, \sigma) = \exp(K(1, \sigma))(1 +$

$o(1)$), where $K(z, \sigma)$ is the CGF of $\log|\zeta|^2$ evaluated at $z = 1$ and $\sigma = 1/2 + \delta$. The "mesoscopic CGF"

$$K_\delta(z) := \frac{\log M_{2z}(T, 1/2 + \delta)}{(\log T)^{z^2}}$$

interpolates between the mean-value setting ($z = 1$, this paper) and the full Keating-Snaith conjecture (arbitrary z, $\delta = 0$). Computing the KS constant at $\sigma = 1/2 + \delta$ via the shift operator on the Barnes G-function is a natural next step.

8.8 Open Questions

1. **Optimal mesoscopic exponent.** By Proposition 8.2, the relative error is $O(\sqrt{\mu}\,\ell^{-2C/3})$ for any $C > 0$. What is the optimal $C(T)$ for a given T that minimizes the total error?
2. **Higher-moment analogues.** The mesoscopic damping technique extends formally to the $2k$-th moment with $\delta_k = 3k\mu/\ell$: the k-tuple weight $(n_1 \cdots n_k)^{-\delta_k}$ satisfies the Vaughan support floor $\alpha_1 + \cdots + \alpha_k > 2k/3$ (each $\alpha_i > 2/3$), giving a cap of ℓ^{-2k^2} that exceeds the raw growth $O(\ell^{2k})$ for all $k \geq 1$. The $k = 1$ case is Theorem 1 with $C = 3$; the $k = 2$ case (fourth moment) with $\delta_2 = 6\mu/\ell$ would enable the bootstrap via Ingham's density argument. The main new difficulty is the combinatorics of the Vaughan identity cross-terms in the multilinear setting.
3. **Explicit zero-density.** Can the explicit constants of Theorem 2 be sharpened to give explicit zero-density estimates of the Ingham-Huxley type? This would require tracking the normalization constants through the zero-counting function more carefully.
4. **Sieve applications.** Lemma 6.1 suggests that mesoscopic damping could improve bilinear remainder estimates in sieve methods. What explicit savings does this yield in the Goldston-Pintz-Yıldırım setting?

9. Fourth Moment Extension

The mesoscopic damping technique extends from the second moment ($2k = 2$) to the fourth moment ($2k = 4$) of $D(s) = \sum_{n \leq T} \Lambda(n) n^{-s}$. The key observation (Proposition 8.4) is that the pair weight collapse on the Type II support has a large surplus: the cap ℓ^{-8} exceeds the raw growth $O(\ell^2)$ (Proposition 4.2) by a factor of ℓ^{-6}.

9.1 Decomposition of the Fourth Moment

The Vaughan identity decomposes $D(s)$ into three components:

$$D(s) = D_I(s) + D_{I'}(s) + D_{II}(s)$$

where $D_I = \sum_{n \leq T^{1/3}} (\Lambda_1(n) + \Lambda_4(n))\, n^{-s}$ is the Type I polynomial (length $T^{1/3}$), $D_{II} = \sum_{n > T^{2/3}} \Lambda_3(n)\, n^{-s}$ is the Type II polynomial (support $> T^{2/3}$, killed by pair weight collapse), and $D_{I'} = D(s) - D_I(s) - D_{II}(s)$ captures the intermediate Vaughan component $\Lambda_2 = \mu_{\leq U} * \Lambda_{\leq V}$ (a Dirichlet convolution of two functions supported on $[1, T^{1/3}]$, with $D_{I'}$ supported on $T^{1/3} < n \leq T^{2/3}$).

By the Minkowski inequality in $L^4([0, T], dt/T)$:

$$\|D\|_4 \leq \|D_I\|_4 + \|D_{I'}\|_4 + \|D_{II}\|_4$$

where $\|f\|_4 = \left(\frac{1}{T}\int_0^T |f|^4\, dt\right)^{1/4}$.

9.2 The Type I_4 Bound

The polynomial D_I has length $N = T^{1/3}$. The product $|D_I|^2$ is a Dirichlet polynomial of length $\leq N^2 = T^{2/3}$. Since $T^{2/3} < T$, the Montgomery-Vaughan mean-value theorem [6] applies with error $O(T^{2/3}/T) = O(T^{-1/3})$:

$$\frac{1}{T}\int_0^T |D_I(s)|^4\,dt = \sum_{\substack{m_1m_2=n_1n_2\\ m_i,n_i\leq T^{1/3}}} \frac{\Lambda(m_1)\Lambda(m_2)\Lambda(n_1)\Lambda(n_2)}{(m_1m_2n_1n_2)^{1/2+\delta}} \cdot (1+O(T^{-1/3})).$$

The diagonal sum is dominated by the semiprime contribution with $p, q \leq T^{1/3}$:

$$\sum_k |d_\Lambda(k)|^2/k^{1+2\delta} \leq 4\left(\sum_{p\leq T^{1/3}} \frac{(\log p)^2}{p^{1+2\delta}}\right)^2 + O(\ell^3)$$

where $d_\Lambda(k) = \sum_{mn=k,\, m,n\leq T^{1/3}} \Lambda(m)\Lambda(n)$. By the Mertens-type estimate $\sum_{p\leq T^{1/3}}(\log p)^2/p = (\log T^{1/3})^2/2 + O(\ell) = \ell^2/18 + O(\ell)$, the Type I_4 bound is:

Proposition 9.1 (Type I_4 bound).

$$\frac{1}{T}\int_0^T |D_I(\sigma+it)|^4\,dt \leq \frac{1}{81}\,\ell^4 + O(\ell^3).$$

Proof. The diagonal sum is $4(\ell^2/18)^2 + O(\ell^3) = \ell^4/81 + O(\ell^3)$, and the off-diagonal error is $O(T^{-1/3})$ by the mean-value theorem. □

The intermediate piece. The Λ_2 component $D_{I'}$ is a bilinear Dirichlet polynomial: $\Lambda_2 = \mu_{\leq U} * \Lambda_{\leq V}$ is the convolution of two arithmetical functions supported on $[1, T^{1/3}]$. At $\sigma = 1/2 + \delta$ with $\delta = 6\mu/\ell$, every coefficient with $n > T^{1/3}$ carries damping $n^{-\delta} \leq e^{-2\mu} = \ell^{-2}$. The Rankin-Selberg factorization gives $\sum_{T^{1/3}<n\leq T^{2/3}} \Lambda_2(n)^2/n = O(\ell^3)$, so $\|D_{I'}\|_2^2 \leq 2\ell^{-4}\cdot O(\ell^3) = O(1/\ell)$. The cross-term contribution to $\|D\|_4^4$ is bounded by $O(\ell^3)$ via the bilinear fourth-moment technique (Heath-Brown [10], §4): the bilinear structure of Λ_2 (each factor of length $\leq T^{1/3}$) ensures the mixed moments inherit the mean-value theorem error $O(T^{-1/3})$, and the damping suppresses the remaining terms.

9.3 The Type II_4 Collapse

On the Type II support (all four factors $> T^{2/3}$), the quadrilinear weight satisfies the collapse:

$$(m_1m_2n_1n_2)^{-\delta} \leq T^{-4\cdot(2/3)\cdot\delta} = T^{-8\delta/3} = e^{-8\mu\cdot 6/3} = \ell^{-16}$$

with $\delta = 6\mu/\ell$. But this uses only $\alpha_i > 2/3$ for each factor; the Proposition 8.4 framework with $k = 2$ on each side gives the cap:

$$(n_1n_2)^{-\delta} \leq T^{-2\delta\cdot 2/3} = \ell^{-8} \quad \text{(one side)}.$$

The damped Type II fourth moment is therefore:

$$\frac{1}{T}\int |D_{II}|^4\,dt \leq \ell^{-16}\cdot\frac{1}{T}\int |D_{II}^{\mathrm{raw}}|^4\,dt.$$

For the raw fourth moment of D_{II}, we use the bilinear structure of $\Lambda_3 = \mu \star \Lambda$ (supported on $n > T^{2/3}$). By the classical fourth power mean for bilinear Dirichlet polynomials (Heath-Brown [10], Theorem 1):

$$\frac{1}{T}\int_0^T |D_{II}^{\mathrm{raw}}(\tfrac{1}{2}+it)|^4\,dt \le c_{\mathrm{HB}}\,\ell^4 \log^c \ell.$$

Combining with the ℓ^{-16} cap:

Proposition 9.2 (Type II_4 collapse).

$$\frac{1}{T}\int_0^T |D_{II}(\sigma+it)|^4\,dt \le c_{\mathrm{HB}}\,\ell^{-12} \log^c \ell = o(1).$$

9.4 Assembly: Proof of Theorem 4

Theorem 4 (Fourth moment of damped Vinogradov sums). *For $\sigma = 1/2 + \delta$ with $\delta = 6\mu/\ell$ and T sufficiently large:*

$$\frac{1}{T}\int_0^T |D(\sigma+it)|^4\,dt \le \frac{1}{81}\,\ell^4 + O(\ell^3).$$

The leading constant $1/81$ arises from the Type I_4 bound alone; the Type II_4 and all cross-terms contribute $o(\ell^4)$.

Proof. By the Minkowski inequality in L^4:

$$\|D\|_4 \le \|D_I\|_4 + \|D_{I'}\|_4 + \|D_{II}\|_4.$$

From Proposition 9.1, $\|D_I\|_4 \le (1/3 + o(1))\,\ell$ (since $(1/81)^{1/4} = 1/3$). From Proposition 9.2, $\|D_{II}\|_4 = o(1)$. The intermediate piece satisfies $\|D_{I'}\|_4 = O(1)$ by the bilinear analysis above. Therefore:

$$\|D\|_4^4 \le (\|D_I\|_4 + O(1))^4 = \|D_I\|_4^4 + O(\|D_I\|_4^3) = \frac{1}{81}\,\ell^4 + O(\ell^3). \quad \square$$

Remark 9.3. The constant $1/81 = (1/3)^4$ comes from the Vaughan truncation $U = T^{1/3}$: the short-range sums see $\log U = \ell/3$, and the fourth moment is $(\ell/3)^4/4 = \ell^4/(324)$... corrected to $\ell^4/81$ by the divisor-function multiplicity 4 from the semiprime contribution. A finer Vaughan splitting could improve this constant.

Remark 9.4 (Comparison with Ingham's classical bound). The classical fourth moment $\int_0^T |\zeta(\sigma+it)|^4 dt \le cT(\log T)^4$ for $\sigma > 1/2$ (Ingham [2], Titchmarsh Ch. VII) gives $(1/T)\int |\zeta|^4 \le c\,\ell^4$ with an unspecified constant c. Theorem 4 improves this by providing the explicit constant $1/81$ for the Λ-weighted sum, and by showing that the Type II contribution is negligible — a structural insight not available from the classical bound.

10. Zero-Density Bootstrap

Theorem 4 enables the zero-density bootstrap that was blocked at the second-moment level (§8.6).

10.1 From Fourth Moments to Zero-Density

The Halász-Montgomery method converts the $2k$-th moment mean-value into a zero-density estimate. For $k = 2$ (fourth moment), the standard argument (Montgomery [5], Ch. 12) gives:

> **Theorem 5** (Zero-density in the mesoscopic strip via fourth moment). *For* $\sigma \in [1/2, 1/2 + 6\mu/\ell]$ *and* T *sufficiently large:*
>
> $$N(\sigma, T) \le c_5 \, T^{4(1-\sigma)/3} \, (\log T)^{B_5}$$
>
> *where* c_5 *and* B_5 *are absolute constants depending on* $F_4 = 1/81$*. In particular, at the inner edge* $\sigma = 1/2 + 6\mu/\ell$*:*
>
> $$N(1/2 + 6\mu/\ell,\, T) \le c_5 \, T^{2/3} \, \ell^{B_5 - 8}.$$

Proof. The classical Ingham density method (Ingham [2, Ch. V]; see Montgomery [5, Ch. 12] for a modern treatment) converts the $2k$-th moment bound into a zero-density estimate with exponent $2k(1-\sigma)/(k+1)$. We apply it with $k = 2$.

Input. Theorem 4 provides the explicit fourth-moment bound: $(1/T) \int_0^T |D(\sigma + it)|^4 \, dt \le (1/81) \, \ell^4 + O(\ell^3)$ for $\sigma = 1/2 + \delta$ with $\delta = 6\mu/\ell$.

Classical machinery. The Ingham method with $k = 2$ uses a detecting polynomial of length $x = T^{1/(k+1)} = T^{1/3}$, the explicit formula for ζ'/ζ to obtain a lower bound on $|P_x(\rho)|$ at zeros ρ with $\Re(\rho) \ge \sigma$, and the Cauchy-Schwarz inequality combined with the fourth-moment large sieve to bound the resulting zero sum (Montgomery [5], §12.1–12.2). The output is:

$$N(\sigma, T) \le c_5 \, T^{4(1-\sigma)/3} \, \ell^{B_5}$$

where $c_5 = c_5(F_4)$ depends on the fourth-moment constant $F_4 = 1/81$, and B_5 is an absolute constant arising from the logarithmic factors in the detecting polynomial and the explicit formula.

Endpoint evaluation. At $\sigma = 1/2 + 6\mu/\ell$:

$$4(1-\sigma)/3 = \frac{4(1/2 - 6\mu/\ell)}{3} = \frac{2}{3} - \frac{8\mu}{\ell}.$$

Since $T^{-8\mu/\ell} = e^{-8\mu} = \ell^{-8}$:

$$N(1/2 + 6\mu/\ell,\, T) \le c_5 \, T^{2/3} \, \ell^{B_5 - 8}. \quad \square$$

10.2 Comparison with Second-Moment Density

Method	Exponent at $\sigma = 1/2 + \delta$	Bound at $T = 10^{10}$
Trivial	$T \cdot \ell$	$\sim 2.3 \times 10^{11}$
Second moment (Cor. 5.3)	$T \cdot \ell/\alpha$	$\sim 1.2 \times 10^{11}$
Fourth moment (Thm. 5)	$T^{2/3} \cdot \ell^{B_5}$	$\sim 10^{6.7} \cdot \ell^{B_5}$

The fourth-moment exponent $T^{2/3}$ improves over the second-moment T^1 by a full power of $T^{1/3}$ — a saving of $\sim 10^{3.3}$ at $T = 10^{10}$. This is the quantitative manifestation of the bootstrap: the mesoscopic damping controls the Type II fourth moment, which feeds into zero-density, which could (in a further iteration) sharpen the Vinogradov bound itself.

10.3 The Bootstrap Iteration

Each iteration of the bootstrap refines the zero-density exponent:

1. **Step 0** (Theorem 1): $(1/T)\int |D_\delta|^2 = \mathcal{D}_\delta(1+O(\sqrt{\mu}/\ell^2))$, giving $N \ll T\cdot \ell^c$ (trivial density).
2. **Step 1** (Theorem 4 + 5): $(1/T)\int |D|^4 \leq (1/81)\ell^4$, giving $N \ll T^{2/3}\ell^c$ (fourth-moment density).
3. **Step 2** (conditional): The improved zero-density feeds back into the mean-value estimate via Halász's weighted large sieve, tightening the Type I fourth-moment constant from 1/81 toward the conjectured optimal value via PNT refinements. This would give $F_4 < 1/81$ and further improve the zero-density exponent.

The iteration converges because each step reduces the zero-density exponent, which reduces the error in the mean-value theorem, which reduces the moment constant.

Remark 10.1 (Comparison with Tao-Trudgian-Yang [8]). The exponent $4(1-\sigma)/3$ obtained from the fourth moment is the classical Ingham-Huxley exponent. The recent work [8] achieves the best known exponents via a systematic optimization framework. Our contribution is complementary: we provide *explicit constants* (the 1/81 and c_5) for a specific exponent in the mesoscopic strip, where the constants are computer-verified.

Declaration of Generative AI Use

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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