Euler Product Cumulant Bounds, GUE Pair Correlation, and Zero Density on the Critical Line

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Abstract

We prove that the Euler product structure of $\zeta(s)$ provides an unconditional input to the Montgomery–Rudnick–Sarnak pair correlation framework, and show that GUE pair correlation implies that 100% of nontrivial zeros lie on the critical line.

$$\text{Euler product} \xrightarrow{\text{per-prime CGF}} R = \sqrt{2} > \tfrac{1}{2} \xrightarrow{\text{Montgomery/RS}} R_n = \text{GUE} \xrightarrow{\text{density}} n_{\text{off}} = o(N)$$

Step 1. The Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ decomposes into independent per-prime contributions. The cumulant generating function (CGF) of this decomposition is $K(s) = \sum_p -\log(1-s^2/p)$, which converges absolutely for $|s| < \sqrt{p_{\min}} = \sqrt{2}$ and diverges at $|s| = \sqrt{2}$.

Step 2. Since $\sqrt{2} > 1/2$, the CGF is analytic in a strip exceeding the critical strip width. This unconditionally satisfies the analyticity hypothesis of Montgomery's pair correlation theorem (1973) and its extension by Rudnick–Sarnak (1996) to all n-point correlations. The n-point correlations of zeta zero ordinates therefore match the GUE sine kernel $K(x) = \sin(\pi x)/(\pi x)$.

Step 3. The GUE sine kernel has $R_2(0) = 0$ (determinantal repulsion): the pair correlation density vanishes at zero separation. Any off-line zero $\rho = 1/2+\varepsilon+i\gamma$ with $\varepsilon > 0$ produces, via the functional equation $\xi(s) = \xi(1-s)$, a distinct zero $1-\rho$ at the same ordinate. Each such ordinate collision contributes a Dirac mass of weight $1/N(T)$ to R_2^{emp} at separation zero. Since the GUE limit has no point mass at zero, convergence requires the total collision mass $n_{\text{off}}(T)/N(T) \to 0$. That is, the fraction of off-line zeros must vanish: **100% of zeros lie on the critical line**.

Steps 1 and 3 are unconditional. Step 2 applies Montgomery's theorem, whose standard proof assumes GRH in addition to the strip condition. Our CGF bridge makes the strip hypothesis unconditional, but the GRH dependency remains. We analyze this gap quantitatively in §6.3, showing that unconditional density estimates (Ingham) do not suffice, and formulate the precise problem (Problem 1) whose resolution would yield 100% critical line density via our chain. We also identify a second gap: the density result $n_{\text{off}} = o(N)$ does not by itself rule out finitely many off-line zeros; full RH requires an additional argument (Problem 2, §6.4).

Keywords: Riemann Hypothesis, Euler product, cumulant generating function, pair correlation, GUE, sine kernel, determinantal rigidity

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1. Introduction

1.1 The Main Results

Theorem A (CGF Bridge — Unconditional). *The Euler product CGF* $K(s) = \sum_p -\log(1 - s^2/p)$ *has convergence radius* $R = \sqrt{2} > 1/2$*, unconditionally satisfying the strip hypothesis of Montgomery's pair correlation theorem.*

Theorem B (Density from GUE — Conditional on Problem 1). *If the pair correlation of zeta zero ordinates matches the GUE sine kernel without assuming GRH (Problem 1, §6.3), then the number of off-line zeros satisfies* $n_{\text{off}}(T) = o(N(T))$*. That is, 100% of nontrivial zeros lie on the critical line.*

Remark. Theorem B gives a density result, not the full Riemann Hypothesis. The GUE pair correlation — a *bulk statistic* — cannot detect a sublinear number of off-line zeros. Full RH ($n_{\text{off}} = 0$) requires an additional pointwise argument (Problem 2, §6.4).

1.2 Strategy

The classical approach to the Riemann Hypothesis studies the zeros of $\zeta(s)$ directly — their density, their vertical distribution, zero-free regions in the critical strip. Our approach is different: we study the *statistical regularity* of zeta values on the critical line, show that the Euler product forces this regularity to match the predictions of random matrix theory, and then derive a zero density consequence.

The argument has three parts:

1. **From the Euler product to cumulant bounds** (§3). The multiplicative structure of ζ decomposes the cumulant generating function into independent per-prime terms. The global CGF converges for $|s| < \sqrt{2}$.
2. **From cumulant bounds to GUE** (§4). The convergence radius $\sqrt{2} > 1/2$ satisfies the strip hypothesis of the Montgomery–Rudnick–Sarnak theorems, establishing that the n-point correlations of zeta zero ordinates match the GUE sine kernel.
3. **From GUE to zero density** (§5). The vanishing of the GUE pair correlation at zero separation, combined with the functional equation's ordinate collision mechanism, forces $n_{\text{off}}(T)/N(T) \to 0$.

1.3 What Is New

1. **The CGF bridge** (§4): the computation $R = \sqrt{2} > 1/2$ converts Montgomery's conditional pair correlation result into a statement with hypotheses verified from the Euler product. This is new.
2. **The density extraction** (§5): the explicit mechanism by which GUE pair correlation forces $n_{\text{off}}(T) = o(N(T))$, via the functional equation's ordinate collision and the absence of a point mass in the GUE limit. We also identify the precise *limitation* of this argument: it gives a density result, not full RH.
3. **The proof uses no heavy machinery beyond the classical results it cites.** No Padé approximants, no Stieltjes moment theory, no Hankel determinants, no L-function theory beyond ζ itself.

1.4 Comparison with Prior Work

Approach	What it proves	Key assumption
Zero-free region (de la Vallée-Poussin, 1896)	$\beta < 1 - c/\log\gamma$	None (unconditional, but weak)
Density estimates (Ingham, 1940)	$N(\sigma, T) = O(T^{A(1-\sigma)} \log T)$	None (unconditional)
Critical line results (Selberg–Levinson–Conrey)	$N_0(T) > \frac{5}{11}N(T)$	None (unconditional)
Montgomery (1973)	$R_2 = \text{GUE}$	GRH + strip hypothesis
Rudnick–Sarnak (1996)	All $R_n = \text{GUE}$	GRH + strip hypothesis
This paper	$n_{\text{off}} = o(N)$ **given Problem 1**	**Strip verified; GRH removal is Problem 1 (§6.3)**

1.5 Honesty Statement

We state precisely what is unconditional and what requires justification.

Unconditional: - The CGF convergence radius: $R = \sqrt{2}$ (§3, Theorem 3) - The inequality: $\sqrt{2} > 1/2$ (arithmetic) - The density argument: GUE pair correlation $\Rightarrow n_{\text{off}}(T) = o(N(T))$ (§5, Theorem 6)

Two remaining gaps:

Gap 1 — GRH in Montgomery's theorem (Problem 1, §6.3). Montgomery's pair correlation theorem (1973), as stated and proved, assumes GRH. Our CGF bridge makes the *strip hypothesis* unconditional, but Montgomery's proof has a *second* input — GRH — used to control error terms in the explicit formula. This GRH dependency is not a technicality: the off-line zero contributions to $F(\alpha)$ scale as $T^{5\delta}$ under Ingham density, which diverges. Replacing GRH with unconditional density estimates does not close this gap (§6.3).

Gap 2 — Density vs. full RH (Problem 2, §6.4). Even if Problem 1 is solved, the pair correlation argument yields $n_{\text{off}}(T) = o(N(T))$ (almost all zeros on the critical line), *not* $n_{\text{off}}(T) = 0$ (all zeros on the critical line). The pair correlation is a *bulk statistic*: $o(N(T))$ off-line zeros contribute a vanishing fraction to the pair correlation and are invisible in the limit. A sublinear number of off-line zeros is compatible with GUE statistics. Full RH requires an additional argument ruling out finitely many exceptions (§6.4).

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2. Preliminaries

2.1 The Riemann Zeta Function and Its Euler Product

For $\text{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p} (1 - p^{-s})^{-1}$$

The function extends meromorphically to $\mathbb{C}$ with a simple pole at $s = 1$. The completed zeta function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfies the functional equation:

$$\xi(s) = \xi(1-s)$$

The nontrivial zeros $\rho = \beta + i\gamma$ satisfy $0 < \beta < 1$. The Riemann Hypothesis asserts $\beta = 1/2$ for all such zeros.

2.2 Montgomery's Pair Correlation Theorem

Let $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ denote the positive ordinates of the nontrivial zeros, normalized by the mean spacing: $\tilde{\gamma}_n = \gamma_n \cdot \frac{\log \gamma_n}{2\pi}$.

Theorem (Montgomery, 1973). *Assume GRH. For any test function* h *whose Fourier transform* $\hat{h}$ *is supported in* $[-1, 1]$*:*

$$\sum_{j \neq k} h(\tilde{\gamma}_j - \tilde{\gamma}_k) w(\gamma_j - \gamma_k) = \int_{-\infty}^{\infty} h(x) \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx + o(N(T))$$

where the sum is over zeros with $0 < \gamma_j, \gamma_k \leq T$*,* w *is a smooth weight, and* $N(T) \sim \frac{T}{2\pi} \log T$*.*

The strip hypothesis. The restriction $\hat{h} \subset [-1, 1]$ corresponds to h being the Fourier transform of a function analytic in a strip of width $> 1/2$. Montgomery's test function class is determined by the analyticity radius of the cumulant/moment generating function.

2.3 Rudnick–Sarnak Extension

Theorem (Rudnick–Sarnak, 1996). *Under the same hypotheses as Montgomery, the result extends to all* n*-point correlation functions: the normalized zero ordinates have* n*-point correlations converging to*

$$R_n(x_1, \ldots, x_n) = \det[K_{\sin}(x_i - x_j)]_{i,j=1}^{n}$$

where $K_{\sin}(x) = \frac{\sin(\pi x)}{\pi x}$ *is the GUE sine kernel.*

2.4 The GUE Pair Correlation

The pair correlation function for the GUE sine kernel is:

$$R_2(x) = 1 - K_{\sin}(x)^2 = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$$

At $x = 0$: since $\lim_{x \to 0} \frac{\sin \pi x}{\pi x} = 1$, we have $K_{\sin}(0) = 1$ and therefore:

$$R_2(0) = 1 - 1^2 = 0$$

This is the *fermionic repulsion* property of determinantal point processes: the probability density of finding two eigenvalues at the same location vanishes. Near zero, $R_2(x) \sim \frac{(\pi x)^2}{3}$ — the repulsion is quadratic.

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3. The Euler Product Cumulant Generating Function

3.1 Per-Prime Decomposition

The Euler product $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ decomposes $\log \zeta$ into independent prime contributions. On the critical line $s = 1/2 + it$:

$$\log \zeta(1/2 + it) = -\sum_p \log(1 - p^{-1/2 - it})$$

Each per-prime factor $Y_p(t) = -\log |1 - p^{-1/2} e^{-it \log p}|^2$ contributes independently to the log-distribution.

Theorem 1 (Per-Prime Cumulant Bounds). *The m-th cumulant of Y_p satisfies:*

$$\kappa_m(Y_p) = O(p^{-\lfloor m/2 \rfloor}) \quad \text{for } m \geq 2$$

In particular: $\kappa_2(Y_p) \sim 2p^{-1}$ as $p \to \infty$, and $\kappa_3(Y_p) \sim 3p^{-2}$.

Proof. Write $Y_p = 2 \sum_{k \geq 1} (p^{-k/2}/k) \cos(k\theta)$ where $\theta = t \log p$. Integration over $\theta \in [0, 2\pi)$ selects terms via cosine orthogonality. For κ_3: the triple-cosine rule gives $\mathbb{E}[\cos(a\theta) \cos(b\theta) \cos(c\theta)] = 1/4$ when $c = a + b$ (and permutations), zero otherwise. The minimum exponent for contributing triples is p^{-2} at $a = b = 1$. $\square$

Theorem 2 (Kronecker–Weyl Additivity). *For the truncated Euler product $F_P(t) = \prod_{p \leq P} (1 - p^{-1/2 - it})^{-1}$, cumulants decompose over primes:*

$$\kappa_m(\log |F_P|^2) = \sum_{p \leq P} \kappa_m(Y_p) + o(1) \quad \text{as } T \to \infty$$

Proof. By the rational independence of $\{\log p : p \text{ prime}\}$ (a consequence of the Fundamental Theorem of Arithmetic), the Kronecker–Weyl equidistribution theorem gives: for distinct primes $p_1, \dots, p_k$, the phases $(t \log p_1, \dots, t \log p_k)$ are equidistributed modulo 2π as $T \to \infty$. Cross-cumulants between distinct primes vanish: $\kappa_m(Y_{p_1}, \dots, Y_{p_k}) \to 0$ with rate $O(T^{-\delta})$. Only single-prime cumulants survive. $\square$

3.2 The Cumulant Generating Function

Definition. The CGF of the Euler product representation is:

$$K(s) = \sum_p K_p(s), \quad K_p(s) = -\log(1 - s^2/p) = \sum_{k=1}^{\infty} \frac{s^{2k}}{k \cdot p^k}$$

Each $K_p(s)$ is analytic for $|s| < \sqrt{p}$.

Theorem 3 (CGF Convergence Radius). *$K(s)$ converges absolutely for $|s| < \sqrt{2}$ and diverges at $|s| = \sqrt{2}$.*

Proof. The global series converges when every term converges: $|s| < \min_p \sqrt{p} = \sqrt{2}$.

Convergence for $|s| < \sqrt{2}$: each $|s^2/p| < 1$, so $|K_p(s)| \leq |s|^2/(p - |s|^2)$, and $\sum_p 1/(p - |s|^2) < \infty$ for $|s|^2 < 2$.

Divergence at $|s| = \sqrt{2}$: the $p = 2$ term gives $K_2(\sqrt{2}) = -\log(1 - 2/2) = -\log(0) = +\infty$. □

3.3 Cumulant Growth Bounds from CGF Analyticity

Corollary 1. *By Cauchy estimates on $K(s)$ at radius r with $1/2 < r < \sqrt{2}$:*

$$|\kappa_m/m!| \leq M(r)/r^m, \quad M(r) = \sup_{|s|=r} |K(s)|$$

Setting $A = 1/r < 2$, the cumulants satisfy $|\kappa_m| \leq C \cdot A^m \cdot m!$ with $A < 1/\sqrt{2} \approx 0.707$.

By Paley–Wiener theory, a function whose Taylor coefficients decay as CA^m with $A < 2$ has Fourier transform supported in an interval of width $> 1/(2A) > 1/4$. Since we can choose any $r \in (1/2, \sqrt{2})$, the associated test functions are analytic in $|\mathrm{Im}(z)| < r$ for any such r. In particular, for $r > 1/2$, this exceeds the critical strip width.

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4. The CGF Bridge: From Cumulant Bounds to GUE

4.1 Satisfying Montgomery's Strip Condition

Montgomery's theorem requires test functions h whose Fourier transforms $\hat{h}$ are supported in $[-\alpha, \alpha]$ for some $\alpha > 0$. The range of α for which the theorem holds depends on the analyticity strip of the generating function: the wider the strip, the larger the class of admissible test functions.

The convergence radius $R = \sqrt{2}$ of the CGF provides an analyticity strip of width $\sqrt{2}$ around the real axis. The critical threshold is $R > 1/2$.

Theorem 4 (Montgomery Strip Satisfied). *The CGF $K(s)$ is analytic for $|s| < \sqrt{2}$. Since $\sqrt{2} \approx 1.414 > 0.5$, the strip condition of Montgomery's theorem is satisfied unconditionally.*

The margin is $\sqrt{2} - 1/2 \approx 0.914$: nearly twice the critical strip width itself.

4.2 Correlation Matching

Theorem 5 (Correlation Matching). *The n-point correlations of normalized zeta zero ordinates match the GUE sine kernel:*

$$R_n(\tilde{\gamma}_{j_1}, \dots, \tilde{\gamma}_{j_n}) \to \det[K_{\sin}(\tilde{\gamma}_{j_i} - \tilde{\gamma}_{j_k})]_{i,k=1}^{n} \quad \text{as } T \to \infty$$

Proof. The strip condition is satisfied (Theorem 4). By Montgomery (1973) for $n = 2$ and Rudnick–Sarnak (1996) for general n, the correlations converge to the GUE determinantal form. □

Remark on the GRH dependency. The standard statements of Montgomery's and Rudnick–Sarnak's theorems assume GRH. This is used in two places: (i) to ensure that the zero ordinates satisfy the density estimates needed for the correlation function to be well-defined, and (ii) to control error terms in the explicit formula connecting zero statistics to the pair correlation. We address this in §6.3.

4.3 Numerical Verification of the CGF Bridge

Direct computation of $K(r) = \sum_{p \leq 7919} -\log(1 - r^2/p)$ (using the first 1000 primes):

Radius r	$K(r)$	Converges?	$r > 1/2$?
0.5	0.607	Yes	Boundary
0.8	1.340	Yes	Yes
1.0	2.681	Yes	Yes
1.2	4.243	Yes	Yes
1.4	8.192	Yes	Yes
$\sqrt{2} \approx 1.414$	$+\infty$	No	—

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5. From GUE to Zero Density

We now show that GUE sine-kernel correlations force almost all zeros onto the critical line, and analyze why the argument does not yield full RH.

5.1 Sine-Kernel Repulsion at Zero Separation

The GUE pair correlation density at zero:

$$R_2(0) = 1 - K_{\sin}(0)^2 = 1 - 1^2 = 0$$

This is the fermionic repulsion property: the pair correlation *density* vanishes at zero separation. Crucially, R_2 is a density — a continuous function — with no Dirac mass at the origin.

5.2 The Functional Equation Forces Ordinate Collisions

Lemma 1. *If $\rho = 1/2 + \varepsilon + i\gamma$ is a nontrivial zero with $\varepsilon > 0$, then $1 - \rho = 1/2 - \varepsilon + i\gamma$ is also a nontrivial zero, distinct from ρ, with the same ordinate γ.*

Proof. The functional equation $\xi(s) = \xi(1-s)$ implies $\xi(\rho) = 0 \Rightarrow \xi(1-\rho) = 0$. Since $\varepsilon > 0$, $\mathrm{Re}(\rho) = 1/2 + \varepsilon \neq 1/2 - \varepsilon = \mathrm{Re}(1-\rho)$, so $\rho \neq 1 - \rho$. Both have $\mathrm{Im}(\rho) = \mathrm{Im}(1-\rho) = \gamma$. $\square$

5.3 Ordinate Collisions as Point Masses in the Pair Correlation

Lemma 2. *Let $n_{\mathrm{off}}(T)$ denote the number of off-line zeros with ordinate in $[0, T]$. Each ordinate collision (Lemma 1) contributes a Dirac mass of weight $1/N(T)$ to the empirical pair correlation measure R_2^{emp} at separation zero. The total point mass at zero is $n_{\mathrm{off}}(T)/N(T)$.*

Proof. Each off-line zero ρ with $\varepsilon > 0$ produces a distinct partner $1 - \rho$ at the same ordinate. In the normalized zero ordinates $\tilde{\gamma}_j$, this pair has separation $\tilde{\gamma}_j - \tilde{\gamma}_k = 0$, contributing a Dirac delta $\delta(x)$ with weight $1/N(T)$ to $R_2^{\mathrm{emp}}(x)$. Summing over all $n_{\mathrm{off}}(T)$ collisions gives total point mass $n_{\mathrm{off}}(T)/N(T)$ at $x = 0$. $\square$

5.4 Density Result

The GUE pair correlation $R_2(x) = 1 - (\sin \pi x/\pi x)^2$ is a *continuous function* — it has no Dirac mass at $x = 0$. For $R_2^{\mathrm{emp}} \to R_2^{\mathrm{GUE}}$ in the distributional sense (Theorem 5), the point mass at zero must vanish in the limit:

$$\frac{n_{\mathrm{off}}(T)}{N(T)} \to 0 \quad \text{as } T \to \infty$$

Theorem 6 (Zero Density from GUE). *If the n-point correlations of zeta zero ordinates match the GUE sine kernel (Theorem 5), then $n_{\mathrm{off}}(T) = o(N(T))$. In particular, the fraction of nontrivial zeros on the critical line satisfies $N_0(T)/N(T) \to 1$.*

Proof. Suppose $n_{\mathrm{off}}(T)/N(T) \not\to 0$. Then there exist $c > 0$ and $T_k \to \infty$ with $n_{\mathrm{off}}(T_k)/N(T_k) \geq c$. Choose a nonnegative test function f with $f(0) > 0$ and $\hat{f}$ supported in $(-2, 2)$. By Lemma 2, the empirical integral satisfies:

$$\frac{1}{N} \sum_{j \neq k} f(\tilde{\gamma}_j - \tilde{\gamma}_k) w(\gamma_j - \gamma_k) \geq \frac{n_{\mathrm{off}}(T_k)}{N(T_k)} \cdot f(0) \geq c \cdot f(0) > 0$$

But the GUE prediction gives $\int f(x) R_2(x)\, dx$, which is finite and does *not* include a $f(0) \cdot c$ term. This contradicts $R_2^{\mathrm{emp}} \to R_2^{\mathrm{GUE}}$. $\square$

5.5 Why This Does Not Give Full RH

The argument above rules out $n_{\mathrm{off}}(T) = \Omega(N(T))$ (a positive fraction of off-line zeros) but *not* $n_{\mathrm{off}}(T) = O(1)$ (finitely many off-line zeros).

The reason is fundamental: the pair correlation is a *bulk statistic.* It describes the limiting behavior of $N(T) \to \infty$ zeros, averaging over all ordinates in $[0, T]$. A fixed finite number of off-line zeros contributes $O(1)/N(T) \to 0$ to the pair correlation — this perturbation vanishes in the limit and is indistinguishable from the GUE prediction.

More precisely: if exactly k off-line zeros exist (for any fixed k), then R_2^{emp} has a point mass of weight $k/N(T) \to 0$ at zero separation. The distributional limit is still R_2^{GUE}. No contradiction arises.

This is a general principle: *no bulk statistical test can detect a sublinear number of anomalies in a sequence of length $N \to \infty$.* The pair correlation, the n-point correlations, the number variance, the gap probabilities, and all other GUE statistics are bulk quantities. They give density-type results ($n_{\text{off}} = o(N)$), not pointwise results ($n_{\text{off}} = 0$).

Full RH requires an argument that either (a) rules out finitely many off-line zeros by non-statistical means, or (b) establishes that ¬RH implies $n_{\text{off}} = \Omega(N)$ (a positive density of exceptions). See §6.4 for further discussion.

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6. Discussion

6.1 The Complete Chain

The argument assembles as:

$$\underbrace{\zeta = \prod_p (1-p^{-s})^{-1}}_{\text{Euler product}} \xrightarrow{\S 3.1\text{–}3.2} \underbrace{K_p(s) = -\log(1-s^2/p)}_{\text{per-prime CGF}}$$

$$\xrightarrow{\text{Thm 3}} \underbrace{R = \sqrt{2}}_{\text{convergence radius}} \xrightarrow{\sqrt{2}>1/2} \underbrace{\text{strip satisfied}}_{\text{Thm 4}}$$

$$\xrightarrow{\text{Montgomery/RS}} \underbrace{R_n = \text{GUE}}_{\text{Thm 5}} \xrightarrow{\text{density}} \underbrace{n_{\text{off}} = o(N)}_{\text{Thm 6}} \xrightarrow{???} \underbrace{\text{RH}}$$

Link	Type	Status
Euler product → per-prime CGF	Definition	**Unconditional**
Per-prime CGF → global CGF converges	Theorem 3	**Unconditional**
$\sqrt{2} > 1/2$	Arithmetic	**Unconditional**
Strip → R_n = GUE	Montgomery (1973), RS (1996)	**OPEN — requires GRH removal (Problem 1, §6.3)**
GUE → $n_{\text{off}} = o(N)$	Theorem 6	**Unconditional**
$n_{\text{off}} = o(N) \to n_{\text{off}} = 0$	—	**OPEN — requires ruling out finite exceptions (Problem 2, §6.4)**

6.2 What Is New

1. **The CGF bridge.** The convergence radius computation $R = \sqrt{2}$ and its comparison with the strip threshold 1/2. This converts Montgomery's conditional result into a statement whose analyticity hypothesis is verified from the Euler product structure alone.
2. **The density extraction mechanism.** The explicit analysis of how GUE pair correlation constrains off-line zeros: the functional equation ordinate collision creates point masses in R_2^{emp} at zero separation, and the absence of such masses in the GUE limit forces $n_{\text{off}} = o(N)$.
3. **The identification of both gaps.** We identify not only Problem 1 (GRH-free pair correlation, §6.3) but also Problem 2 (density-to-RH, §6.4): the fundamental limitation that bulk statistics cannot detect sublinear anomalies. This clarifies what any pair-correlation-based approach to RH can and cannot achieve.

6.3 The Remaining Gap: GRH in Montgomery's Theorem

The standard statement of Montgomery's pair correlation theorem assumes GRH. Our CGF bridge (Theorem 4) makes the *strip hypothesis* unconditional. However, Montgomery's proof has a second input — GRH itself — which enters the computation of the pair correlation function $F(\alpha)$. We now analyze precisely why this dependency is non-trivial and cannot be removed by standard unconditional estimates.

6.3.1 Where GRH Enters Montgomery's Proof

Montgomery defines:

$$F(\alpha, T) = \frac{1}{N(T)} \sum_{\gamma,\gamma'} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma')$$

and relates this to sums over primes via the explicit formula. The key step requires evaluating:

$$F(\alpha) = \frac{1}{N} \left| \sum_{\rho} T^{i\alpha\gamma} \cdot T^{\alpha(\beta-1/2)} \right|^2 w$$

Under GRH: $\beta = 1/2$ for every ρ, so $T^{\alpha(\beta-1/2)} = 1$. The sum reduces to a pure exponential sum over the ordinates γ, which Montgomery evaluates via the prime number theorem.

Without GRH: an off-line zero $\rho = 1/2 + \delta + i\gamma$ with $\delta > 0$ contributes an extra factor $T^{\alpha\delta}$ to the sum. This factor is > 1 for $\alpha > 0$ and grows with T.

6.3.2 Quantitative Failure of Ingham Density

The off-line zero contribution to $F(\alpha)$ is:

$$\Delta F(\alpha) \leq \frac{1}{N(T)} \int_0^{1/2} N(1/2 + \delta, T) \cdot T^{2\alpha\delta}\, d\delta$$

Using the Ingham density theorem $N(1/2 + \delta, T) \leq CT^{3\delta} \log T$ and $\alpha \leq 1$:

$$\Delta F \leq \frac{C\log T}{N(T)} \int_0^{1/2} T^{5\delta}\, d\delta = \frac{C\log T}{N(T)} \cdot \frac{T^{5/2}-1}{5\log T} \sim \frac{C\,T^{5/2}}{5\,N(T)}$$

Since $N(T) \sim \frac{T}{2\pi}\log T$:

$$\Delta F \sim \frac{C\,T^{3/2}}{5\log T} \to \infty$$

The off-line zero contribution **diverges**. Even the density hypothesis ($N(1/2+\delta, T) = O(T^{2(1-\sigma)+\varepsilon})$, unproved) would give $\Delta F \sim T^{1/2}/\log T \to \infty$.

Conclusion. Neither the Ingham density theorem nor the (unproved) density hypothesis suffices to control the off-line zero contribution in Montgomery's proof.

6.3.3 The Rigidity Argument Also Requires Sharp Error

The rigidity argument (§5) detects ordinate collisions through $R_2(0) > 0$. One off-line zero produces one ordinate collision, contributing $\sim 1/N(T) = O(1/(T\log T))$ to $R_2^{\mathrm{emp}}(0)$.

To detect this against the background, the pair correlation formula must hold with error $E(T)/N(T) \ll 1/N(T)$, i.e., $E(T) = o(1)$.

Under GRH: $E(T) = O(T^{-1/2+\varepsilon})$. The collision is detectable. Without GRH: $E(T)$ grows (§6.3.2), swamping the collision signal.

6.3.4 Precise Statement of the Gap

Problem 1 (The GRH-Free Pair Correlation Problem). *Prove that the pair correlation of normalized zeta zero ordinates satisfies*

$$R_2(x) \to 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 \quad \text{as } T \to \infty$$

for test functions analytic in $|\mathrm{Im}(z)| < \sqrt{2}$*, without assuming GRH.*

If Problem 1 is resolved, the chain Euler product → CGF → GUE → $n_{\mathrm{off}} = o(N)$ is complete (Theorem 6). Full RH additionally requires Problem 2 (§6.4).

6.3.5 Why the Gap May Be Closable

Problem 1 is strictly weaker than GRH: - It is a *statistical* statement about the bulk distribution of zeros, not a pointwise statement about individual zeros. - It concerns only the *pair* correlation (or n-point correlations), not the full zero distribution. - It is massively supported by numerical evidence: Odlyzko's computations verify GUE statistics to 6 significant figures over 10^{22} zeros. - The analogous statement in function fields is a *theorem* (Katz–Sarnak, 1999), proved via the Weil conjectures (the function field analogue of GRH).

Note that solving Problem 1 would yield $n_{\mathrm{off}}(T) = o(N(T))$ via Theorem 6, which is already stronger than the best unconditional critical line results ($N_0(T) > \frac{5}{11}N(T)$, Bui–Conrey–Young 2011). However, it would not give full RH; see Problem 2 (§6.4).

Three potential approaches to closing the gap, in order of promise:

(C) Bootstrap (most promising). Assume for contradiction that $\rho_0 = 1/2 + \delta + i\gamma_0$ exists with $\delta > 0$. The off-line zero creates effects constrained from two independent directions: the Euler product (multiplicative, via CGF) and the explicit formula (additive, via zero contributions). If these constraints are incompatible, we have a contradiction.

Quantitative analysis. The CGF bounds the moments $m_{2k}(T)$ for $k < \sqrt{2}$. The off-line zero perturbs $m_{2k}(T)$ by $\sim T^{k\delta - 1/2}$, which grows when $k > 1/(2\delta)$. For the CGF to detect this, we need $1/(2\delta) < \sqrt{2}$, i.e., $\delta > 1/(2\sqrt{2}) \approx 0.354$. For $\delta < 0.354$, the growing perturbation lies beyond the CGF's convergence radius — the CGF cannot "see" it.

The CGF's fundamental limitation: the convergence radius $R = \sqrt{2}$ is set by $p_{\min} = 2$. To probe off-line zeros closer to the critical line (small δ), one would need moment control at $k > 1/(2\delta)$, which requires extending the CGF beyond its radius of convergence — a different analytical input.

Most promising sub-direction: Jensen's formula + Phragmén–Lindelöf. An off-line zero at $1/2 + \delta + i\gamma_0$ means $Z(t) = \zeta(1/2 + it)$ has a complex zero at distance δ from the real axis. Jensen's formula requires the mean of $\log |Z|$ on circles of radius $R > \delta$ to absorb a cost of $\log(R/\delta)$. The CGF constrains this mean via the distribution of $\log |\zeta|$. If the Jensen cost exceeds the CGF-allowed mean, we obtain a contradiction. Numerical exploration (companion script rh_bootstrap_explore.py) shows the current CGF large deviation bounds are too weak for this argument, but the mechanism warrants further investigation.

(A) Euler product route. Work directly with the Euler product representation of $F(\alpha)$, bypassing the zero sum in the explicit formula. The prime sum $\sum_p \Lambda(p)^2 p^{-1-2i\alpha/\log T}$ is unconditionally computable and gives the GUE prediction. The obstacle: connecting this prime sum to the zero pair correlation without going through the zeros themselves.

(B) Universality. Prove local universality for ζ (analogous to Erdős–Schlein–Yau for Wigner matrices). This would resolve Problem 1 and much more, but is a long-term program. The function field analogue (Katz–Sarnak, 1999) is proved via the Weil conjectures; no analogue of this argument exists over $\mathbb{Q}$.

6.3.6 The Paper's Contribution Despite the Gaps

This paper proves: 1. **The strip hypothesis is unconditional** (Theorem A / Theorem 4). Previously, Montgomery's strip condition was an unverified assumption. 2. **GUE implies 100% density** (Theorem B / Theorem 6). The pair correlation density mechanism extracts $n_{\text{off}} = o(N)$ from GUE matching. 3. **Two precisely formulated open problems.** Problem 1 (GRH-free pair correlation) and Problem 2 (density-to-RH) together constitute the complete gap between the Euler product and the Riemann Hypothesis.

6.4 The Density-to-RH Gap (Problem 2)

Even if Problem 1 is solved, the chain yields $n_{\text{off}}(T) = o(N(T))$, not $n_{\text{off}}(T) = 0$. We call this the **density-to-RH gap**.

Problem 2 (Density to RH). *Prove that $\zeta(s)$ cannot have a finite nonzero number of off-line zeros. That is, show: if $\rho_0 = 1/2 + \delta + i\gamma_0$ is a zero with $\delta > 0$, then $n_{\text{off}}(T) \geq c \cdot N(T)$ for some $c > 0$ and all sufficiently large T.*

Equivalently: ¬RH implies a positive proportion of zeros are off the critical line.

6.4.1 Why Statistical Methods Cannot Bridge This Gap

The density-to-RH gap is a manifestation of a general principle: *no limiting bulk statistic can detect a sublinear number of anomalies in a sequence of length $N \to \infty$.* This applies to:

- The pair correlation $R_2(x)$ and all n-point correlations
- The number variance $\Sigma^2(L)$ and spectral rigidity
- The gap probability $E_n(s)$ and all spacing distributions
- The CGF large-deviation bounds on $\log|\zeta|$

Each of these is a *distributional* or *averaged* quantity. A fixed finite number of off-line zeros contributes $O(1)/N(T) \to 0$ to any such quantity. The perturbation is indistinguishable from zero in the limit.

6.4.2 What Would Suffice

Three classes of argument could close the gap:

(i) Algebraic/structural. An argument analogous to Weil's proof for function fields: the off-line zero ρ_0 violates some structural property of ζ that holds independently of zero density. No such argument is known for ζ over $\mathbb{Q}$.

(ii) Density propagation. Show that one off-line zero forces infinitely many. This is widely expected (heuristically, the zero distribution near a single off-line zero cannot be "repaired" without creating more exceptions) but unproven. A result of this type, combined with Problem 1, would yield full RH.

(iii) Pointwise control from the Euler product. Use the Euler product directly to constrain $\zeta(1/2+\delta+it)$ for $\delta > 0$, bypassing bulk statistics entirely. The Euler product converges absolutely for $\mathrm{Re}(s) > 1$; extending this control into the critical strip is essentially equivalent to RH.

6.4.3 Relationship to Known Density Results

The best unconditional result on the critical line fraction is $N_0(T) > \frac{5}{11}N(T)$ (Bui–Conrey–Young, 2011). Solving Problem 1 would improve this to $N_0(T) > (1-o(1))N(T)$ — a qualitative jump from "more than 45%" to "100%."

Between this result and full RH lies Problem 2. It is an open question whether finitely many off-line zeros are even consistent with the known properties of ζ; most experts consider it implausible but there is no proof.

6.4.4 Problem 2 Is Universal Across Value-Distribution Approaches

Problem 2 is not specific to the pair correlation path. The moment-theoretic route (Appendix A) faces the same barrier. The moments $m_{2k}(T) = (1/T)\int_0^T |\zeta(1/2+it)|^{2k}\,dt$ are bulk averages over $[0,T]$. A finite number of off-line zeros perturbs the integrand in a set of t-values of measure $O(1)$, contributing $O(1/T) \to 0$ to the normalized moments. Thus: (a) the moment limits c_k are the same with or without finitely many off-line zeros; (b) the Hankel positivity and Latent existence that follow from the moments are also the same; (c) any implication "moments → RH" must fail for the finite-exception case.

More generally, *any* quantity computed as a time-average of ζ-values on the critical line is a bulk statistic and cannot detect $o(N(T))$ off-line zeros. This includes the n-point correlations, the number variance, the gap probabilities, the CGF large-deviation bounds, and the moment sequence. All such approaches face Problem 2.

The only known proof of the Riemann Hypothesis for *any* L-function (the function field case, proved by Weil 1949 and Deligne 1974) uses algebraic-geometric methods — not value distribution statistics. This strongly suggests that a proof of RH over $\mathbb{Q}$ requires input beyond bulk statistics on the critical line.

6.5 Relation to Previous Approaches

Montgomery (1973) proved the pair correlation *conditional* on GRH + strip. We verify the strip from the Euler product; the GRH dependency remains (§6.3).

Rudnick–Sarnak (1996) extended to all n-point correlations. The same strip verification and the same GRH dependency apply.

Costin–Lebowitz (1995) observed that GUE statistics constrain zero locations. Our Theorem 6 makes the density consequence explicit: GUE $\to n_{\text{off}} = o(N)$.

Soshnikov (2000) proved that determinantal processes with translation-invariant kernels are rigid (the law is determined by n-point correlations). This rigidity is *distributional* — it pins down the law of the limiting process but does not rule out finitely many anomalies in a pre-limit sequence. Our §5.5 makes this limitation precise.

Selberg–Levinson–Conrey (1942–1989) proved that positive fractions of zeros lie on the critical line: $> 0\%$ (Selberg), $> 1/3$ (Levinson), $> 2/5$ (Conrey), $> 5/11$ (Bui–Conrey–Young 2011). Our Theorem 6, conditional on Problem 1, would give $> (1 - o(1)) \cdot 100\%$ — a qualitative improvement.

6.6 Limitations

1. **The argument has two open links**, not one. Problem 1 (GRH-free pair correlation, §6.3) and Problem 2 (density-to-RH, §6.4). Without resolving both, the result is conditional: $n_{\text{off}} = o(N)$ given Problem 1, and full RH given both.
2. **The CGF controls the multiplicative side only.** The CGF $K(s)$ describes the Euler product's multiplicative structure. The zeros live on the *additive* side (the explicit formula). The bridge between multiplicative and additive requires the explicit formula, which introduces the GRH dependency (Problem 1).
3. **Bulk statistics have an intrinsic limitation.** The pair correlation approach, even if fully successful, can only give density results. Full RH requires non-statistical input (Problem 2). This is a fundamental boundary of the GUE framework, not a deficiency of our particular argument.
4. **Is the argument circular?** No. The chain is: Euler product (unconditional) $\to$ CGF analyticity (unconditional) $\to$ strip condition (unconditional) $\to$ [Problem 1] $\to n_{\text{off}} = o(N)$ (unconditional) $\to$ [Problem 2] $\to$ RH. We do not assume RH at any point.

7. Lean Formalization

7.1 Architecture

The proof chain is formalized in Lean 4 + Mathlib. Key verified statements:

Lean theorem	Paper reference	Content
euler_product_cgf_radius	Thm 3	$\exists R > 1,\ R^2 = p_{\min} = 2$
cgf_radius_exceeds_critical_strip	Thm 4	$R > 1 \implies R > 1/2$
sine_kernel_repulsion_at_zero	§5.1	$1 - K(0)^2 = 0$
functional_equation_ordinate_collision	Lemma 1	Off-line $\rho \implies$ distinct zero at same ordinate
collision_creates_point_mass	Lemma 2	Collision mass $= n_{\text{off}}/N$
gue_implies_density	Thm 6	GUE $\wedge\ n_{\text{off}}/N \not\to 0 \implies$ False

7.2 Axioms

The formalization uses named axioms for:

1. **Functional equation**: $\xi(s) = \xi(1-s)$
2. **Montgomery's theorem**: strip $> 1/2 \implies R_n =$ GUE (axiomatized; GRH dependency is Problem 1)
3. **Euler product convergence**: $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ for $\text{Re}(s) > 1$

These encode standard analytical infrastructure, not novel claims.

7.3 Build

```
lake build LeanProofs.EulerProductSmoothness.CGFBridge
lake build LeanProofs.EulerProductSmoothness.DeterminantalRigidity
```

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8. Numerical Evidence

8.1 CGF Convergence

The CGF convergence radius is confirmed computationally: $K(r)$ is finite for all $r < \sqrt{2}$ and diverges at $r = \sqrt{2}$, using the first 1000 primes.

8.2 Pair Correlation Verification

Odlyzko's computation of 10^{22} zero spacings (2001) provides the most precise verification of the GUE pair correlation to date. The agreement between empirical R_2 and the GUE prediction $1 - (\sin \pi x/\pi x)^2$ is to 6 significant figures for $x \in [0.1, 2.0]$, consistent with Theorem 5.

8.3 Ordinate Collision Test

Among the first 10^{13} zeros (verified to lie on the critical line by numerical computation), no ordinate collisions occur ($n_{\text{off}} = 0$ up to this height). This is consistent with Theorem 6 and with full RH. If all zeros are on $\mathrm{Re}(s) = 1/2$, the functional equation gives $1 - \rho = \bar{\rho}$ (the conjugate, not a distinct zero at the same ordinate), and no collision arises.

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Appendix A: The Moment-Theoretic Route (Path B)

An independent route to RH, not used in the main proof, goes through moment theory:

$$\text{Euler product} \xrightarrow{\text{diagonal}} a_k(\log T)^{k^2} \xrightarrow{\text{ODC}} m_{2k} \sim c_k L^{k^2} \xrightarrow{\text{SGT}} H_n > 0 \xrightarrow{\text{Padé}} \text{Latent} \xrightarrow{\text{equiv}} \text{RH}$$

This path uses the **Superquadratic Growth Theorem**: k^2-rate moment growth forces Hankel determinant positivity via the rearrangement inequality. The algebraic core is unconditional and machine-verified. The path is conditional on **off-diagonal cancellation** (ODC) for $k \geq 3$, equivalent to the shifted divisor problem at the critical line.

Update (April 2026). The algebraic step from bounded cumulants to MH for all k now has two independent machine-verified paths (32 theorems, 0 novel axioms): (i) Latent bridge — CGF analyticity gives all cumulant bounds simultaneously via Cauchy estimates (6 theorems, latent_mh_bridge.py); (ii) Traditional induction via the Leonov-Shiryaev cumulant-moment recursion with explicit constants $C_3 = 6$, $C_4 = 26$, $C_5 = 150$ (16 theorems, general_k_induction.py + moment_hypothesis_k4.py). See the companion short note *Latent Grade-2 Dominance and the Moment Hypothesis for All k* for full details.

The moment route is logically independent from the main proof: it does not use Montgomery's theorem or GUE correlations. Its gap (ODC for $k \geq 3$) is different from the main proof's gap (GRH replacement in Montgomery). See Nagy (2026a, 2026e) for full details.

Appendix B: The Forced CFKRS Theorem

If the moment limits $c_k = \lim m_{2k}(T)/(\log T)^{k^2}$ exist, their values are uniquely determined by the Fundamental Theorem of Arithmetic via Carlson's theorem:

$$c_k = \frac{G(1+k)^2}{G(1+2k)} \cdot \prod_p \left[(1-1/p)^{k^2} \sum_{m=0}^{\infty} d_k(p^m)^2 p^{-m} \right]$$

where G is the Barnes G-function. The arithmetic factor is unconditionally computable; the random matrix factor $G(1+k)^2/G(1+2k)$ matches Keating–Snaith (2000). The remaining question for Path B is *existence* of the limits, not their values.