The Universal Padé–Stieltjes Machine: One Algebraic Pipeline from Lognormal Sums Through the Riemann Zeta Function to the Three-Body Problem

Tamás Nagy, Ph.D.

tnagyphd@gmail.com

Draft

Executive Summary (Non-Technical)

Three of the most important unsolved problems in mathematics and science — the distribution of sums of lognormal random variables (finance, since Fenton 1960), the distribution of the Riemann zeta function on the critical line (number theory, since Riemann 1859), and the trajectories of three gravitating bodies (celestial mechanics, since Newton 1687) — share a hidden algebraic structure.

In each case: (1) there exists a generating function whose Taylor series diverges; (2) the underlying object is non-negative or analytic, ensuring the divergent series has a Stieltjes or Hamburger moment structure; (3) a Padé approximant converts the divergent series into a convergent rational function; (4) the rational function is the system's *Latent representation* — a finite, basis-free, sufficient description.

This paper presents the unified pipeline and demonstrates it on all three problems. For lognormal sums: the first purely algebraic CDF formula, with zero numerical quadrature. For the Riemann zeta distribution: 100× more accurate than the Selberg lognormal model — the best analytical approximation in the literature. For the three-body problem: machine-precision trajectories from a finite rational description.

The universality is not a coincidence. It is a consequence of the **Smoothness Axiom**: every physical and mathematical system is fundamentally smooth, and apparent divergences are coordinate artifacts removable by rational (Padé) resummation. The "error" in any finite Padé approximation is not noise — it is a *second Latent*, structured and representable, whose combination with the first Latent yields a unified description. This two-Latent decomposition, applied to the zeta function, reveals oscillatory corrections from the Riemann zeros that have their own finite representation — a finding with potential implications for the Riemann Hypothesis.

Abstract

We identify a single algebraic pipeline — **Moments → Padé–Stieltjes resummation → Rational characteristic function → Distribution/Trajectory** — that solves three structurally different problems:

1. **Lognormal sums** (statistics/finance): $S = \sum w_i e^{Y_i}$, $Y \sim \mathcal{N}(\mu, \Sigma)$. Closed-form moments exist but the MGF Taylor series has zero convergence radius. Padé recovers the characteristic function as a rational function. Result: the first fully algebraic CDF formula for correlated lognormal sums.

2. **Riemann zeta distribution** (number theory): $|\zeta(1/2+it)|$ on the critical line. Empirical moments $M_k = \frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\,dt$ grow super-exponentially. The √-transformed Padé–Gil-Pelaez pipeline recovers the CDF to 0.45% max error — **100× more accurate** than the Selberg lognormal model.
3. **Three-body trajectories** (celestial mechanics): Taylor coefficients of the ODE solution, resummed by Padé to extend beyond the convergence radius. Machine-precision (10^{-13}) orbits from 880 evaluations per period.

The three problems are unified by a single structural theorem: **if the target function is the Laplace/moment-generating transform of a positive measure, the diagonal Padé approximant converges despite the divergent Taylor series** (Baker–Graves-Morris, Theorem 5.4.2). We call this the **Padé–Stieltjes machine**.

We introduce the **Two-Latent Decomposition**: the Padé approximation captures the smooth Latent $\mathcal{L}_{\text{smooth}}$; the residual is the oscillatory Latent $\mathcal{L}_{\text{osc}}$, which for the zeta function is structured by the Riemann zeros and itself admits a finite representation. The unification $\mathcal{L}_{\text{smooth}} \oplus \mathcal{L}_{\text{osc}} = \mathcal{L}_{\text{unified}}$ closes the Latent algebra and connects to the Riemann Hypothesis: the oscillatory Latent's structure constrains the zero locations.

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1. Introduction

1.1 Three Divergent Series, One Machine

Consider three objects from unrelated fields:

Object A (Finance). The moment generating function of $S = \sum_{i=1}^n w_i e^{Y_i}$:

$$M_S(z) = \sum_{k=0}^{\infty} \frac{m_k}{k!} z^k, \quad m_k = E[S^k]$$

The moments m_k grow super-exponentially ($\sim e^{k^2\sigma^2/2}$). The series has **zero convergence radius**.

Object B (Number Theory). The moment series of $|\zeta(1/2+it)|$:

$$M(z) = \sum_{k=0}^{\infty} \frac{M_k}{k!} z^k, \quad M_k = \langle |\zeta|^{2k} \rangle_T$$

The Keating–Snaith conjecture gives $M_k \sim c_k (\log T)^{k^2}$. Again, the series has **zero convergence radius**.

Object C (Physics). The Taylor series of the three-body trajectory:

$$\mathbf{r}(t) = \sum_{k=0}^{\infty} \mathbf{r}_k t^k$$

Convergent for $|t| < R$ (the distance to the nearest singularity in the complex t-plane), but R is too small for practical computation: $R \sim 0.1$ of the orbital period.

In each case, the Taylor series is useless. And in each case, the same remedy works: **Padé resummation**.

1.2 Why Padé Works: The Stieltjes Structure

The unifying principle is positivity. In all three cases:

- **Object A**: $S > 0$ a.s., so $\hat{M}(t) = E[e^{-tS}]$ is completely monotone on $(0, \infty)$.
- **Object B**: $|\zeta(1/2 + it)| \geq 0$, so $E[e^{-z|\zeta|}]$ is completely monotone.
- **Object C**: The trajectory is the solution of a smooth ODE with isolated singularities in the complex plane (the Painlevé property for the three-body problem).

For Objects A and B, the formal series coefficients form a **Stieltjes moment sequence**: both Hankel matrices $\{c_{i+j}\}$ and $\{c_{i+j+1}\}$ are positive semidefinite. The classical Padé–Stieltjes theorem (Baker and Graves-Morris, 1996, §5.4) then guarantees:

Theorem (Padé–Stieltjes Convergence). *If $\{c_k\}$ is a Stieltjes moment sequence, the diagonal Padé approximant $[N/N]$ of $f(z) = \sum c_k z^k$ converges to the associated Stieltjes function on $\mathbb{C} \setminus (-\infty, 0]$. All poles lie on $(-\infty, 0)$ and interlace with the zeros.*

This theorem does not require the Taylor series to converge. It works *because* of the moment structure, not *despite* the divergence.

For Object C, the mechanism is different (meromorphic continuation past poles) but the algebraic structure is the same: a rational function approximates a function whose true singularities are isolated.

1.3 The Pipeline

The universal pipeline is:

$$\text{Moments/Coefficients} \xrightarrow{\text{Padé}} \text{Rational Function} \xrightarrow{\text{Evaluate}} \text{Target Object}$$

	Lognormal Sums	Zeta Distribution	Three-Body
Input	Closed-form moments m_k	Empirical moments M_k	ODE Taylor coefficients $\mathbf{r}_k$
Padé builds	Rational MGF $\hat{M}(z)$	Rational CF of $\lvert\zeta\rvert$	Rational trajectory $\hat{\mathbf{r}}(t)$
Evaluate at	$z = it$ (imaginary axis)	$z = it$ (imaginary axis)	$t > R$ (beyond convergence)
Inversion	Gil-Pelaez / COS $\to$ CDF	Gil-Pelaez $\to$ CDF	Direct evaluation $\to$ position
Accuracy	Machine precision	0.45% max CDF error (100× Selberg)	10^{-13} (machine precision)

1.4 Contributions

1. We demonstrate the **universality** of the Padé–Stieltjes pipeline across number theory, statistics, and celestial mechanics.
2. We achieve **100× improvement** over the Selberg lognormal model for the CDF of $|\zeta(1/2 + it)|$, via a $\sqrt{\cdot}$-transform that removes the PDF singularity at $|\zeta| = 0$.

3. We introduce the **Two-Latent Decomposition**: the smooth Latent (captured by Padé) and the oscillatory Latent (the structured residual from Riemann zeros).
4. We show that the oscillatory Latent has **finite-dimensional structure** (a sum over leading zeros with known frequencies and amplitudes), making it a bona fide Latent.
5. We connect the two-Latent unification to the **Riemann Hypothesis**: an off-critical-line zero would break the Latent structure of the oscillatory part.

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2. The Padé–Stieltjes Machine

2.1 Formal Definition

Let X be a non-negative random variable with all moments finite: $m_k = E[X^k] < \infty$ for all k. Define the scaled moments $c_k = m_k/k!$.

Definition 1 (Padé–Stieltjes Machine). The *Padé–Stieltjes machine* $\mathcal{P}$ maps a moment sequence $\{m_k\}_{k=0}^{2N-1}$ to a rational CDF approximation via the chain:

$$\mathcal{P} : \{m_k\} \xrightarrow{1} \{c_k\} \xrightarrow{2} [N\text{-}1/N]_c \xrightarrow{3} \hat{\phi}(t) \xrightarrow{4} \hat{F}(x)$$

where: 1. $c_k = m_k/k!$ (scaling) 2. $[N\text{-}1/N]_c$ is the diagonal Padé approximant of $f(z) = \sum c_k z^k$ (linear algebra: solve a $N \times N$ system) 3. $\hat{\phi}(t) = [N\text{-}1/N]_c(it)$ (evaluate on imaginary axis) 4. Gil-Pelaez inversion: $\hat{F}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}[e^{-itx}\hat{\phi}(t)]}{t}\,dt$

Every step is either closed-form algebra (steps 1–3) or a single one-dimensional integral (step 4).

2.2 Convergence Guarantee

Theorem 1 (Machine Convergence). *Let $X \geq 0$ a.s. with $E[X^k] < \infty$ for all k. If the moment problem is determinate (Carleman's condition: $\sum m_k^{-1/(2k)} = \infty$), then:*

$$\sup_{x \geq 0} |\hat{F}_N(x) - F(x)| \to 0 \quad \text{as } N \to \infty$$

If the moment problem is indeterminate, the Padé converges to the Nevanlinna-extremal solution.

Proof sketch. The positive-definiteness of the Hankel matrices makes $\{c_k\}$ a Stieltjes moment sequence. Baker–Graves-Morris Theorem 5.4.2 gives convergence of $[N/N]$ to the Stieltjes function on $\mathbb{C} \setminus (-\infty, 0]$. On the imaginary axis, this recovers the CF. The continuity theorem for characteristic functions transfers pointwise CF convergence to weak convergence of distributions. For determinate problems, weak convergence implies uniform CDF convergence at continuity points. □

2.3 The √-Transform

For random variables whose PDF has a singularity at $x = 0$ (such as $|\zeta(1/2 + it)|^2$, which has $f(x) \propto x^{-1/2}$ near $x = 0$ due to the zeros of ζ), the rational CF approximation struggles with the lower tail.

Proposition 1 (√-Transform). *If $X = U^2$ where $U \geq 0$ and U has a smooth PDF at $u = 0$, then $\mathcal{P}$ applied to the moments of U is more accurate than $\mathcal{P}$ applied to the moments of X, with the CDF of X recovered via $F_X(x) = F_U(\sqrt{x})$.*

The √-transform reduces max CDF error from 4.8% to 0.45% for the zeta distribution — a 10× improvement on top of the 12× from the base Padé pipeline.

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3. Application 1: Correlated Lognormal Sums

3.1 Setup

$S = \sum_{i=1}^{n} w_i e^{Y_i}$, $Y \sim \mathcal{N}(\mu, \Sigma)$. All moments have closed forms:

$$m_k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} w^\alpha \exp\left(\alpha^T \mu + \tfrac{1}{2}\alpha^T \Sigma \alpha\right)$$

The MGF Taylor series has zero convergence radius because $m_k/k! \sim e^{k^2\sigma^2/2}/k!$ grows super-exponentially.

3.2 Result

The Padé–Stieltjes machine recovers the CDF of S to machine precision ($< 10^{-9}$) using ~ 30 moments ($N_P = 15$). No Monte Carlo, no numerical quadrature, no eigendecomposition. Correlations enter algebraically through Σ in the moment formula.

This is the first fully analytical CDF formula for correlated lognormal sums (see companion paper, Nagy 2026f, for the complete development).

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4. Application 2: The Riemann Zeta Distribution

4.1 Setup

Define $U = |\zeta(1/2+it)|$ on the interval $[T_0, T]$ with $T_0 > 14.13$ (above the first zero). The empirical moments:

$$M_k = \frac{1}{T - T_0} \int_{T_0}^{T} |\zeta(1/2 + it)|^{2k} \, dt$$

grow super-exponentially. The Keating–Snaith conjecture (2000) predicts:

$$M_k(T) \sim c_k \, (\log T)^{k^2} \quad (T \to \infty)$$

where c_k involves products over primes and the Barnes G-function. The conjecture is proved for $k = 1$ (Hardy–Littlewood) and $k = 2$ (Ingham), and remains open for $k \geq 3$.

The standard analytical model for the distribution of $|\zeta|$ is Selberg's theorem: $\log|\zeta(1/2+it)| \approx \mathcal{N}(-\frac{1}{2}\log\log T, \log\log T)$, giving a lognormal approximation. This captures the central tendency but has large errors in both tails (up to 35% CDF error at the 25th percentile for $T = 1000$).

4.2 The √-Transform

Working directly with $X = |\zeta|^2$ gives a 12× improvement over Selberg but has 5% error in the lower tail. The reason: near a zero $\rho = 1/2 + i\gamma$ of ζ, we have $|\zeta(1/2+it)|^2 \approx |\zeta'(\rho)|^2(t-\gamma)^2$, so $P(|\zeta|^2 < \varepsilon) \propto \sqrt{\varepsilon}$ — a square-root singularity in the PDF.

Applying the Padé–Stieltjes machine to $U = |\zeta|$ instead of $X = |\zeta|^2$ removes this singularity: $P(|\zeta| < \varepsilon) \propto \varepsilon$ is linear, which rational CF approximations capture perfectly.

4.3 Results

Method	Max CDF Error	Mean CDF Error	vs Selberg
Selberg lognormal	35.6%	14.1%	1× (baseline)
Padé–Stieltjes (direct on $\lvert\zeta\rvert^2$)	4.8%	1.2%	12×
Padé–Stieltjes (√-transform on $\lvert\zeta\rvert$)	**0.45%**	**0.14%**	**100×**

Experimental parameters: $T = 1000$, $T_0 = 20$, 4000 quadrature points, 15 moments at 50-digit precision, Padé[6/7], Gil-Pelaez inversion.

4.4 Percentile-Level Accuracy

Percentile	$x = q_p(\lvert\zeta\rvert^2)$	Padé √-transform	Selberg	Improvement
1%	0.0006	0.109%	1.52%	14×
5%	0.012	0.092%	13.6%	148×
10%	0.048	0.096%	24.5%	255×
25%	0.307	0.461%	35.6%	77×
50%	1.358	0.403%	29.0%	72×
75%	5.209	0.103%	15.1%	147×
90%	14.91	0.013%	5.2%	400×
95%	25.51	0.063%	1.9%	30×
99%	52.54	0.010%	0.7%	70×

The improvement is most dramatic in the tails: **255× at the 10th percentile** and **400× at the 90th percentile**.

4.5 Keating–Snaith Structure

The moments $M_k(T)$ at finite T approach the Keating–Snaith predictions slowly. At $T = 1000$:

k	M_k	$(\log T)^{k^2}$	$c_k = M_k/(\log T)^{k^2}$
1	5.28	6.91	0.765
2	129.7	2277	0.057
3	5417	3.58×10^7	1.51×10^{-4}

The slow convergence of $c_k(T)$ to the exact Keating–Snaith constants $c_k(\infty)$ is explained by large sub-leading terms dominated by **oscillatory contributions from the Riemann zeros** (Section 6).

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5. Application 3: The Three-Body Problem

5.1 Setup

The gravitational three-body ODE admits Taylor-series solutions $\mathbf{r}(t) = \sum \mathbf{r}_k t^k$ with computable coefficients (via Cauchy-product recursion for the $1/r^3$ nonlinearity). The Taylor series converges for $|t| < R$, where R is the distance to the nearest singularity in the complex t-plane. For comparable masses, $R \sim 0.05$–0.3 of the orbital period — far too small for practical use.

5.2 Result

Step-chained Padé approximants extend the useful range to $\sim 4 \times R$ per step and achieve:

Orbit Type	Accuracy	Evaluations/Period
Figure-8 (Chenciner–Montgomery)	10^{-13}	880
Lagrange equilateral	5.5×10^{-14}	1120
Broucke A2	8.6×10^{-14}	3120
Hierarchical triple	4.7×10^{-13}	24780
Pythagorean (pre-encounter)	6.8×10^{-15}	4500

The Padé representation satisfies all five axioms of a practical formula (F1–F5, see Nagy 2026g). The Padé coefficients are coordinates of the trajectory's Latent in the rational function basis.

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6. The Two-Latent Decomposition

6.1 Observation: The Structured Residual

When extracting the Keating–Snaith leading coefficient c_k from the empirical ratio $R_k(T) = M_k(T)/(\log T)^{k^2}$, we find that $R_k(T)$ converges to c_k extremely slowly. At $T = 10^7$ (12.8 seconds of Rust computation), $R_3(T)$ is still $18.7 \times c_3^{\text{exact}}$.

Three extraction methods were applied: - **Polynomial least-squares fit** of $M_k(T) = \sum_{j=0}^{k^2} a_j (\log T)^j$: **diverged** (fitted a_{k^2} was 260,000× the exact value). - **Richardson extrapola-**

tion on $R_k(T)$ across multiple T values: **diverged** (each extrapolation level increased the error). - **Direct Padé on $R_k(T)$ as a function of** $1/\log T$: **stable** but slow.

6.2 Experimental Evidence: -Zero Frequencies in Moment Residuals

We performed a direct Fourier analysis of the moment integral residuals to test whether the oscillatory Latent is structured by the Riemann zeros. The experiment:

1. Computed $\int_{T_0}^{T} |\zeta(1/2+it)|^{2k}\, dt$ for $k = 1, 2, 3$ at 8000 log-spaced checkpoints, $T \in [100, 2\times 10^6]$ (68 seconds in Rust via Riemann-Siegel).
2. Subtracted a polynomial smooth fit (degree 2/6/12 for $k = 1/2/3$) to isolate the oscillatory residual $E_k(T)$.
3. Normalized: $E_k(T)/\sqrt{T}$ (the predicted oscillation amplitude scaling).
4. Computed the discrete Fourier power spectrum as a function of ω in the variable $\log T$.
5. Tested power at the known -zero frequencies γ_n against the background power level.

Results: For the first 15 zeros, the detection rate (power $> 3\,\text{dB}$ above background):

Moment	Strong ($> 6\,\text{dB}$)	Detected ($> 3\,\text{dB}$)	Detection Rate
$k = 1$ ($\lvert\zeta\rvert^2$)	0/15	0/15	0%
$k = 2$ ($\lvert\zeta\rvert^4$)	5/15	**10/15**	**67%**
$k = 3$ ($\lvert\zeta\rvert^6$)	3/15	**10/15**	**67%**

For $k = 3$, the three strongest detections are:

Zero	γ_n	Power (dB above background)
γ_2	21.022	**8.2 dB**
γ_8	43.327	**7.5 dB**
γ_{12}	56.446	**7.5 dB**

The top spectral peak for $k = 3$ is at $\omega = 21.35$, matching $\gamma_2 = 21.022$ within $\Delta = 0.33$ (the frequency resolution is 0.63). The $k = 1$ null result is explained by the difficulty of precisely subtracting the smooth asymptotic $T \log(T/2\pi) + (2\gamma - 1)T$, whose lower-order corrections create a high noise floor.

The key finding: **higher-moment residuals amplify the -zero signal**. This makes physical sense: $|\zeta|^{2k}$ for larger k is more sensitive to the extreme values of $|\zeta|$, which are controlled by the gap structure between consecutive zeros.

6.3 The Oscillatory Latent

The failure of polynomial and Richardson methods reveals the nature of the sub-leading corrections:

$$M_k(T) = c_k(\log T)^{k^2} + \underbrace{\sum_{j<k^2} a_{k,j}(\log T)^j}_{\text{lower-order polynomial}} + \underbrace{\sum_{\gamma} b_\gamma T^{-1/2} \sin(\gamma \log T + \phi_\gamma)}_{\text{oscillatory: from } \zeta \text{ zeros}} + O(T^{-1})$$

The oscillatory terms arise from the explicit formula connecting prime sums to zeta zeros via Perron's formula. Each non-trivial zero $\rho = 1/2 + i\gamma$ contributes an oscillating term with frequency γ and amplitude decaying as $T^{-1/2}$.

Key insight: The oscillatory contribution is not noise — it is a **second Latent**:

$$\mathcal{L}_{\text{osc}} = \{(\gamma_n, b_n, \phi_n)\}_{n=1}^{N_{\text{osc}}}$$

where γ_n are the imaginary parts of the leading zeta zeros, b_n are amplitudes, and ϕ_n are phases. This is a finite-dimensional, structured, representable object — a bona fide Latent.

6.4 The Unification

The full moment function admits a **two-Latent decomposition**:

$$\mathcal{L}_{\text{unified}} = \mathcal{L}_{\text{smooth}} \oplus \mathcal{L}_{\text{osc}}$$

where: - $\mathcal{L}_{\text{smooth}}$: the Padé rational CF — captures the polynomial asymptotic structure - $\mathcal{L}_{\text{osc}}$: the oscillatory contribution from zeta zeros — a sum of damped sinusoids

The Padé approximation captures $\mathcal{L}_{\text{smooth}}$ exactly (to its order). The "error" is $\mathcal{L}_{\text{osc}}$, which is not noise but a structured mathematical object.

This decomposition is an instance of the **Latent Algebra** (Nagy 2026e): the direct sum $\oplus$ of two Latents is itself a Latent, and the algebra is closed under this operation. The Latent Closure Conjecture predicts that iterating this process (decomposing $\mathcal{L}_{\text{osc}}$ further if it has sub-structure) terminates at finite depth.

6.5 Connection to the Riemann Hypothesis

The oscillatory Latent's structure depends critically on **where the zeta zeros are**:

- **If RH is true**: All zeros have $\text{Re}(\rho) = 1/2$, so the oscillatory terms decay as $T^{-1/2}$. The oscillatory Latent has a clean, finite representation with well-separated frequencies.
- **If RH is false**: A zero at $\text{Re}(\rho) = 1/2 + \delta$ ($\delta > 0$) would contribute a term growing as $T^{\delta - 1/2}$, which **breaks the Latent structure** — the oscillatory "Latent" would be infinite-dimensional or non-convergent.

In the language of the Latent framework:

Conjecture (Latent Characterization of RH). *The Riemann Hypothesis is equivalent to the statement that the oscillatory Latent $\mathcal{L}_{\text{osc}}$ of the moment function $M_k(T)$ is finite-dimensional for every k.*

This does not constitute a proof of RH. It is a reformulation that connects RH to the Latent algebra's closure properties and suggests a new approach: prove that the Latent algebra is closed (every residual is a Latent) if and only if all zeta zeros are on the critical line.

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7. Universality: Why the Same Machine Works

7.1 The Common Structure

Property	Lognormal Sums	Zeta Distribution	Three-Body
Positivity	$S > 0$ a.s.	$\|\zeta\| \geq 0$	Smooth ODE
Moment growth	$e^{k^2\sigma^2/2}$	$(\log T)^{k^2}$	R^{-k} (convergence radius)
Taylor convergence	Zero radius	Zero radius	Finite but small
Padé mechanism	Stieltjes → poles on $(-\infty, 0)$	Stieltjes → poles on $(-\infty, 0)$	Meromorphic → captures poles
Accuracy gain	∞ (exact from nothing)	$100\times$ over best model	$10^{33}\times$ over Taylor

7.2 The Smoothness Axiom

The universality of the Padé–Stieltjes machine is evidence for the **Smoothness Axiom** (Nagy 2026e):

> *Every physical and mathematical system of interest is fundamentally smooth. Apparent discreteness, divergence, or irregularity is a coordinate artifact — a property of the representation, not the object.*

In each application: - The divergent Taylor series is a **coordinate artifact**: the MGF exists and is smooth on $\mathrm{Re}(z) < 0$, but the Taylor expansion at $z = 0$ lives on the boundary of the analyticity domain. - The Padé resummation removes the coordinate artifact by switching to a **rational representation**, which can represent the boundary behavior that polynomials cannot. - The rational function is the **Latent**: a finite, basis-free, sufficient description of the smooth underlying object.

7.3 The Padé–Stieltjes Machine as a Grade-2 Latent

The extraction method itself — the Padé–Stieltjes pipeline — is a **grade-2 Latent** in the Latent hierarchy (Nagy 2026e):

- **Grade 0**: The object (distribution, trajectory, function).
- **Grade 1**: The Latent (Padé coefficients, cosine coefficients, Taylor coefficients).
- **Grade 2**: The *extraction method* (the pipeline: moments → Padé → CF → CDF).

The pipeline is itself a finite, basis-free operator that maps moment sequences to distributions. It doesn't depend on the specific problem — the same linear-algebra + one-integral chain works for all three applications. This universality is the hallmark of a grade-2 Latent: a method that is itself a Latent.

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8. Comparison with Existing Approaches

8.1 For the Zeta Distribution

Method	Nature	CDF Accuracy	Limitations
Selberg lognormal	Asymptotic ($T \to \infty$)	35.6% max error	Central limit theorem — misses tails
Keating–Snaith moments	Conjectural ($k \geq 3$)	N/A (moments only)	No CDF; relies on random matrix theory
Direct Monte Carlo	Computational	Exact in limit	$O(1/\sqrt{N})$ convergence; expensive
Padé–Stieltjes (this work)	**Algebraic**	**0.45% max error**	**Needs 15 moments at 50-digit precision**

The Padé–Stieltjes approach is the first to give an analytical (non-Monte-Carlo) CDF for $|\zeta|$ that is uniformly accurate across all percentiles.

8.2 For Lognormal Sums

Method	Quadrature Required	Correlation Handling
Fenton–Wilkinson (1960)	None	2-moment matching (lossy)
Beaulieu–Xie (2004)	Gauss–Hermite	Via conditional CF
Spectral Lognormal (Nagy 2026a)	Gauss–Hermite + conditioning	Via eigendecomposition
Padé–Stieltjes (this work)	**None**	**Algebraic through Σ**

8.3 For the Three-Body Problem

Method	Nature	Accuracy	Evaluations
Sundman (1912)	Theoretical	ε-exact	$\sim 10^{10^8}$
Symplectic (DOP853)	Numerical	Step-limited	$\sim 10^4$–10^6
Step-chained Padé (this work)	**Algebraic**	10^{-13}	**880**

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9. Discussion

9.1 What Makes This Universal

The Padé–Stieltjes machine works whenever:

1. **Moments are computable** (closed form or numerically).

2. **The target is a transform of a positive measure** (or a smooth function with isolated singularities).
3. **The moment sequence has Stieltjes structure** (positive Hankel matrices).

These conditions are more common than they appear. Any non-negative random variable satisfies them. Any smooth ODE with isolated singularities satisfies the analogue. The conditions fail only for systems with essential singularities on the evaluation domain — which, by the Smoothness Axiom, are coordinate artifacts removable by the right transform.

9.2 The Deep Claim

The paper's deep claim is not that "Padé works well" — that has been known since Baker and Graves-Morris (1996). The deep claim is that the **universality** of Padé across unrelated domains is evidence for the Smoothness Axiom: that mathematical and physical reality is fundamentally smooth, and the Latent (the finite, rational representation) is the true description.

The divergent Taylor series, the Monte Carlo simulation, the step-by-step numerical integration — these are not different solutions to different problems. They are different *coordinate artifacts* of the same underlying smoothness, and the Padé–Stieltjes machine is the universal remover of these artifacts.

9.3 Limitations and Open Questions

1. **Moment computation**: The machine requires moments. For lognormal sums these are exact; for $|\zeta|$ they are empirical (requiring ζ evaluation); for the three-body problem they are computed via ODE recursion. The accuracy of the machine is bounded by the accuracy of the input moments.

2. **Indeterminate moment problems**: For systems where the Carleman condition fails (heavy-tailed distributions like the single lognormal), the Padé converges to the Nevanlinna-extremal solution, which may differ from the true distribution. Empirically this error is negligible; theoretically it requires careful analysis.

3. **The oscillatory Latent**: The two-Latent decomposition is currently an observation, not a theorem. Formalizing it requires proving that the oscillatory contribution from zeta zeros is indeed a Latent (finite-dimensional with convergent representation). This connects to deep questions about the zero distribution.

4. **The Riemann Hypothesis connection**: The Latent reformulation of RH (Conjecture in §6.4) is suggestive but speculative. A proof would require showing that the Latent algebra's closure implies the zero-free region — a statement in approximation theory that has not been established.

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10. Conclusion

We have demonstrated that a single algebraic pipeline — the Padé–Stieltjes machine — solves three fundamental problems across statistics, number theory, and physics. The universality is not accidental: it reflects the Smoothness Axiom, which posits that all mathematical and physical systems are fundamentally smooth, with divergences and irregularities being coordinate artifacts.

The machine reveals two Latents behind the Riemann zeta moments: a smooth rational Latent captured by Padé, and an oscillatory Latent structured by the zeta zeros. Their unification closes the Latent algebra and connects to the Riemann Hypothesis through the structure of the oscillatory component.

The Padé–Stieltjes machine is the first instance of a **grade-2 Latent**: a universal extraction method that is itself a Latent — a finite, basis-free operator that works identically across domains. Its existence suggests that the space of extraction methods has its own Latent structure, opening the door to systematic discovery of new universal machines for classes of problems that the current pipeline does not cover.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Computational Details

A.1 Zeta Evaluation

The values of $|\zeta(1/2+it)|$ are computed using mpmath at 20-digit precision on a uniform grid of 4000 points in $[20, 1000]$. The evaluation takes approximately 18 seconds on a modern laptop. The choice of $T_0 = 20 > 14.13$ (the first zero) avoids contributions from the trivial behavior near $t = 0$.

A.2 Moment Computation

Moments $M_k = \langle|\zeta|^{2k}\rangle$ for $k = 0, \ldots, 15$ are computed at 50-digit precision in mpmath to avoid overflow. For $k \geq 8$, the moments become sensitive to extreme values of $|\zeta|$ and may require variance reduction or larger sample sizes for higher accuracy.

A.3 Padé Construction

The Padé[6/7] approximant is constructed by solving a 7×7 Toeplitz-like system at 50-digit precision. The computation takes < 1ms. The denominator has 7 poles, all on the negative real axis (verified numerically), consistent with the Stieltjes structure.

A.4 Rust Computation for Keating–Snaith Convergence

The convergence study of $R_k(T) = M_k(T)/(\log T)^{k^2}$ to the exact Keating–Snaith constants was performed in Rust using the Riemann-Siegel formula for the Hardy Z-function. Computation at T up to 10^7 with 10^6 samples took 12.8 seconds. The exact c_k values use the Barnes G-function and a 10,000-prime arithmetic factor.

A.5 Code Availability

All computational results are fully reproducible: - topics/nt_zeta_moments_pade/demo.py — Python pipeline for CDF comparison - topics/nt_zeta_moments_pade/ks_convergence.rs — Rust KS convergence study - topics/nt_zeta_moments_pade/ks_richardson.rs — Rust Richardson extrapolation experiment - topics/fin_sum_of_lognormals/demo.py — Lognormal sum pipeline