$D_\{}\{}infty$ and the Goldbach Convergence Hierarchy

Why Binary Goldbach Is the Threshold of Additive Number Theory

One number — 0.046 — explains why Goldbach is hard, Vinogradov is easy, and twin primes stay on the prime-counting (divergent) side

Dr. Tamás Nagy

`tnagyphd@gmail.com`

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Two guitar strings played together produce interference. One guitar string read twice produces the same note. This is why Goldbach sits in the convergent convolution channel, while twin primes stay on the correlative (prime-counting) side.

Abstract

We identify the total zero energy $D_\infty = \sum_\rho 1/|\rho|^2 = 2 + \gamma - \log(4\pi) \approx 0.046$ as the invariant that **organizes** the difficulty landscape of additive prime problems in this framework. For a k-fold additive problem (representing n as a sum of k primes), each independent summand contributes a factor of $1/\rho$ to the zero sum in the explicit-formula picture, so the natural tail is heuristically $\sum 1/|\rho|^k$. This sum diverges for $k = 1$ (prime counting) and converges for $k \geq 2$ (Goldbach and beyond). Binary Goldbach sits at the first convergent scale for primes—the borderline where the tail is summable but the constants are tight.

We present three layered claims. First, the *Convolution Convergence Theorem* (conditional sketch): under RH, the triangle inequality and absolute convergence of $\sum_\rho 1/|\rho|^2$ yield the explicit bound $|E(n)| \leq 2D_\infty\sqrt{n} + O(1)$ for the standard zero-sum form of the Goldbach error; comparing this to the Hardy–Littlewood main term shows $|E(n)| < S(n)$ for all **sufficiently large** even n, while any uniform statement down to $n = 4$ still requires a separate treatment of the main-term approximation for small n (or appeal to numerical verification). Second, the *k-fold Damping Hierarchy*: classical ternary Goldbach (Vinogradov; completed by Helfgott) is **consistent with** $D_\infty^{(3)} \approx 0.0012$ lying deep in the convergent regime, where even the Vinogradov–Korobov zero-free region suffices for large-n analytic approaches. Third, the *Additive–Correlative Duality*: twin primes, prime k-tuples, and other **fixed-shift** correlations are *correlative* problems whose standard zero-sum models retain the $\sum 1/|\rho|$ scaling of prime counting—without the Goldbach convolution damping.

The number $D_\infty \approx 0.046$ is small enough that, once n is large enough for the Hardy–Littlewood main term to dominate the error bound $|E(n)| \ll \sqrt{n}$, the zero sum has limited headroom: in that asymptotic regime the zeta zeros cannot overwhelm the main term. **Small** n are outside the domain of that asymptotic comparison and are handled separately in any complete argument (computation or sharper sieve input).

Keywords: Goldbach conjecture, zero energy, convolution damping, additive–correlative duality, Waring problem

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1. Introduction

1.1 A Number

The nontrivial zeros of the Riemann zeta function satisfy $\zeta(\rho) = 0$ with $0 < \mathrm{Re}(\rho) < 1$. From the Hadamard product representation of $\xi(s)$:

$$D_\infty = \sum_\rho \frac{1}{\rho(1-\rho)} = 2 + \gamma - \log(4\pi) \approx 0.04619$$

where $\gamma \approx 0.5772$ is the Euler–Mascheroni constant. Under the Riemann Hypothesis, $1 - \rho = \bar{\rho}$, so $D_\infty = \sum 1/|\rho|^2$. This sum converges — and its value is extraordinarily small.

This paper argues that D_∞ is the single number that organizes the difficulty landscape of additive prime problems in this framework. It explains why binary Goldbach is hard, ternary Goldbach is easier in the k-fold picture, why many Waring-type problems permit more summands (hence more damping), and why twin primes are structurally correlative rather than convolution-type. The mechanism is convolution damping: each independent summand in an additive problem contributes a factor of $1/\rho$ to the zero sum in the explicit formula. The transition from divergent ($k = 1$, prime counting) to convergent ($k = 2$, Goldbach) occurs at exactly the binary threshold.

1.2 The Observation

The prime counting function $\psi(x) = \sum_{n \le x} \Lambda(n)$ has the explicit formula

$$\psi(x) = x - \sum_\rho \frac{x^\rho}{\rho} + O(\log x)$$

where the sum runs over nontrivial zeros. The zero sum contribution at height T is controlled by $\sum_{|\gamma| \le T} 1/|\rho|$, which grows as $\sim (\log T)^2/(4\pi)$. It diverges. This is the fundamental difficulty of prime number theory: the zeros of ζ exert collective influence that grows without bound.

The Goldbach representation $r(n) = \sum_{p+q=n} 1$ has its own explicit formula, derived from the circle method:

$$r(n) = \mathfrak{S}(n) \cdot \frac{n}{\log^2 n} + E(n), \quad E(n) \sim -2\,\mathrm{Re} \sum_\rho \frac{n^\rho}{\rho(\rho-1)}$$

The extra factor of $1/(\rho - 1)$ in the denominator comes from the convolution $P \star P$. For zeros on the critical line ($|\rho| \sim |\gamma|$ for large $|\gamma|$), this adds one power of $1/|\rho|$ to each term. The controlling sum is now $\sum 1/|\rho|^2 = D_\infty \approx 0.046$. It converges.

One power of $1/|\rho|$. That is the entire difference between a divergent problem and a convergent one, between 284 years without a proof and a short conditional comparison once the explicit formula is in place.

1.3 Plan

Section 2 derives the damping mechanism from the convolution integral and computes D_∞ numerically. Section 3 gives a five-line **conditional sketch** for Goldbach (comparing explicit-formula error to the Hardy–Littlewood main term in the asymptotic regime). Section 4 establishes the k-fold hierarchy, recovers Vinogradov's ternary theorem, and extends the principle to Waring's problem. Section 5 identifies the twin prime obstruction and states the additive–correlative duality as a structural principle. Section 6 discusses the threshold interpretation and the unconditional frontier.

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2. The Convolution Damping Mechanism

2.1 Where the Extra $1/\rho$ Comes From

Consider two arithmetic functions f and g with explicit formulas of the form

$$f(x) = \mathrm{main}_f(x) + \sum_\rho a_\rho \cdot x^\rho, \quad g(x) = \mathrm{main}_g(x) + \sum_\rho b_\rho \cdot x^\rho.$$

Their Cauchy convolution $(f \star g)(n) = \sum_{j+k=n} f(j)g(k)$ integrates the product of oscillatory terms over the independent variable j:

$$\int_0^n x^{\rho_1}(n-x)^{\rho_2}\, dx = n^{\rho_1+\rho_2+1} \cdot B(\rho_1+1, \rho_2+1)$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ is the Beta function. For ρ_1, ρ_2 on the critical line with $|\rho_j| \gg 1$, Stirling's approximation gives $B(\rho_1+1, \rho_2+1) \sim 1/\rho_2$. The integration over the independent variable j produces a damping factor of $1/\rho_2$ — one power of decay per independent summand.

For the Goldbach convolution $r(n) = (P \star P)(n)$ with P the prime indicator, the error inherits two zero sums (one per copy of P), and the Beta function mechanism converts the product into a single sum with $1/(\rho(\rho-1))$ per zero. The result:

$$\sum_\rho \frac{1}{|\rho(\rho-1)|} \le \sum_\rho \frac{1}{|\rho|^2} = D_\infty < \infty.$$

For a single prime sum (no convolution), there is no independent variable to integrate over, and each zero contributes $1/|\rho|$. The sum $\sum 1/|\rho|$ diverges.

2.2 The Guitar Analogy

A single vibrating string has harmonics at frequencies γ_k with amplitudes $a_k \sim 1/\gamma_k$. The total displacement $\sum |a_k| \sim \sum 1/\gamma_k$ diverges — the pointwise amplitude can be arbitrarily large. But the energy $\sum |a_k|^2 \sim \sum 1/\gamma_k^2$ converges: a single string has finite energy.

When two strings are played together (convolution = independent summation), the joint amplitude involves products $a_j \cdot a_k$, and the total is controlled by $(\sum |a_k|^2)^{1/2}$ rather than $\sum |a_k|$. This is why

Goldbach (two primes summing) has a convergent zero sum, while prime counting (one prime) has a divergent one.

The analogy is not decorative — it captures the mechanism exactly. The two primes p and $q = n - p$ in a Goldbach pair are the two independent strings. The integration over p (or equivalently over the free variable in the convolution) produces the Beta function, which is the Fourier-analytic manifestation of destructive interference between independent oscillators.

When one string is read twice at a fixed offset — as in the twin prime count $\sum_{p \le x} \mathbf{1}_{p+2 \in \mathbb{P}}$ — there is no second independent variable. The string is correlated with itself. No interference, no damping. The zero sum retains $\sum 1/|\rho|$, and the problem is as hard as prime counting.

2.3 Computing D_∞

The Hadamard product identity gives the exact value. We verify it numerically by direct summation over nontrivial zeros.

Under RH, $\rho_k = 1/2 + i\gamma_k$ and $|\rho_k|^2 = 1/4 + \gamma_k^2$. Using the first 500 zeros:

Zeros used	Partial sum	Percentage of D_∞
10	0.02707	58.6%
50	0.03708	80.3%
100	0.03997	86.5%
200	0.04207	91.1%
500	0.04389	95.0%

Individual terms decay as $2/\gamma_k^2$: the product $\gamma_k^2 \cdot (2/|\rho_k|^2) \to 2.0000$ within four decimal places for $k \ge 50$, consistent with the zero-density asymptotics $n(t) \sim \log(t/2\pi e)/(2\pi)$.

The tail estimate via Euler–Maclaurin summation against the smooth density: at $\gamma_{500} \approx 811.2$, the tail is $\approx 2.300 \times 10^{-3}$, against the exact missing amount 2.298×10^{-3} — an overestimate of 0.09%. Corrected total: $D_\infty^{\mathrm{num}} = 0.04619$, agreeing with the analytic value to relative error 4.6×10^{-5}.

The number is real, it is small, and it governs everything that follows.

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3. The Simplest Conditional Proof Sketch for Goldbach

Under the Riemann Hypothesis, the cleanest way to package the comparison is the following five-line **sketch**. Each step must be read with its analytic scope: Steps 3–5 compare an explicit-formula error bound to the **Hardy–Littlewood main term** $S(n) \sim \mathfrak{S}(n)\, n/\log^2 n$, which is **asymptotic** as $n \to \infty$, not a certified pointwise lower bound at fixed small n.

Step 1 (Explicit formula). The Goldbach error satisfies $E(n) = -2\,\mathrm{Re}\sum_{\gamma>0} n^\rho/(\rho(\rho-1)) + O(1)$.

Step 2 (Triangle inequality). $|E(n)| \le 2\sqrt{n} \cdot \sum_\gamma 1/|\rho(\rho-1)| \le 2D_\infty\sqrt{n} + O(1) \approx 0.092\sqrt{n}$.

Step 3 (Main term, asymptotic regime). For even n, $\mathfrak{S}(n) \ge 2C_2 > 1.32$, and the Hardy–Littlewood prediction gives $S(n) \sim \mathfrak{S}(n)\, n/\log^2 n$. For **large** n this exceeds any fixed multiple of $\sqrt{n}$.

Step 4 (Comparison). There exists N_0 such that $0.092\sqrt{n} < 1.32 \cdot n/\log^2 n$ for all $n \geq N_0$ (elementary calculus). Hence $|E(n)| < S(n)$ for all even $n \geq N_0$ along this pipeline, **provided** the explicit formula for $E(n)$ and the main-term formula for $S(n)$ are used in the ranges where they are valid as stated.

Step 5 (Conclusion in the asymptotic window). For even $n \geq N_0$, $|E(n)| < S(n)$ gives $r(n) = S(n) + E(n) > 0$ under the same analytic hypotheses. The initial segment $n < N_0$ is finite and is handled by sharper bounds or by numerical verification (e.g. Oliveira e Silva et al., 2014, to very large n). $\square$

No energy decomposition. No pair correlation. No density hypothesis. The structural input is the explicit formula for $r(n)$, the convergence of $\sum 1/|\rho|^2$, and the smallness of $D_\infty \approx 0.046$.

Remark (why not $n = 4$ in Step 4). The inequality $S(n) \geq 1.32\, n/\log^2 n$ is a convenient **asymptotic** normalization, not a theorem that $r(n)$ exceeds that expression at $n = 4$. At $n = 4$ one has $r(4) = 1$ by inspection, while $1.32 \cdot 4/\log^2 4 \approx 2.75$ is only the shape of the main term at infinity. A claim of "all $n \geq 4$ from one inequality" without a finite check would confuse prediction with uniform lower bounds.

The contrast with Hardy and Littlewood (1923) remains: their GRH route yields $|E(n)| = O(n^{1/2+\varepsilon})$, which is not sharp enough to pin down a **small** constant multiple of $\sqrt{n}$ in the Goldbach error. The point of D_∞ is that the convergent zero sum supplies an explicit **coefficient** in an $O(\sqrt{n})$ bound once the explicit formula is in force—not that the asymptotic main term can be applied at $n = 4$.

Three contrasts with Hardy–Littlewood (1923) **along this conditional pipeline**:

Dimension	Hardy–Littlewood	This sketch
Hypothesis	GRH (all Dirichlet L-functions)	RH (ζ only), where applicable
Scope	Sufficiently large n	Asymptotic window $n \geq N_0$; finite n by sharper bounds or computation
Mechanism	Minor arc bound	Triangle inequality on absolutely convergent zero sum

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4. The k-fold Damping Hierarchy

4.1 The Hierarchy Table

For a k-fold additive representation $R_k(n) = \#\{(p_1, \dots, p_k) : p_1 + \cdots + p_k = n\}$, each additional independent prime summand supplies one extra factor of $1/|\rho|$ in the zero-sum tail (heuristically, one more Beta-function damping in the convolution chain). The controlling series is therefore $\sum 1/|\rho|^k$ in the notation of the table below. The full picture:

k	Sum	Value	Convergent?	Problem
1	$\sum 1/\lvert\rho\rvert$	$\sim \log^2 T$	No	Prime counting
2	$\sum 1/\lvert\rho\rvert^2$	$D_\infty \approx 0.046$	**Yes**	**Goldbach**
3	$\sum 1/\lvert\rho\rvert^3$	$D_\infty^{(3)} \approx 0.0012$	Yes (fast)	Ternary Goldbach
≥ 4	$\sum 1/\lvert\rho\rvert^k$	Tiny	Yes (trivial)	Waring-type

Binary Goldbach ($k = 2$) is the threshold: the first convergent case, hence the hardest tractable additive problem.

4.2 Recovering Vinogradov

The classical explanation of Vinogradov's 1937 theorem is that the circle method works for three primes because the major arc contribution dominates the minor arc when there are three summands. The convergence hierarchy gives a sharper explanation.

Under RH, the ternary error satisfies $|E_3(n)| \leq 2D_\infty^{(3)} \cdot n \approx 0.0024n$, while $S_3(n) \sim \mathfrak{S}_3(n) \cdot n^2/(2\log^3 n)$. The ratio $|E_3|/S_3 \to 0$ with enormous margin — the triple convolution pushes $D_\infty^{(3)}$ so deep into the convergent regime that even the crude VK zero-free region suffices.

Specifically: the Vinogradov–Korobov region gives $\beta < 1 - c/(\log T)^{2/3}$. For the ternary error:

$$|E_3(n)| \leq C \cdot n^2 \cdot \exp(-c(\log n)^{1/3})$$

while $S_3(n) \sim n^2/\log^3 n$. The ratio decays super-polynomially, so $|E_3| < S_3$ for all sufficiently large n — unconditionally. For binary Goldbach, $D_\infty \approx 0.046$ is convergent but *barely so*: the VK zero-free region approaches $\beta = 1$ as $1/(\log T)^{2/3}$, which is too slow to overcome the $\log^2 n$ growth of $1/S(n)$.

The difference between "easy" and "hard" is not qualitative — it is the distance from the convergence threshold. Ternary Goldbach sits at $D_\infty^{(3)} \approx 0.0012$, a factor of 38 below D_∞. That factor is the margin that VK needs and gets.

4.3 The Universal Additive Principle

The same mechanism governs all additive representation problems, not just primes. For k-th powers with s summands, the representation count $R_{k,s}(n) = \#\{(x_1, \dots, x_s) : x_1^k + \cdots + x_s^k = n\}$ has an explicit formula with zero sum controlled by $\sum 1/|\rho|^{2s}$, converging for $s \geq 2$.

Informal principle (Universal convergence — expository). *Additive counting problems $R(n) = \#\{x_1 + \cdots + x_s = n : x_i \in A\}$ with well-behaved sets A are often modeled by explicit formulas whose zero-sum tails gain decay as s increases. As a* ***heuristic organization*** *(not a single proved meta-theorem):*

1. *For $s \geq 2$, the tail is plausibly controlled by convergent zeta-zero sums of the shape $\sum 1/|\rho|^{2s}$ in parallel Waring-type setups.*
2. *A problem-dependent "critical exponent" $c(A)$ (density of A in the additive sense) heuristically governs how large s must be for unconditional minor-arc technology.*

3. *Binary Goldbach ($s = 2$ with $A = \mathbb{P}$) sits at the first convergent scale for primes and is the hardest additive prime problem in this narrative.*

The classical results of Vinogradov (1937), Hua (1938), and Wooley (2012) illustrate increasing summands and improved mean-value control; they are standard anchors for this picture, not corollaries of a literal universal theorem stated here:

Problem	Set A	$c(A)$	Threshold s	Status
Binary Goldbach	Primes	1	$s = 2$ (threshold)	Conditional on RH
Ternary Goldbach	Primes	1	$s = 3 > 2$	Theorem (Vinogradov, 1937; Helfgott, 2013, all odd n)
Waring (k-th powers)	$\{n^k\}$	k	$s > 2k$ (classical target)	Deep results; Wooley (2012) is one landmark in the Vinogradov mean-value program
Smooth numbers	$\{n : P^+(n) \leq y\}$	< 1	$s = 2$ (easy)	Unconditional

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5. The Twin Prime Obstruction

5.1 Convolution vs. Correlation

At first glance, twin primes resemble Goldbach — both involve pairs of primes. But the algebraic structure is fundamentally different, and the difference determines everything.

Goldbach is a *convolution*: $r(n) = \sum_{p+q=n} 1$. The two primes range independently over all pairs that sum to n. In the generating function framework, $G(z) = P(z)^2$ — a product of independent copies.

Twin primes are a *correlation*: $\pi_2(x) = \sum_{p \leq x} \mathbf{1}_{p+2 \in \mathbb{P}}$. The two primes p and $p + 2$ are locked — the second is determined by the first. The explicit formula involves $\sum_\rho x^\rho \cdot e^{2i\gamma \log x}/\rho$, where the phase $e^{2i\gamma}$ comes from the fixed shift by 2. There is no second independent variable, no integral to produce a Beta function, no damping.

5.2 The Damping Mechanism

In the convolution $\int_0^n f(x)f(n-x)\,dx$, the integration over the independent variable x produces $B(\rho_1 + 1, \rho_2 + 1) \sim 1/\rho_2$. In the correlation $\sum_n f(n)f(n+h)$, the summation variable n is shared. No independent integral, no Beta function, no damping.

Structural claim (convolution vs. correlation). *For additive convolutions, each independent summand contributes an extra $1/|\rho|$-scale damping in the explicit-formula tail. For correlations with a fixed shift h, the diagonal nature of the sum blocks the same Beta-function mechanism.*

Consequence (heuristic). *Twin-prime counting is a correlation problem; standard formulations keep a prime-counting-type (divergent) zero-sum scale. This is* ***not*** *a proof that the twin-prime conjecture is as difficult as RH or primes in short intervals—only that it does not inherit the binary Goldbach convolution damping.*

5.3 The Additive–Correlative Duality

Additive number theory splits into two regimes:

	Additive (convergent)	Correlative (divergent)
Structure	Independent variables summing to n	Locked variables at fixed distance
Generating function	$F(z)^k$ (product)	$F(z) \cdot \overline{F(\bar{z})}$ (autocorrelation)
Zero sum	$\sum 1/\lvert\rho\rvert^k$	$\sum 1/\lvert\rho\rvert$
Convergent?	Yes for $k \geq 2$	No
Examples	Goldbach, Vinogradov, Waring	Twin primes, k-tuples, prime gaps
Analogy	Two strings → interference	One string twice → same note
Relation to RH	RH suffices for some routes (e.g. §3 sketch)	RH-scale analytic input common; **not** known to be equivalent to RH

This duality explains *why* some problems in additive number theory are tractable and others are not. It is not about the size of the numbers or the density of the primes. It is about whether the algebraic structure introduces independent integration (convolution) or preserves the locked structure of the primes (correlation).

The explanatory power extends beyond individual problems to the structure of the field. Many flagship additive results — ternary Goldbach (Vinogradov; completed by Helfgott), major advances in Waring's problem, representations by smooth numbers — align with the "more summands / more damping' ' picture. Problems that remain fully open in their classical forms — notably binary Goldbach and twin primes — occupy the threshold or correlative columns of the table. (Prime gaps: bounded gaps are now a theorem, while infinitude of twin primes remains open.) The classification is a narrative device, not a substitute for theorems in each problem family.

6. Discussion

6.1 The Threshold Interpretation

Binary Goldbach is the *threshold* of additive number theory in the sense of this note: it is the first prime case where the convolution tail is modeled by a **convergent** $\sum 1/|\rho|^2$-type series. Below the threshold ($k = 1$: prime counting, twin primes as correlation), the natural zero-sum models grow like $\sum 1/|\rho|$. Above ($k \geq 3$: ternary Goldbach, many Waring setups), the tail decays faster. At $k = 2$, the sum converges — barely. The margin $D_\infty \approx 0.046$ motivates why RH-grade control is plausibly relevant, while Vinogradov-type zero-free regions still leave binary Goldbach conditional.

This perspective **organizes** the classical hierarchy. Vinogradov's ternary theorem (1937), together with later refinements culminating in Helfgott's complete proof for all odd integers, is consistent with the $k = 3$ row sitting deep in the convergent regime in this heuristic picture. The difficulty of binary Goldbach is not merely a failure of technique — it is structural: $k = 2$ is the first convergent scale for primes. Twin primes remain open not because the community neglected them, but because the correlation setup lacks the convolution damping that produces a $\sum 1/|\rho|^2$ tail for Goldbach.

6.2 The Unconditional Frontier

The convergent regime begins at $k = 2$, but any unconditional proof requires more than convergence — it requires quantitative control of the convergent tail relative to the main term growth rate. The five-line sketch (§3) uses RH. Without RH, the analytic behavior of the zero sum is subtler (conditional vs. absolute convergence, truncation errors), and converting a summatory formula to a pointwise Goldbach statement needs additional input beyond this expository note.

Three routes are discussed in the companion working paper (*The Goldbach Conjecture as a Latent Positivity Theorem*). First: extend rigorous zero-height verification (Platt and Trudgian, 2021, verify RH for $0 < \gamma \leq 3 \times 10^{12}$, i.e. on the order of 10^{13} zeros) and propagate the resulting analytic control. Second: prove a density-type bound strong enough to make the relevant off-line zero sums manageable in that program (ibid., §14). Third: bound the Perron truncation constant C_P for $(-\zeta'/\zeta)^2$ sharply enough for the five-layer strategy (ibid., §29).

Quantitative envelopes, costs, and the exact logical dependencies are **not** reproved here; this note isolates the D_∞ hierarchy. The companion paper is the appropriate place for those claims.

6.3 What D_∞ Does Not Explain

The convergence hierarchy governs the *structural tractability* of additive problems — whether the zero sum converges at all. It does not determine the *quantitative* difficulty within the convergent regime. Two problems at $k = 2$ may have very different effective constants, depending on the singular series, the main term growth rate, and the specific arithmetic of the set A. Goldbach for primes ($c(\mathbb{P}) = 1$) is harder than the corresponding problem for smooth numbers ($c(A) < 1$), even though both sit at $k = 2$.

Nor does the hierarchy address problems that are neither additive nor correlative in the strict sense — for instance, the distribution of primes in short intervals, or the Siegel zero problem. These require different structural invariants.

6.4 Conclusion

The total zero energy $D_\infty = 2 + \gamma - \log(4\pi) \approx 0.046$ is the Goldbach constant in this framework. It organizes the simplest conditional comparison (§3), the k-fold difficulty hierarchy (§4), the classical ternary-Goldbach story (§4.2), the expository "universal'' principle (§4.3), and the twin-prime obstruction narrative (§5). The constant is exact from the Hadamard product and matches direct zero-sum numerics to several decimal places.

The question that 284 years of number theory has been circling — why is binary Goldbach so much harder than ternary? — has a one-line answer: because $\sum 1/|\rho|^2$ converges but $\sum 1/|\rho|$ does not, and $k = 2$ is where the transition happens.

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