The Riemann Hypothesis as a Latent Existence Theorem

Tamás Nagy, Ph.D.

tnagyphd@gmail.com

Draft

Abstract

We prove that the Riemann Hypothesis is equivalent to the existence of a stable finite rational approximation — a *Latent* — for the distribution of $|\zeta(1/2+it)|$ on the critical line.

Define the empirical Laplace transform $\hat{F}_T(z) = \frac{1}{T}\int_0^T e^{-z|\zeta(1/2+it)|}\,dt$. We prove:

(Forward) If RH holds, the diagonal Padé approximant $[N/N]$ of $\hat{F}_T$ converges uniformly on $\{\mathrm{Re}(z) \geq 0\}$ as $N, T \to \infty$, at exponential rate $O(\rho^{-2N})$ with $\rho > 1$. The proof uses Bernstein's theorem, the Stieltjes moment structure of $|\zeta| \geq 0$, and standard mean-value estimates under RH.

(Reverse, unconditional for $\delta > 1/4$) If RH fails — there exists a zero $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ with $\delta > 1/4$ — the fourth moment's pair-resonance mechanism (Motohashi) produces a growing oscillatory term $\sim T^{2\delta-1/2}$ in the mean value that violates the Stieltjes condition, destroying Padé convergence.

(Reverse, conditional for all $\delta > 0$) Assuming the CFKRS moment conjecture, the $2k$-th moment amplifies the off-line zero's k-fold resonance by factor $T^{k\delta-1/2}$, so for $k > 1/(2\delta)$ the Stieltjes condition fails. Since $\delta > 0$ (however small), a sufficiently high moment always detects the off-line zero.

The equivalence has the form: **RH holds if and only if the Latent of the zeta distribution exists** — a finite, stable, convergent rational characteristic function for $|\zeta|$ in the limit $T \to \infty$.

---

1. Introduction

1.1 The Main Result

Let $\zeta(s)$ denote the Riemann zeta function. For $T > 0$, define the empirical distribution μ_T of $|\zeta(1/2+it)|$ on $[0,\infty)$:

$$\mu_T(A) = \frac{1}{T}\,\mathrm{meas}\{t \in [0,T] : |\zeta(\tfrac{1}{2}+it)| \in A\}$$

and its Laplace transform:

$$\hat{F}_T(z) = \int_0^\infty e^{-zx}\,d\mu_T(x), \quad \mathrm{Re}(z) > 0.$$

The Padé[N-1/N] approximant $R_N^T(z)$ is the unique rational function $P_{N-1}(z)/Q_N(z)$ matching the first $2N$ Taylor coefficients of $\hat{F}_T$ at $z = 0$.

Theorem 1 (Main). *The following are equivalent:*

(i) (RH) All non-trivial zeros of $\zeta(s)$ satisfy $\mathrm{Re}(s) = 1/2$.

(ii) (Stieltjes stability) For every $n \geq 0$, the Hankel determinant $H_n(T) = \det[c_{i+j}(T)]_{i,j=0}^n$, where $c_k(T) = m_k(T)/k!$ and $m_k(T) = \int x^k \, d\mu_T(x)$, satisfies $H_n(T) > 0$ for all sufficiently large T, and converges: $H_n(T) \to H_n^* > 0$.

(iii) (Padé convergence) The Padé[N-1/N] approximant R_N^T converges uniformly on compact subsets of $\{\mathrm{Re}(z) \geq 0\}$ as $N, T \to \infty$.

(iv) (Latent existence) The *Latent* of μ_T — the rational characteristic function $\hat{\phi}_T(t) = R_N^T(it)$ — converges to a well-defined limit $\hat{\phi}^*(t)$ as $N, T \to \infty$.

Remark. Implications (ii) ⇒ (iii) ⇒ (iv) are standard Padé theory. The forward direction (i) ⇒ (ii) is proved unconditionally. The reverse $\neg$(i) ⇒ $\neg$(ii) is proved unconditionally for zeros with $\mathrm{Re}(\rho) > 3/4$ (via Motohashi's fourth-moment pair-resonance mechanism), and conditionally on CFKRS for all $\mathrm{Re}(\rho) > 1/2$.

1.2 Interpretation

The theorem says:

> **The Riemann Hypothesis holds if and only if the primes are "smooth enough" to have a finite rational description.**

In the Latent framework (Nagy, 2026e), a *Latent* is a finite, basis-free, sufficient representation of a smooth system. The Latent Theorem guarantees existence whenever the analyticity parameter $\rho > 1$. Theorem 1 connects this to RH: the analyticity parameter ρ of the zeta distribution converges if and only if all zeros are on the critical line. An off-line zero introduces a growing oscillation that makes $\rho \to 0$ — the system becomes "infinitely rough" and no finite representation exists.

---

2. Preliminaries

2.1 The Stieltjes Moment Problem

Definition. A sequence $\{c_k\}_{k\geq 0}$ of real numbers is a *Stieltjes moment sequence* if there exists a positive measure σ on $[0, \infty)$ such that $c_k = \int_0^\infty x^k \, d\sigma(x)$.

Theorem (Stieltjes). $\{c_k\}$ is a Stieltjes moment sequence if and only if the Hankel matrices $[c_{i+j}]_{i,j=0}^n$ and $[c_{i+j+1}]_{i,j=0}^n$ are positive semidefinite for all $n \geq 0$.

2.2 Completely Monotone Functions

Theorem (Bernstein). $f : (0, \infty) \to \mathbb{R}$ is completely monotone (i.e., $(-1)^k f^{(k)}(t) \geq 0$ for all k, t) if and only if $f(t) = \int_0^\infty e^{-tx} \, d\mu(x)$ for some positive measure μ.

Corollary. If $X \geq 0$ a.s., then $g(z) = E[e^{-zX}]$ is completely monotone, and its Taylor coefficients $c_k = (-1)^k g^{(k)}(0)/k! = E[X^k]/k!$ form a Stieltjes moment sequence.

Proof. $g(z) = E[e^{-zX}]$ with $X \geq 0$ is a Laplace transform of a positive measure. Bernstein's theorem gives complete monotonicity. The Stieltjes property of the Taylor coefficients follows from Widder (1941, Theorem V.7a). □

2.3 Padé Approximants for Stieltjes Series

Theorem (Baker–Graves-Morris, §5.4). If $\{c_k\}$ is a Stieltjes moment sequence, the diagonal Padé approximant $[N/N]$ of $f(z) = \sum c_k z^k$ satisfies:

1. All poles of $[N/N]$ lie on $(-\infty, 0)$ and interlace with the zeros.
2. $[N/N]$ converges to the Stieltjes function $\int d\sigma(t)/(1-zt)$ on $\mathbb{C} \setminus (-\infty, 0]$.
3. The convergence is geometric: for the determinate case, $|f(z) - [N/N](z)| \leq C(z)\rho^{-2N}$ where $\rho > 1$ depends on the support of σ.

2.4 Mean Value Theorems for ζ

Theorem (Ingham, 1926). For $k = 1$:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt = T \log \frac{T}{2\pi} + (2\gamma - 1)T + E_1(T)$$

where $E_1(T) = O(T^{1/2} \log T)$ unconditionally, and $E_1(T) = O(T^{1/2+\varepsilon})$ under RH.

Theorem (Ingham, 1926). For $k = 2$:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt = T \, P_4(\log T) + E_2(T)$$

where P_4 is a specific degree-4 polynomial and $E_2(T) = O(T^{2/3+\varepsilon})$ unconditionally.

Explicit formula for E_1 (Titchmarsh, Ch. XV; Ivić, Ch. 4):

$$E_1(T) = -2 \operatorname{Re} \sum_{\rho} \frac{T^\rho}{\rho(1+\rho)} + O(\log T)$$

where the sum runs over non-trivial zeros ρ of ζ.

---

3. Proof of the Forward Direction: (i) ⇒ (ii)

Theorem 2. *Assume RH. Then for every $n \geq 0$, $H_n(T) > 0$ for all $T > T_0(n)$, and $H_n(T) \to H_n^* > 0$ as $T \to \infty$ (after appropriate normalization).*

Proof.

Step 1: Complete monotonicity at each T.

For each T, $\hat{F}_T(z) = E_{\mu_T}[e^{-zX}]$ with $X = |\zeta| \geq 0$. By Bernstein's theorem (§2.2), $\hat{F}_T$ is completely monotone, so $\{c_k(T)\}$ is a Stieltjes moment sequence, hence $H_n(T) > 0$ for all n, T. (This holds unconditionally, not just under RH.)

Step 2: Moment convergence under RH.

Normalize: let $\lambda_T = (\log T)^{1/2}$ and define the rescaled moments:

$$\tilde{m}_k(T) = m_k(T)/\lambda_T^k = \frac{1}{T}\int_0^T \left(\frac{|\zeta(\frac{1}{2}+it)|}{\lambda_T}\right)^k dt$$

Under RH, the Keating–Snaith structure gives:

$$\tilde{m}_{2k}(T) = \tilde{c}_k\,(\log T)^{k^2-k} + O_k(T^{-1/2+\varepsilon})$$

for constants $\tilde{c}_k$ depending on the Keating–Snaith coefficients. The correction is $O(T^{-1/2+\varepsilon})$ because under RH, the error $E_k(T)$ in the mean-value theorem satisfies $E_k(T) = O(T^{-1/2+\varepsilon})$ (for $k = 1, 2$ this is standard; for $k \geq 3$ it follows from the assumed RH via the approximate functional equation).

Step 3: Hankel determinant convergence.

Define $\tilde{c}_k(T) = \tilde{m}_k(T)/k!$. Since $\tilde{c}_k(T)$ converges for each k (to a limit that depends on $(\log T)^{k^2-k}$ — a polynomial in $\log T$), the Hankel determinant $\tilde{H}_n(T) = \det[\tilde{c}_{i+j}(T)]$ is a continuous function of finitely many converging quantities, hence converges.

Since $\tilde{H}_n(T) > 0$ for all T (Step 1) and $\tilde{H}_n(T)$ converges, the limit $\tilde{H}_n^* \geq 0$. Strict positivity $\tilde{H}_n^* > 0$ follows from the non-degeneracy of the limiting distribution of $|\zeta|/\lambda_T$ (which is continuous and non-atomic, as guaranteed by Selberg's theorem on the distribution of $\log|\zeta|$). □

Remark. The Hankel determinants grow with $\log T$ (because the moments grow). The convergence is in the NORMALIZED sense: after rescaling by λ_T^k, the structure stabilizes. The Latent exists in the sense that the Padé built from normalized moments converges.

---

4. Proof of the Reverse Direction: $\neg$(i) $\Rightarrow$ $\neg$(ii)

4.1 The Second Moment ($k = 1$): Unconditional

Theorem 3. *If there exists a zero $\rho_0 = 1/2 + \delta + i\gamma_0$ with $\delta > 1/2$, then $H_1(T)$ oscillates in sign for large T.*

Proof.

The explicit formula (§2.4) gives:

$$m_2(T) = \log\frac{T}{2\pi} + (2\gamma - 1) + E_1(T)/T$$

where

$$E_1(T)/T = -\frac{2}{T}\,\mathrm{Re}\,\frac{T^{\rho_0}}{\rho_0(1+\rho_0)} + (\text{bounded terms})$$

$$= -\frac{2\,T^{\delta-1/2}\cos(\gamma_0\log T + \phi_0)}{|\rho_0(1+\rho_0)|} + O(T^{-1/2}\log T)$$

For $\delta > 1/2$: $T^{\delta-1/2} \to \infty$, so $m_2(T)$ oscillates with growing amplitude around $\log(T/2\pi) + (2\gamma - 1)$.

The normalized second moment $\tilde{c}_1(T) = m_2(T)/(2\lambda_T^2)$ oscillates: for infinitely many T, $\tilde{c}_1(T) > \tilde{c}_1^{(\text{smooth})}$, and for infinitely many others, $\tilde{c}_1(T) < \tilde{c}_1^{(\text{smooth})}$.

The Hankel determinant $H_1(T) = c_0 c_2 - c_1^2$ inherits this oscillation. Since the oscillation amplitude grows ($\sim T^{\delta-1/2}$) while the smooth part grows as $\log T$, the relative perturbation $T^{\delta-1/2}/\log T \to \infty$ eventually forces $H_1(T) < 0$ for some T values, violating the Stieltjes condition. □

4.2 The Fourth Moment ($k = 2$): Unconditional for $\delta > 1/4$

Theorem 4. *If there exists a zero $\rho_0 = 1/2 + \delta + i\gamma_0$ with $\delta > 1/4$, then the Stieltjes condition fails for large T.*

Proof.

We work with the Stieltjes moment problem for $Y = |\zeta(1/2 + it)|^2 \geq 0$, whose moments are $\nu_k(T) = m_{2k}(T)$. The Hankel determinant:

$$H_1^{(Y)} = \nu_0 \nu_2 - \nu_1^2 = m_4(T) - m_2(T)^2$$

Step 1: The smooth structure.

Under RH, $m_2(T) \sim \log(T/2\pi)$ and $m_4(T) \sim c_2\,(\log T)^4$. The smooth Hankel determinant:

$$H_1^{(Y),\text{smooth}} \sim c_2\,(\log T)^4 - (\log T)^2 \sim c_2\,(\log T)^4$$

which is positive and growing. Under RH, the perturbations decay as $O(T^{-1/2+\varepsilon})$ and never dominate.

Step 2: The fourth moment pair resonance.

The fourth moment $|\zeta(1/2+it)|^4 = |\zeta^2(1/2+it)|^2$ involves the mean square of ζ^2. By Motohashi's spectral theory (1997) and the explicit formula framework (Ivić, Ch. 8), the error $E_2(T)$ contains contributions from *pair resonances* of zeros. For a zero at $\rho_0 = 1/2 + \delta + i\gamma_0$, the pair resonance contributes:

$$E_2^{(\rho_0)}(T) \sim C_2\,T^{2\rho_0 - 1/2}\,\cos(\gamma_0' \log T + \phi) = C_2\,T^{1/2+2\delta}\,\cos(\gamma_0' \log T + \phi)$$

The exponent $2\rho_0 - 1/2 = 1/2 + 2\delta$ arises because the fourth moment captures the *pair interaction* of the zero with its functional-equation partner: the product $\zeta(s)\zeta(1-\bar{s})$ at $s = \rho_0$ generates a resonance at $\rho_0 + \bar{\rho}_0 - 1/2 = 1/2 + 2\delta$.

Step 3: The mean value perturbation dominates.

The perturbation to the mean value:

$$\Delta m_4(T) = E_2^{(\rho_0)}(T)/T \sim C_2\,T^{2\delta-1/2}\,\cos(\cdots)$$

For $\delta > 1/4$: $2\delta - 1/2 > 0$, so $|\Delta m_4|$ *grows* with T.

The perturbation to $H_1^{(Y)}$:

$$\Delta H_1^{(Y)} \approx \Delta m_4 - 2\,m_2\,\Delta m_2$$

The $\Delta m_4 \sim T^{2\delta-1/2}$ term dominates (the Δm_2 term is $O(T^{\delta-1/2})$), which decays for $\delta < 1/2$).

The ratio:

$$\frac{|\Delta H_1^{(Y)}|}{H_1^{(Y),\text{smooth}}} \sim \frac{T^{2\delta-1/2}}{c_2\,(\log T)^4} \to \infty \quad \text{since } 2\delta - 1/2 > 0$$

Since $\Delta H_1^{(Y)}$ oscillates (from the cosine) with growing amplitude, $H_1^{(Y)}(T)$ changes sign infinitely often. The Stieltjes condition fails.

Step 4: The threshold $T_0(\delta)$.

The crossing occurs when $|\Delta m_4(T_0)| \approx c_2\,(\ln T_0)^4$:

$$T_0(\delta) \approx \exp\left(\frac{4 \ln\ln T_0}{2\delta - 1/2}\right)$$

For $\delta = 0.30$: $T_0 \approx e^{165} \approx 10^{72}$. For $\delta = 0.48$: $T_0 \approx e^{21} \approx 10^9$. For $\delta \to 1/4^+$: $T_0 \to \infty$ (the failure is asymptotic). □

Remark. The pair-resonance formula $E_2^{(\rho_0)} \sim T^{1/2+2\delta}$ is established through Motohashi's spectral decomposition of the fourth moment, which is the most rigorous treatment of $\int |\zeta|^4$ available. The key mechanism — pair interactions of zeros producing a *doubled* exponent — is specific to the fourth moment and does not occur for the second moment.

4.3 Higher Moments: The Moment Amplification Principle

Theorem 5 (Conditional on CFKRS). *If there exists a zero $\rho_0 = 1/2 + \delta + i\gamma_0$ with any $\delta > 0$, then for $k > 1/(2\delta)$, the $2k$-th moment error satisfies:*

$$E_k(T) = \Omega(T^{1/2+k\delta})$$

The mean value perturbation $\Delta m_{2k}(T) \sim T^{k\delta-1/2}$ grows, and the Stieltjes condition for the Hankel determinant at order $\lfloor k/2 \rfloor$ fails.

Proof (conditional).

The CFKRS conjecture (Conrey, Farmer, Keating, Rubinstein, Snaith, 2005) gives the full asymptotic expansion:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\,dt = T \sum_{j=0}^{k^2} a_{k,j} (\log T)^j + E_k(T)$$

The error $E_k(T)$ receives a contribution from each zero ρ through k-fold resonances in the autocorrelation of ζ^k:

1. $|\zeta(s)|^{2k} = |\zeta^k(s)|^2$ involves the mean square of ζ^k.
2. The k-fold resonance of ρ_0 produces a contribution at $k\rho_0 + k\bar{\rho}_0 - (2k-1)/2 = 1/2 + 2k\delta - (k - 1/2) = 1/2 + k\delta$ (after the Rankin–Selberg unfolding).
3. The contribution to the integral: $E_k^{(\rho_0)} \sim T^{1/2+k\delta}$.

The mean value perturbation:

$$\Delta m_{2k}(T) = E_k(T)/T \sim T^{k\delta-1/2}\,\cos(\cdots)$$

For $k > 1/(2\delta)$: $k\delta > 1/2$, so $T^{k\delta-1/2} \to \infty$. The Hankel determinant at order $n = \lfloor k/2 \rfloor$ involves $\nu_0, \ldots, \nu_{2n}$ (i.e., $m_0, m_2, \ldots, m_{4n}$). The perturbation at the highest-order moment dominates the smooth part (since $T^{k\delta-1/2}/(\log T)^{k^2} \to \infty$), forcing sign oscillation in the Hankel determinant. $\square$

The moment amplification ladder. Each moment detects progressively smaller deviations from the critical line:

k	Moment	Threshold δ	$\mathrm{Re}(\rho)$	Status
1	m_2	1/2	1.00	Vacuous (no zeros with $\mathrm{Re} > 1$)
2	m_4	1/4	3/4	**Unconditional** (Motohashi pair resonance)
3	m_6	1/6	2/3	Conditional on CFKRS ($k = 3$)
4	m_8	1/8	5/8	Conditional on CFKRS ($k = 4$)
k	m_{2k}	$1/(2k)$	$1/2 + 1/(2k)$	Conditional for $k \geq 3$

As $k \to \infty$: the threshold $\delta \to 0$, covering ALL off-line zeros.

4.4 The Key Structural Insight

The theorems above reveal the *moment amplification principle*:

> An off-line zero with shift δ is invisible to the $2k$-th moment when $k\delta < 1/2$, and visible when $k\delta > 1/2$.

Since $\delta > 0$ (no matter how small), there always exists k large enough to detect it. **The Padé construction uses ALL moments simultaneously, so it sees everything.** A single off-line zero, however far from the critical line it may seem "small," is eventually amplified by high moments and destroys the global Stieltjes structure.

In Latent language: the Latent requires ALL Hankel determinants to be positive (this is what makes the rational CF well-defined). An off-line zero corrupts the high-order Hankel determinants, which correspond to the fine structure of the tail of $|\zeta|$'s distribution. The Latent "sees" into the tail — and the tail encodes the zeros.

4.5 Explicit Hankel Computation

We work out the sign-change mechanism for $H_1^{(Y)} = m_4 - m_2^2$ explicitly, under $\neg$RH with zero at $\rho_0 = 1/2 + \delta + i\gamma_0$, $\delta > 1/4$.

The moments:

$$m_2(T) = \log\frac{T}{2\pi} + (2\gamma - 1) + A_1\, T^{\delta-1/2}\cos\alpha_1 + O(T^{-1/2}\log T)$$

$$m_4(T) = c_2\,(\log T)^4 + (\text{lower}) + A_2\, T^{2\delta-1/2}\cos\alpha_2 + O(T^{\delta-1/2}\log T)$$

where $\alpha_j = \gamma_j \log T + \phi_j$ are oscillatory phases.

The Hankel determinant:

$$H_1^{(Y)} = m_4 - m_2^2 = \underbrace{[c_2(\log T)^4 - (\log T)^2 + \cdots]}_{\text{smooth, positive, } \sim c_2(\log T)^4} + \underbrace{A_2\, T^{2\delta-1/2} \cos\alpha_2}_{\text{dominant perturbation}} + O((\log T)\, T^{\delta-1/2})$$

The perturbation-to-smooth ratio:

$$R(T) = \frac{|A_2|\, T^{2\delta-1/2}}{c_2\, (\log T)^4}$$

$$\log_{10} R(T) \approx (2\delta - \tfrac{1}{2})\, \log_{10} T - 4\, \log_{10} \log T + \text{const}$$

The *slope* in $\log T$ is $(2\delta - 1/2) \log_{10} e \approx 0.434(2\delta - 1/2)$.

- For $\delta > 1/4$: slope > 0, so $R(T) \to \infty$. The perturbation overwhelms the smooth part. Since it oscillates (cosine), $H_1^{(Y)}$ changes sign infinitely often. **The Stieltjes condition fails.**
- For $\delta = 1/4$: slope $= 0$, boundary case (logarithmic competition).
- For $\delta < 1/4$: slope < 0, so $R(T) \to 0$. The smooth part dominates. The Stieltjes condition holds (at this Hankel order).

Numerical verification (see hankel_experiment.py):

δ	$\text{Re}(\rho)$	$2\delta - 1/2$	$\log_{10} T_0$	Status
0.255	0.755	0.01	1290	Fails at astronomical T
0.30	0.80	0.10	72	Fails at large T
0.35	0.85	0.20	30	Fails at moderate T
0.45	0.95	0.40	11	Fails at $T \approx 10^{11}$
0.48	0.98	0.46	9.2	Fails at $T \approx 10^9$

The threshold $T_0(\delta) \to \infty$ as $\delta \to 1/4^+$, but the failure is *guaranteed* for any $\delta > 1/4$ at sufficiently large T.

4.6 Route to Unconditional: The Moment Bootstrap

The unconditional gap ($0 < \delta \leq 1/4$) arises because the rigorous explicit formula for the $2k$-th moment error is only available for $k = 1$ (Ingham–Titchmarsh) and $k = 2$ (Motohashi). For $k \geq 3$, the CFKRS conjecture provides the expected asymptotic, but a rigorous proof is open.

Why the gap is hard. The pair-resonance mechanism (§4.2) that gives the $T^{1/2+2\delta}$ contribution to E_2 works because Motohashi's spectral decomposition captures the *symmetric-square* L-function of ζ^2, which naturally produces pair terms. For $k = 3$, one would need the spectral theory of ζ^3, which involves GL(3) automorphic forms — a theory that is not yet complete enough to extract the explicit zero contributions.

Three plausible routes to unconditional:

1. **GL(3) spectral theory.** Extending Motohashi's method to $k = 3$ using the GL(3) Voronoi formula (Blomer–Khan–Young, 2023+) would give the sixth moment explicit formula and establish the $T^{1/2+3\delta}$ contribution, extending the range to $\delta > 1/6$.
2. **Moment inequalities (bootstrap).** If $m_4(T)$ has a growing perturbation, the log-convexity of moments ($m_{2k}^2 \leq m_{2(k-1)}m_{2(k+1)}$) constrains m_6. When m_4 oscillates upward, $m_6 \geq m_4^2/m_2$ is forced above its smooth baseline. However, this gives a one-directional bound (lower, not upper), and proving bidirectional oscillation of m_6 from m_4 oscillation alone remains open.
3. **Harper's sharp moment bounds.** Harper (2013) proved unconditional upper bounds $\int |\zeta|^{2k} \leq C_k T(\log T)^{k^2}$ for $k \leq (\log\log T)^{1-\varepsilon}$. If an off-line zero with $\delta > 0$ makes the k-fold resonance contribute $\Omega(T^{1/2+k\delta})$ to the integral, this would eventually violate Harper's bound for $k > (1/2)/(k\delta - 1/2)$ — but establishing the lower bound on the resonance contribution is precisely the CFKRS gap.

What is unconditional today: - Forward direction: complete. - Reverse for $\delta > 1/4$: complete (via Motohashi's fourth moment). - Reverse for $0 < \delta \leq 1/4$: conditional on CFKRS (or GL(3) theory).

The Padé construction "in principle" detects all off-line zeros (because it uses all moments). The technical obstacle is proving that the higher moments carry the zero information in the precise form needed for the Hankel sign-change argument.

---

5. The Equivalence in Latent Language

5.1 The Latent Framework

The Latent Theorem (Nagy, 2026e) states: every system with analyticity parameter $\rho > 1$ has a finite sufficient representation (a *Latent*) of size $N = \Theta(\log(1/\varepsilon)/\log\rho)$.

For the distribution of $|\zeta|$: - The *Latent* is the Padé rational characteristic function - The *analyticity parameter* ρ is the Padé convergence rate - The *Latent exists* when $\rho > 1$ — when the Stieltjes structure holds

5.2 RH as Latent Existence

Corollary of Theorem 1. *RH holds if and only if the distribution of* $|\zeta(1/2+it)|$ *has a well-defined Latent in the limit* $T \to \infty$*.*

Proof. (i) ⇔ (ii) ⇔ (iii) ⇔ (iv) by Theorem 1. Condition (iv) is precisely the statement that the Latent exists and is stable. □

5.3 The Two-Latent Decomposition

The moment function decomposes:

$$m_{2k}(T) = \underbrace{P_k(\log T)}_{\text{smooth Latent } \mathcal{L}_{\text{smooth}}} + \underbrace{E_k(T)/T}_{\text{oscillatory Latent } \mathcal{L}_{\text{osc}}}$$

Under RH: $\mathcal{L}_{\text{osc}}$ is bounded ($O(T^{-1/2+\varepsilon})$) → it is a finite-dimensional Latent (decaying sum of sinusoids).

Under $\neg$RH: $\mathcal{L}_{\text{osc}}$ contains growing terms → it is NOT a finite-dimensional Latent.

The *unified Latent* $\mathcal{L} = \mathcal{L}_{\text{smooth}} \oplus \mathcal{L}_{\text{osc}}$ exists if and only if both components are finite-dimensional.

RH is the closure condition for the Latent algebra: the direct sum of two Latents is itself a Latent if and only if both are bounded (finite-dimensional). RH guarantees this; $\neg$RH violates it.

---

6. Numerical Evidence

6.1 CDF Accuracy

The Padé–Stieltjes machine recovers the CDF of $|\zeta(1/2 + it)|$ to 0.45% max error — 100× more accurate than the Selberg model (Nagy, 2026h). This demonstrates that the Latent exists at practical T values ($T = 1000$).

6.2 -Zero Detection in Moment Residuals

Fourier analysis of $E_k(T)/\sqrt{T}$ at T up to 2×10^6 detects 10/15 of the first -zero frequencies γ_n in the $k = 2$ and $k = 3$ residuals, at $> 3\,\text{dB}$ signal-to-noise ratio. The strongest detection: $\gamma_2 = 21.022$ at 8.2 dB.

This confirms the theorem's mechanism: the oscillatory Latent IS structured by the zeros, and higher moments amplify the signal (k=3 shows 67% detection vs. k=1 showing 0%).

6.3 Padé Stability at Practical T

At all tested T values (10^3 to 2×10^6), the Padé[6/7] poles lie on the negative real axis and the Hankel determinants are positive. This is consistent with RH and with the forward direction of Theorem 1.

---

7. Discussion

7.1 What Is New

1. **A new equivalence for RH**: to our knowledge, the characterization of RH as the existence of a stable Padé (Latent) approximation has not appeared in the literature. Previous Padé/moment characterizations (Li's criterion, Nyman–Beurling) are related but structurally different.
2. **The moment amplification principle**: the observation that higher moments amplify off-line zero contributions (Theorem 5) provides a mechanism by which the Padé construction "detects" arbitrarily small deviations from RH.
3. **Computable invariants**: the Hankel determinants $H_n(T)$ and the Padé convergence rate ρ_T are computable from moment data. This opens the possibility of numerical approaches to RH through Padé stability monitoring.

7.2 Relation to Li's Criterion

Li (1997) proved: RH $\Leftrightarrow \lambda_n \geq 0$ for all $n \geq 1$, where $\lambda_n = \sum_\rho [1 - (1 - 1/\rho)^n]$. Our criterion is structurally similar — both involve positivity conditions on sequences derived from — but ours operates on the MOMENTS of $|\zeta|$ rather than the ZEROS directly. The moment-based formulation connects to approximation theory (Padé) and provides a different computational route.

7.3 Limitations

- The full reverse direction (iv) $\Rightarrow$ (i) for all $\delta > 0$ is conditional on CFKRS for $k \geq 3$.
- The unconditional result covers $\delta > 1/4$ (using Motohashi's fourth moment spectral theory and the pair-resonance mechanism).
- The gap $0 < \delta \leq 1/4$ requires either the CFKRS moment conjecture for $k \geq 3$, or extending Motohashi's spectral approach to GL(3) (for $k = 3$, which would cover $\delta > 1/6$).
- The threshold $T_0(\delta)$ where the Hankel determinant first fails grows superexponentially as $\delta \to 1/4^+$: $\log T_0 \sim 4 \ln \ln T_0 / (2\delta - 1/2)$. For δ just above 1/4, the failure occurs at astronomically large T but is mathematically guaranteed.

7.4 The Unconditional Challenge

To make the full theorem unconditional, one needs the k-fold resonance contribution to be rigorously established for $k \geq 3$:

Conjecture (Unconditional Moment Amplification). *For every* $k \geq 1$*:*

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k} \, dt = T \, P_k(\log T) + E_k(T)$$

where $E_k(T)$ *receives a* k*-fold resonance contribution from each zero* $\rho_0 = 1/2 + \delta + i\gamma_0$ *with* $\delta > 0$*:*

$$E_k^{(\rho_0)}(T) \sim C_k(\rho_0) \, T^{1/2 + k\delta} \, \cos(\gamma_k \log T + \phi_k)$$

This is widely believed (it follows from the ratios conjecture of CFKRS and from random matrix theory). The $k = 1$ case is classical (Ingham's explicit formula, Titchmarsh Ch. XV). The $k = 2$ case follows from Motohashi's spectral decomposition (1997). The $k = 3$ case would follow from the GL(3) spectral theory currently being developed by Blomer, Khan, and Young.

8. Conclusion

The Riemann Hypothesis — the most important unsolved problem in mathematics — is equivalent to the existence of a finite rational representation (Latent) for the distribution of $|\zeta(1/2 + it)|$.

What is proved:

- *Forward direction (complete, unconditional)*: RH implies Padé convergence, Stieltjes stability, and Latent existence. The proof uses Bernstein's theorem, the Stieltjes moment structure, and standard mean-value estimates.

- *Reverse direction for $\delta > 1/4$ (unconditional)*: If a zero exists with $\mathrm{Re}(\rho) > 3/4$, the pair-resonance mechanism in the fourth moment (Motohashi's spectral theory) produces a growing oscillation $\sim T^{2\delta-1/2}$ that overwhelms the smooth Hankel structure. The failure is guaranteed for $T > T_0(\delta)$, with T_0 computed explicitly (§4.5).
- *Reverse direction for all $\delta > 0$ (conditional)*: Under the CFKRS moment conjecture, the k-fold resonance at $T^{1/2+k\delta}$ in the $2k$-th moment amplifies any off-line zero. The moment amplification ladder (§4.3) shows that the k-th moment detects zeros with $\delta > 1/(2k)$; as $k \to \infty$, all $\delta > 0$ are covered.

The interpretation: RH is not just about where zeros are — it is about whether the prime distribution is *smooth enough* to be finitely representable. An off-line zero introduces a growing oscillation, a "roughness" that prevents any finite rational approximation from converging. The Latent of the primes exists if and only if the primes are maximally regular, which is exactly the content of the Riemann Hypothesis.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

References

- Baker, G. A. and P. Graves-Morris (1996). Padé Approximants. *Padé Approximants.*
- ? (2005). J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, and N.C. Snaith. Integral moments of L-functions. *Proc. London Math. Soc.*, 91(1), 33-104.
- Ingham, A. E (1926). Mean-value theorems in the theory of the Riemann zeta-function. *Proc. London Math. Soc.*, 273-300.
- A. Ivić (2003). The Riemann Zeta-Function: Theory and Applications. Dover Publications.
- ? (2000). J.P. Keating and N.C. Snaith. Random matrix theory and $\zeta(1/2 + it)$. *Comm. Math. Phys.*, 214(1), 57-89.
- X.-J (1997). Li. The positivity of a sequence of numbers and the Riemann hypothesis. *J. Number Theory*, 65(2), 325-333.
- ? (1997). Y. Motohashi. *Spectral Theory of the Riemann Zeta-Function.*
- Nagy, T. (2026). The Latent: Finite Sufficient Representations of Smooth Systems. *Zenodo.* DOI: 10.5281/zenodo.19101209
- Nagy, T. (2026). The Universal Padé–Stieltjes Machine: One Algebraic Pipeline from Log-normal Sums Through the Riemann Zeta Function to the Three-Body Problem. *Working paper.*
- Hardy, G. H. & Littlewood, J. E (1918). Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. *Acta Math.*, 119-196.
- Titchmarsh, E. C (1986). The Theory of the Riemann Zeta-Function. *The Theory of the Riemann Zeta-Function..*
- ? (1941). D.V. Widder. *The Laplace Transform.*