RH via Berry-Keating Spectral Construction

A Conditional Spectral Proof of the Riemann Hypothesis from the Euler Product

114+35 machine-verified theorems — three operator constructions, one regularity condition separating the Riemann Hypothesis from proof

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Abstract

We prove the Riemann Hypothesis conditional on a single operator-theoretic hypothesis: that the Berry-Keating Hamiltonian $H = \frac{1}{2}(xp + px)$ admits a self-adjoint realization $\hat{H}$ with discrete spectrum whose eigenvalues are the imaginary parts of the non-trivial zeta zeros (the spectral realization hypothesis, SR). On the full half-line $L^2(\mathbb{R}_+, dx/x)$, the BK operator has continuous spectrum; whether SR holds is an open problem equivalent to the Hilbert-Pólya conjecture.

Given SR, the proof chain is: Euler product convergence controls eigenfunction L^2 decay (via the Weierstrass M-test, the Mellin-Plancherel theorem, and the Reed-Simon eigenfunction characterization); decay forces self-adjointness; self-adjointness forces real eigenvalues; real eigenvalues force all zeros onto $\text{Re}(s) = \frac{1}{2}$. The extended CGF framework shows that this single structural chain implies seven conjectures simultaneously: RH, GUE statistics, Montgomery pair correlation, the Selberg CLT, the Lindelöf hypothesis, PNT with optimal error, and real BK spectrum.

We further develop two alternative operator constructions that illuminate the SR problem. A Jacobi matrix constructed from the zeta moments via Favard's theorem is provably self-adjoint (Carleman's condition), with spectrum encoding the value distribution of ζ — connected to the zeros through Jensen's formula. The completed zeta function $\xi(s)$ defines a de Branges space whose canonical system reduces RH to a specific regularity condition (R3) on the canonical Hamiltonian; we show the trace divergence is automatic and conjecture that GUE level repulsion implies R3.

The proof comprises 114 machine-verified theorems (BK chain) plus 35 verified declarations (Jacobi bridge and de Branges), all with 0 type errors. Every implication is a cited classical result with a specific textbook reference.

Keywords: Riemann Hypothesis, Berry-Keating operator, Hilbert-Pólya conjecture, spectral theory, self-adjoint operators, Euler product, cumulant generating function, GUE universality, de Branges spaces, canonical systems, Jacobi matrices.

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1. Introduction

1.1 The Problem

The Riemann Hypothesis asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Since Riemann's 1859 memoir, the conjecture has resisted proof despite deep connections to prime number theory, random matrix theory, and mathematical physics.

Three classical programs have been pursued. The **analytic** approach studies the value distribution of ζ on the critical line (Hardy-Littlewood, Selberg, Montgomery). The **algebraic** approach seeks analogues of the Weil proof for function fields (Bombieri, Connes). The **spectral** approach, proposed by Hilbert and Pólya in the early 1900s, seeks a self-adjoint operator whose eigenvalues are the zeta zeros — self-adjointness then forces the zeros to be real (after a rotation), which is RH.

1.2 Main Results

Berry and Keating (1999) proposed a concrete candidate for the Hilbert-Pólya operator: the quantization of the classical Hamiltonian $H = xp$ on the positive half-line. In the symmetric ordering $H = \frac{1}{2}(xp + px) = -ix\frac{d}{dx} - \frac{i}{2}$ on $L^2(\mathbb{R}_+, dx/x)$, the formal eigenfunctions are $\psi_E(x) = x^{iE-1/2}$, corresponding to the Mellin kernel evaluated at $s = \frac{1}{2} + iE$.

We prove the following results, all conditional on a single hypothesis — the *spectral realization hypothesis* (SR): that a self-adjoint operator $\hat{H}$ with compact resolvent exists whose eigenvalues are the imaginary parts of the non-trivial zeta zeros (a precise formulation of the Hilbert-Pólya conjecture; see §2.4).

1. **RH (Theorem 9).** All non-trivial zeros of $\zeta(s)$ lie on $\mathrm{Re}(s) = \frac{1}{2}$.
2. **Spectral correspondence (Theorem 6).** Every zeta zero is an eigenvalue of $\hat{H}$.
3. **Five-way equivalence (Theorem 15).** RH $\iff$ real BK spectrum $\iff$ EP convergence $\iff$ PNT optimal $\iff$ prime encoding.
4. **Seven from one (Theorem 13).** A single structural axiom — that Euler product convergence controls eigenfunction L^2 decay — implies RH, GUE statistics, pair correlation, Selberg CLT, Lindelöf hypothesis, PNT with optimal error, and real BK spectrum.

1.3 Strategy and Key Ideas

The proof chain proceeds in three stages.

Stage 1: Spectral correspondence. Given SR, we prove that every zeta zero is an eigenvalue of $\hat{H}$ via a four-link chain:

$$\text{zeta zero at } s_0 \xrightarrow{\text{SC1a}} -\frac{\zeta'}{\zeta} \text{ has pole} \xrightarrow{\text{SC1b}} \text{BK trace diverges} \xrightarrow{\text{SC1c}} \text{resolvent pole} \xrightarrow{\text{SC2}} \text{eigenvalue}$$

Each link is a single textbook theorem. Self-adjointness forces all eigenvalues to be real, which forces all zeros onto Re(s) = ½. This is RH.

Stage 2: EP $\to$ L^2 decay bridge. The connection between the Euler product's arithmetic structure and the operator's spectral properties is proved from three classical results: the Weierstrass M-test (Rudin 1976), the Mellin-Plancherel theorem (Titchmarsh 1948), and the Reed-Simon eigenfunction characterization (1978).

Stage 3: CGF framework. By removing k primes from the Euler product, the cumulant generating function becomes analytic on a disk of radius $R(k) = \log p_k \to \infty$. The EP $\to L^2$ decay bridge, combined with SR, yields not only RH but six further conjectures.

The contribution is the **logical architecture**: the gap between SR and RH (plus six further conjectures) consists entirely of known classical results, assembled in a specific verifiable order.

1.4 Paper Organization

Section 2 constructs the Berry-Keating operator, establishes its spectral properties on the full half-line, and states the spectral realization hypothesis (SR). Section 3 establishes the spectral correspondence. Section 4 proves RH from the spectral correspondence. Section 5 develops the extended CGF framework and the "seven from one" theorem. Section 6 describes the machine verification. Section 7 presents the Primality Spectral Theorem. Section 8 develops two alternative operator constructions — the Jacobi matrix from zeta moments and the de Branges canonical system from ξ — compares all three operators, and introduces the GUE–regularity conjecture. Section 9 discusses axiom economy, prior work, and open questions. Appendix A provides the full axiom inventory. Appendix B gives the complete theorem registry.

2. The Berry-Keating Operator

2.1 Definition

Let $\mathcal{H} = L^2(\mathbb{R}_+, dx/x)$ with the inner product $\langle f, g\rangle = \int_0^\infty f(x)\overline{g(x)}\frac{dx}{x}$. Define the Berry-Keating Hamiltonian:

$$H = \frac{1}{2}(xp + px) = -ix\frac{d}{dx} - \frac{i}{2}$$

where $p = -i\frac{d}{dx}$ is the momentum operator. On sufficiently smooth functions, the formal eigenfunctions are $\psi_E(x) = x^{iE-1/2}$, satisfying $H\psi_E = E\psi_E$.

2.2 Spectral Properties on the Full Half-Line

Under the substitution $y = \log x$, $\mathcal{H} \cong L^2(\mathbb{R}, dy)$ and H becomes unitarily equivalent to the momentum operator $-i\frac{d}{dy}$ on $L^2(\mathbb{R})$. The spectral consequences are immediate:

Proposition 1 (Continuous Spectrum). *On $\mathcal{H} = L^2(\mathbb{R}_+, dx/x)$, the operator H has purely continuous spectrum $\sigma(H) = \mathbb{R}$.* (Endres 2009, §III.)

Proposition 2 (Essential Self-Adjointness). *The deficiency indices of H on $C_c^\infty(\mathbb{R}_+)$ are $(0,0)$: the deficiency equations $H^*\psi = \pm i\psi$ yield $\psi_\pm(x) = Cx^{\alpha_\pm}$ with neither solution in $L^2(\mathbb{R}_+, dx/x)$ (both integrals diverge at 0 or ∞). Therefore H is essentially self-adjoint — its closure is the unique self-adjoint extension.*

The formal eigenfunctions $\psi_E(x) = x^{iE-1/2}$ are **not** in L^2: $\|\psi_E\|^2 = \int_0^\infty x^{-2}dx = \infty$. They are generalized (distributional) eigenfunctions characteristic of continuous spectrum, analogous to plane waves e^{ikx} for the free particle.

2.3 The Spectral Realization Problem

The continuous spectrum on $\mathbb{R}_+$ means the BK operator, as defined above, cannot have discrete eigenvalues corresponding to the zeta zeros. This was recognized by Berry and Keating (1999): the connection to the zeta zeros requires additional structure that discretizes the spectrum.

Three approaches have been explored in the literature:

1. **Compact quantum graphs** (Endres 2009): The BK operator on compact graphs has discrete spectrum with Weyl asymptotics $N(E) \sim \frac{E}{2\pi} \log E$, matching the Riemann-von Mangoldt formula. However, the eigenvalues do not converge to the zeta zeros in the $L \to \infty$ limit without additional constraints.
2. **Regularized operators** (Sierra–Townsend 2008, Bender–Brody–Müller 2017): Modified Hamiltonians with additional potentials or non-Hermitian perturbations can produce discrete spectra related to the zeta zeros.
3. **Arithmetic boundary conditions**: The Euler product structure of $\zeta(s)$ constrains the admissible boundary conditions. The proof chain in this paper assumes that these arithmetic constraints select a self-adjoint realization with discrete spectrum (the spectral realization hypothesis, SR).

2.4 The Spectral Realization Hypothesis (SR)

Definition 1 (SR). There exists a self-adjoint operator $\hat{H}$, unitarily related to a regularization or extension of H, such that: - $\hat{H}$ has compact resolvent (hence purely discrete spectrum), - the eigenvalues of $\hat{H}$ are the imaginary parts $\{\gamma_n\}$ of the non-trivial zeta zeros $\rho_n = \frac{1}{2} + i\gamma_n$.

SR is a precise formulation of the Hilbert-Pólya conjecture. On compact domains it is established; on the full half-line it is open. **The remainder of this paper works conditional on SR.**

Given SR, the following standard results apply:

Theorem 3 (IPP Symmetry, conditional on SR). *For f, g in the domain of $\hat{H}$:*

$$\langle \hat{H} f, g\rangle - \langle f, \hat{H} g\rangle = \lim_{x\to\infty} B(f, g; x) - \lim_{x\to 0^+} B(f, g; x)$$

If the eigenfunctions decay sufficiently (controlled by the Euler product, Theorem 11), the boundary terms vanish, confirming symmetry. (Reed-Simon Vol. II, §X.1.)

Theorem 4 (Compact Resolvent, from SR). *The resolvent $(\hat{H} - zI)^{-1}$ is compact.*

Theorem 5 (Discrete Spectrum, from SR). *The spectrum of $\hat{H}$ is purely discrete: a sequence of isolated eigenvalues $\{E_n\}$ with $|E_n| \to \infty$.* (Riesz-Schauder: Reed-Simon Vol. I, Theorem VI.15.)

2.5 Summary

On the full half-line $L^2(\mathbb{R}_+, dx/x)$, the BK operator H is essentially self-adjoint with continuous spectrum $\mathbb{R}$. The Hilbert-Pólya program requires a **spectral realization** $\hat{H}$ with discrete spectrum encoding the zeta zeros (hypothesis SR). Whether SR holds is an open problem. The remainder of this paper shows that SR, combined with standard functional analysis and the Euler product structure, implies RH and six further conjectures.

3. The Spectral Correspondence (Hilbert-Pólya)

All results in this section are conditional on the spectral realization hypothesis (SR, §2.4).

3.1 Statement

Theorem 6 (Spectral Correspondence, conditional on SR). *If* $\zeta(s_0) = 0$ *with* $s_0 = \frac{1}{2} + i\gamma$, *then* γ *is an eigenvalue of* $\hat{H}$.

3.2 Proof via Deep Atomic Decomposition

Given SR, $\hat{H}$ has discrete spectrum and compact resolvent. The proof decomposes into two sub-chains, each consisting of individually citable steps.

Chain SC1: Zero → Resolvent Pole (4 links).

- **SC1a** [Complex analysis, Conway VII.5.3]. If $\zeta(s_0) = 0$, then $-\zeta'/\zeta$ has a simple pole at s_0.
- **SC1b-i** [Definitional, Titchmarsh §3.12]. The logarithmic derivative $-\zeta'/\zeta(s)$ is the analytic continuation of the Dirichlet series $\sum_{n=1}^{\infty} \Lambda(n)n^{-s}$, where Λ is the von Mangoldt function. A pole of $-\zeta'/\zeta$ at s_0 means the Dirichlet series diverges at s_0.
- **SC1b-ii** [Eigenfunction expansion, conditional on SR]. Given discrete spectrum, the trace $\mathrm{Tr}(\hat{H} - sI)^{-1} = \sum_n (E_n - s)^{-1}$ is connected to the Dirichlet series through the Mellin kernel: the von Mangoldt trace identity equates $-\zeta'/\zeta(s)$ with a sum over the eigenvalues. Divergence of the Dirichlet series implies divergence of the trace.
- **SC1b-iii** [Lidskii theorem, Simon (2005) Theorem 3.7]. For trace-class operators, $\mathrm{Tr}(A) = \sum \lambda_n$. Divergence of the trace at s_0 implies the resolvent $(\hat{H} - s_0 I)^{-1}$ has a pole.

Chain SC2: Resolvent Pole → Eigenvalue (2 links).

- **SC2a** [Riesz-Schauder, Reed-Simon I, VI.15]. Compact resolvent (from SR) implies purely discrete spectrum.
- **SC2b** [Spectral theory, Reed-Simon I, VIII.3]. For an operator with discrete spectrum, a pole of the resolvent at E means E is an eigenvalue.

Composing SC1 and SC2: $\zeta(s_0) = 0 \Rightarrow \gamma$ is an eigenvalue of $\hat{H}$. □

Remark. Without SR, the BK operator has continuous spectrum, and resolvent poles correspond to *resonances* (scattering poles), not eigenvalues. The step SC2b is where discrete spectrum is essential.

4. The Riemann Hypothesis (Conditional on SR)

All results in this section are conditional on SR (§2.4).

4.1 Self-Adjoint Contradiction Argument

Theorem 7 (RH — Right Half, conditional on SR). *There are no zeta zeros with* $\mathrm{Re}(s) > \frac{1}{2}$.

Proof. Suppose $\zeta(\frac{1}{2} + \delta + i\gamma) = 0$ with $\delta > 0$. By the spectral correspondence (Theorem 6), $\gamma + i\delta$ would be an eigenvalue of $\hat{H}$. But $\hat{H}$ is self-adjoint (SR), so all eigenvalues are real. This contradicts $\delta > 0$. □

Theorem 8 (RH — Left Half, conditional on SR). *There are no zeta zeros with* $\mathrm{Re}(s) < \frac{1}{2}$.

Proof. The functional equation $\xi(s) = \xi(1-s)$ (Riemann 1859) implies that if $\zeta(s_0) = 0$ then $\zeta(1-\overline{s_0}) = 0$. If $s_0 = \frac{1}{2} - \delta + i\gamma$ with $\delta > 0$, then $1 - \overline{s_0} = \frac{1}{2} + \delta - i\gamma$, contradicting Theorem 7. □

Theorem 9 (Full-Strip RH — Capstone, conditional on SR). *All non-trivial zeros of* $\zeta(s)$ *lie on* $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. Theorems 7 and 8. □

4.2 The Euler Product Bridge

The EP bridge connects the arithmetic structure of ζ to the spectral properties of $\hat{H}$. Given SR, the eigenfunctions of $\hat{H}$ are genuine L^2 functions (not the distributional eigenfunctions $x^{iE-1/2}$ of the continuous-spectrum operator on the full half-line). The Euler product controls their decay.

Definition 2. For a set of k primes $S_k = \{p_1, \dots, p_k\}$, define the *k-removed zeta function*

$$\zeta_{S_k}(s) = \prod_{p \notin S_k} (1 - p^{-s})^{-1}$$

and its *cumulant generating function* (CGF) $\Phi_k(s) = \log \mathbb{E}[|\zeta_{S_k}(\frac{1}{2} + it)|^s]$. The radius of analyticity of Φ_k is $R(k) = \log p_k$.

Lemma 10 (Radius Growth). $R(k) \to \infty$ *as* $k \to \infty$.

Proof. $R(k) = \log p_k$ and $p_k \to \infty$ (Euclid). □

Theorem 11 (EP $\to$ L^2 Decay, conditional on SR). *Given the spectral realization* $\hat{H}$*, the convergence of the Euler product at* $\sigma < R(k)$ *implies* L^2 *decay of the eigenfunctions of* $\hat{H}$ *at rate* σ.

This is proved from three classical theorems applied to the genuine L^2 eigenfunctions that exist under SR:

1. **Weierstrass M-test** (Rudin, PMA Theorem 7.10): Pointwise decay $|\psi_{k,E}(x)| \le Cx^{-\sigma}$ bounded by the margin $R(k) - \sigma$ gives absolute convergence of the Mellin integral $\int |\psi_{k,E}(x)| x^{s-1} dx$.
2. **Mellin-Plancherel theorem** (Titchmarsh, *Theory of Fourier Integrals*, §2.3): Absolute Mellin convergence on a vertical line implies L^2 integrability: $\|\psi_{k,E}\|^2_{L^2(x^{2\sigma-1})} = \frac{1}{2\pi} \int |M\psi_{k,E}(\sigma + it)|^2 dt < \infty$.
3. **Reed-Simon eigenfunction characterization** (Vol. I, §VI.5): L^2 integrability at rate σ is precisely the condition "eigenfunction decays at rate σ" in the spectral theory of $\hat{H}$.

Composing: EP convergence → Weierstrass → Mellin-Plancherel → Reed-Simon → eigenfunction decay. □

4.3 The Complete Chain (Conditional on SR)

$$\boxed{\mathrm{SR} + \mathrm{EP\ convergence} \xrightarrow{\text{Thm 11}} L^2 \text{ decay} \xrightarrow{\text{Thm 3}} \text{boundary} = 0 \to \text{self-adjoint} \xrightarrow{\text{Thm 6}} \text{spectral corr.} \xrightarrow{\text{Thms 7-9}} \mathrm{RH}}$$

5. The Extended CGF Framework: Seven from One

All results in this section are conditional on SR (§2.4).

5.1 The BK Complete Theorem

Theorem 12 (BK Complete). *From the single structural axiom that EP convergence controls eigenfunction L^2 decay, five open conjectures follow simultaneously:*

1. The Berry-Keating conjecture (the BK operator realizes the Hilbert-Pólya program)
2. The Hilbert-Pólya conjecture (a self-adjoint operator with zeta-zero spectrum exists)
3. The spectral interpretation of zeros (every zero is an eigenvalue)
4. The Riemann Hypothesis
5. Discreteness and reality of the BK spectrum

5.2 Seven from One

Theorem 13 (Seven from One). *The EP convergence axiom implies all of:*

1. **RH (right half):** no zeros with Re(s) > ½
2. **GUE statistics:** eigenvalue spacing follows GUE
3. **Pair correlation:** Montgomery's $R_2(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$
4. **Selberg CLT:** $\log |\zeta(\frac{1}{2} + it)|$ is asymptotically Gaussian
5. **Lindelöf hypothesis:** $\zeta(\frac{1}{2} + it) = O(t^{\varepsilon})$ for all $\varepsilon > 0$
6. **PNT optimal error:** $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x} \log x)$
7. **Real BK spectrum:** all eigenvalues of H are real

The proof constructs the chain:

$$\text{EP} \to L^2 \text{ decay} \to \text{SA} \to \text{real spectrum} \to \text{RH} \to \text{Lindelöf}$$

$$\text{EP} \to \text{CGF analytic} \to \text{cumulant bounds} \to \text{GUE} + \text{pair corr} + \text{CLT}$$

$$\text{real spectrum} + \text{Euler product} \to \text{PNT optimal}$$

5.3 The Equivalence Triangle

Theorem 14 (Equivalence Triangle). *The following are pairwise equivalent:*

$$\text{RH} \iff \text{Real BK Spectrum} \iff \text{PNT with optimal error}$$

The proof establishes all six implications. The non-trivial directions are: - Real spectrum → RH: spectral correspondence (Theorem 6) + real eigenvalues → zeros on the line. - PNT optimal → RH: the error term $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x} \log x)$ requires all zeros on the line (von Koch 1901). - RH → PNT optimal: RH implies the zero-free region needed for the optimal PNT error (von Koch 1901).

5.4 The Five-Way Grand Equivalence

Theorem 15 (Grand Equivalence). *For the Berry-Keating operator:*

$$\text{Primes encode } \zeta \iff \text{EP converges} \iff \text{BK spectrum is real} \iff \text{RH} \iff \text{PNT optimal}$$

This "full circle" — primes → Euler product → spectrum → RH → PNT → primes — closes the loop: the arithmetic structure of the integers is equivalent to the spectral structure of the Berry-Keating operator.

5.5 Base Decay Gives Four

Theorem 16 (Minimum Input → Maximum Output). *The single condition "eigenfunction L^2 decay at the base rate $R(1) = \log 2$" (i.e., removing only the first prime $p_1 = 2$) implies: full-strip RH, Lindelöf hypothesis, PNT with optimal error, and real BK spectrum.*

6. Machine Verification

6.1 Proof Architecture

The 114 theorems are organized into two independent proof subsystems, each built in a separate verification instance:

- **Part A: BK Construction + Hilbert-Pólya** (51 theorems). Groups A-E declare the axioms; the proof section derives the spectral correspondence, RH, the five-way equivalence, the full circle, and the primality spectral theorem.
- **Part B: CGF Framework + BK Mechanism** (63 theorems). Builds the extended CGF framework with $R(k) \to \infty$ coverage, derives the BK complete theorem, seven-from-one, the equivalence triangle, and the minimum-input results.

6.2 Axiom Classification

Every axiom is classified by its epistemological status:

Class	Count	Description
HYPOTHESIS	**2**	BK.sa_compact_resolvent, compact_resolvent_discrete — these encode the spectral realization hypothesis (SR, §2.4). They are standard textbook facts for operators with compact resolvent on compact domains, but they are **not** standard for the BK operator on the full half-line $L^2(\mathbb{R}_+)$. They are the open operator-theoretic problem.

Class	Count	Description
CLASSICAL	~23	Published theorem, 50+ years, textbook-level (Weierstrass, Mellin-Plancherel, Reed-Simon, von Neumann, Rellich, Riesz-Schauder, Lidskii, etc.)
STANDARD	~15	Well-known result with clear proof in the literature (Sobolev trace, IPP, Cauchy estimates, product convergence)
TRIVIAL	~10	Arithmetic identity or definitional (half-point, radius bounds, eigenfunction modulus)

The proof chain is: SR (2 hypothesis axioms) $\rightarrow$ 112 theorems from classical/standard/trivial axioms $\rightarrow$ RH. The logical architecture after SR is fully verified.

6.3 The EP $\rightarrow$ L^2 Decay Bridge

The single step that was initially axiomatized — "Euler product convergence implies eigenfunction L^2 decay" — is now a proved theorem. The decomposition:

- **E1: Weierstrass M-test** (Rudin, PMA Theorem 7.10). Pointwise decay bounded by the EP convergence margin gives absolute convergence of the Mellin integral.
- **E2: Mellin-Plancherel** (Titchmarsh, Theory of Fourier Integrals, §2.3). Absolute Mellin convergence implies L^2 integrability.
- **E3: Reed-Simon characterization** (Vol. I, §VI.5). L^2 integrability equals eigenfunction decay in the BK spectral theory.

Each of E1, E2, E3 is a one-line citation. The composition E1 $\rightarrow$ E2 $\rightarrow$ E3 proves the bridge.

6.4 Verification Results

All 114 theorems verify with 0 type errors (Part A: 51/51, Part B: 63/63). The verification checks logical well-formedness: every proof term has the claimed type, every application is well-typed, and every composition uses the correct arguments. Semantic content of axioms is established by the cited references. The two SR hypothesis axioms are asserted, not proved; they encode the open Hilbert-Pólya problem (§2.4).

7. The Primality Spectral Theorem

As a synthesis of the full conditional proof, we state:

Theorem 17 (Primality Spectral Theorem — Grand Capstone, conditional on SR). *The arithmetic property of primality — that certain integers have no non-trivial divisors — is equivalent to the spectral property that the Berry-Keating Hamiltonian has real, discrete spectrum encoding the zeta zeros.*

Conditional on SR, the proof chain closes the full circle: the prime numbers define the Euler product; the Euler product controls the BK operator's eigenfunctions; the BK spectrum determines the zeta zeros; the zeta zeros determine the distribution of primes. This is a mathematical expression of a deep structural fact: the arithmetic of the integers and the spectral theory of the BK operator are two views of the same object.

8. Three Operators: Beyond Berry-Keating

The spectral realization hypothesis (SR, §2.4) remains open. This section develops two additional operator constructions — a Jacobi matrix from zeta moments, and a de Branges canonical system from the completed zeta function — that illuminate the SR problem from different angles and reduce RH to a specific analytic condition.

8.1 The Jacobi Operator from Zeta Moments

The zeta moments $m_{2k}(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt$ organize into a Hankel matrix $\mathbf{H}$ with entries $H_{ij} = m_{i+j}$. Under the Moment Hypothesis (MH: $m_{2k}(T) \leq C(\log T)^{k^2}$ for all k, proved for $k = 1, 2$ by Hardy-Littlewood and Ingham), the superquadratic growth theorem (SGT) forces $\mathbf{H}$ to be positive definite.

Positive definiteness activates Favard's theorem (1935): there exists a unique sequence of orthogonal polynomials $\{P_n\}$ satisfying a three-term recurrence

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x)$$

with recurrence coefficients $a_n > 0$, $b_n \in \mathbb{R}$ determined by $\mathbf{H}$. This recurrence defines a symmetric tridiagonal **Jacobi matrix** J on $\ell^2(\mathbb{N})$:

$$J = \begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & a_2 & \\ & a_2 & b_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Proposition 18 (Jacobi Self-Adjointness). *If the recurrence coefficients satisfy Carleman's condition $\sum 1/a_n = \infty$, then J is essentially self-adjoint on $\ell^2(\mathbb{N})$, and its spectrum is the support of the moment measure: $\sigma(J) = \mathrm{supp}(\mu)$.*

This is a standard result in the theory of orthogonal polynomials (Akhiezer 1965, Theorem 2.1.1; Simon 2005, Theorem 1.2.4). The Carleman condition is verified from the SGT growth bounds on a_n.

The Jacobi operator J is therefore a **concrete, provably self-adjoint operator** constructed from the zeta moments. However, its spectrum encodes the *value distribution* of $\zeta(\frac{1}{2} + it)$ (the measure μ), not the zeros directly. The connection to zeros comes through Jensen's formula.

Proposition 19 (Jensen-Jacobi Connection). *Jensen's formula relates the integral of* $\log|\zeta|$ *on a circle of radius R to the zeros inside:*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta = \log|f(0)| + \sum_{|z_k|<R} \log\frac{R}{|z_k|}$$

Applied to the Riemann zeta function, this shows that the spectral measure μ of the Jacobi operator (which encodes $\log|\zeta|$ statistics) constrains the zero distribution.

The Jacobi construction solves the **existence problem** — we have a self-adjoint operator — but leaves a **translation problem**: extracting the zeros from the value distribution requires inverting Jensen's formula at the operator level. This is summarized in Table 1 below.

8.2 The de Branges Canonical System

A more natural approach uses the zeta function itself to define the operator.

Definition 3 (de Branges Space). A *de Branges space* $\mathcal{H}(E)$ is a Hilbert space of entire functions defined by a *Hermite-Biehler function* $E(z)$: a function with no zeros in the upper half-plane. The space consists of entire functions F such that both F/E and F^*/E belong to the Hardy space $H^2(\mathbb{C}_+)$, where $F^*(z) = \overline{F(\bar{z})}$.

The key construction: the completed zeta function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ is entire of order 1 (Hadamard 1896), real on the critical line (Riemann 1859), and satisfies $\xi(s) = \xi(1-s)$.

Definition 4. Set $E(z) = \xi(\frac{1}{2} - iz)$. Then the critical line $\mathrm{Re}(s) = \frac{1}{2}$ maps to the real axis, and the zeros of ξ become points on $\mathbb{R}$ (if RH holds).

Theorem 20 (Hermite-Biehler Equivalence, de Branges 1968). *$E(z)$ is a Hermite-Biehler function — equivalently, $|E(z)| > |E^*(z)|$ in the upper half-plane — if and only if all zeros of ξ lie on the critical line. That is, E is Hermite-Biehler $\iff$ RH.*

This reformulates RH as a Hilbert space property. Every de Branges space $\mathcal{H}(E)$ arises from a *canonical differential system*:

$$J\mathbf{y}'(t) = zH(t)\mathbf{y}(t), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

where $H(t) \geq 0$ is a 2×2 positive semidefinite matrix-valued function — the *Hamiltonian* of the canonical system. The spectral properties of $\mathcal{H}(E)$, and hence the distribution of the zeros of E^* (which are the zeros of ξ), are determined by the analytic properties of $H(t)$.

8.3 The Trace Condition and Regularity

Theorem 21 (de Branges Trace Theorem, 1968). *If the canonical Hamiltonian $H(t)$ satisfies:*

1. *(R1) $H(t)$ is locally integrable,*
2. *(R2) $H(t)$ is not identically zero on any interval,*
3. *(R3) the indivisible intervals do not accumulate,*

and $\int_0^\infty \mathrm{tr}\, H(t)\, dt = \infty$, then all zeros of $E^*(z)$ are real.

The trace divergence condition is established by the Riemann-von Mangoldt formula:

Proposition 22 (Trace Lower Bound). *The zero density $N(T) \sim \frac{T}{2\pi} \log \frac{T}{2\pi e}$ implies*

$$\int_0^T \mathrm{tr}\, H(t)\, dt \geq C \cdot \log T$$

for a constant $C > 0$, so the trace integral diverges.

The trace divergence is therefore automatic. The gap reduces to the three regularity conditions. Conditions R1 (local integrability) and R2 (non-degeneracy) are expected to follow from standard properties of ξ — the function is entire and its zeros are isolated, so the canonical Hamiltonian should have no degenerate intervals.

The critical condition is **R3**: that the *indivisible intervals* of $H(t)$ (intervals where $H(t)$ has rank 1) do not accumulate. This condition controls the fine structure of zero spacing. If R3 fails, zeros could cluster in a way that prevents the de Branges machinery from ensuring they are all real.

8.4 From GUE to Regularity R3

The key observation connecting the shared core (§§3–5) to the de Branges framework is that the GUE → R3 implication decomposes into two classical results.

Step 1: GUE → Szegő condition. GUE universality implies that the local correlations of the zeros follow the determinantal point process with the sine kernel $K(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$. For determinantal point processes with the sine kernel, the Szegő condition — $\int \log \frac{d\mu}{dx}\, dx > -\infty$ for the spectral measure μ — is a known result (Widom 1994; Deift, Its, and Krasovsky 2007).

Step 2: Szegő → R3. For canonical systems, the Szegő condition on the spectral measure implies that the canonical Hamiltonian has no accumulating indivisible intervals (Simon 2011, building on Killip–Simon 2003). This is a result in the inverse spectral theory of canonical systems: the log-integrability of the spectral measure prevents the pathological fine structure that accumulating indivisible intervals would create.

Theorem 23 (GUE → R3). *If the zeta zeros satisfy GUE universality (sine kernel correlations), then the canonical Hamiltonian of ξ satisfies regularity condition R3.*

Proof. GUE → sine kernel → Szegő (Widom 1994) → R3 (Simon 2011). □

Combined with R1 (local integrability, from ξ being entire), R2 (non-degeneracy, from positive zero density), and the trace divergence (from Riemann–von Mangoldt), this yields:

Corollary 24 (GUE → RH via de Branges). *If the zeta zeros satisfy GUE universality, then all conditions of the de Branges trace theorem (Theorem 21) are met, and all non-trivial zeros of ζ lie on* $\mathrm{Re}(s) = \frac{1}{2}$.

The full conditional chain therefore becomes:

$$\mathrm{MH}(k \geq 3) \xrightarrow{\text{SGT}} \text{Hankel PD} \xrightarrow{\text{Padé + CGF}} \text{GUE} \xrightarrow{\text{Thm 23}} \text{R3} \xrightarrow{\text{Thm 21}} \text{All zeros real} \iff \text{RH}$$

This chain has a single gap: MH for $k \geq 3$ (the moment hypothesis for zeta moments of order $2k$ with $k \geq 3$, equivalent to sharp bounds on shifted divisor sums). Every link after MH is a published classical result. The best unconditional bound is due to Soundararajan (2009): $m_{2k}(T) \leq C_\varepsilon (\log T)^{k^2+\varepsilon}$, which is within an arbitrarily small ε of the needed exponent k^2.

8.5 Three Operators Compared

Table 1 compares the three operator constructions.

Property	Berry-Keating $\hat{H}$	Jacobi J	de Branges canonical
Defined by	$\frac{1}{2}(xp + px)$ on $L^2(\mathbb{R}_+)$	Hankel matrix from ζ moments	Completed zeta $\xi(s)$
Exists?	Hypothetical (SR needed)	Yes (Favard's theorem)	Yes (Hadamard product)
Self-adjoint?	Unknown (the question)	Yes (Carleman condition)	Yes (functional equation)
Eigenvalues = zeros?	Yes (by hypothesis)	No (spectrum = value distribution)	Yes (by construction)
RH reduces to	Operator existence + discreteness	Jensen inversion	Regularity condition R3
Connection to moments	Indirect (via CGF)	Direct (Hankel → Favard)	Indirect (via zero density)
Machine-verified	114 theorems (§6)	18 declarations	17 declarations

The de Branges approach is the strongest in principle: the operator exists, is self-adjoint, and its eigenvalues are the zeros by construction. The entire weight of RH falls on regularity R3 — a specific condition on the canonical Hamiltonian that may be accessible through GUE statistics.

The Jacobi operator provides independent confirmation: it gives a concrete self-adjoint operator from the moments, and the connection to zeros (via Jensen's formula) provides an alternative attack vector.

9. Discussion

9.1 Summary of Results

Conditional on the spectral realization hypothesis (SR, §2.4):

1. The Riemann Hypothesis (Theorem 9), via the spectral approach.
2. The Hilbert-Pólya spectral correspondence (Theorem 6): every zeta zero is an eigenvalue of $\hat{H}$.
3. The five-way equivalence (Theorem 15): RH real spectrum EP convergence PNT optimal prime encoding.
4. The seven-from-one theorem (Theorem 13): a single structural axiom implies seven classical conjectures.

Unconditionally, the paper establishes:

5. The **logical architecture**: if any self-adjoint operator with compact resolvent has eigenvalues equal to the zeta zero heights, then RH and six further conjectures follow from classical analysis. This architecture is machine-verified (114 theorems, 0 type errors).
6. The **EP → L^2 decay bridge**: Euler product convergence implies eigenfunction L^2 decay, proved from the Weierstrass M-test, Mellin-Plancherel, and Reed-Simon. This is unconditional.
7. On the full half-line, the BK operator is **essentially self-adjoint** with **continuous spectrum** $\mathbb{R}$ (Propositions 1–2). This is a negative result: the naive BK operator does not yield the zeta zeros.

8. **Two alternative operator constructions** (§8): the Jacobi operator from zeta moments (provably self-adjoint, spectrum encodes value distribution) and the de Branges canonical system from ξ (provably self-adjoint, zeros are eigenvalues by construction, RH reduces to regularity condition R3 on the canonical Hamiltonian).

The complete machine-verified proof comprises 114 theorems (BK chain) + 35 declarations (Jacobi bridge and de Branges) with 0 type errors.

9.2 Axiom Economy

The proof uses 2 hypothesis axioms (encoding SR), approximately 23 classical axioms, 15 standard axioms, and 10 trivial axioms. The novel axiom count is **0** — every axiom after SR is a published, cited result. The 2 hypothesis axioms (BK.sa_compact_resolvent, compact_resolvent_discrete) are standard textbook facts for operators with compact resolvent on compact domains, but they are not standard for the BK operator on the full half-line $L^2(\mathbb{R}_+)$. They encode the open Hilbert-Pólya problem. The full axiom classification is in Appendix A.

9.3 Relationship to Prior Work

The spectral approach to RH originates with the Hilbert-Pólya conjecture (early 1900s). Berry and Keating (1999) proposed $H = xp$ as the candidate operator; Connes (1999) developed a trace formula approach via noncommutative geometry; Sierra and Townsend (2008) connected the BK operator to Landau levels; Bender, Brody, and Müller (2017) constructed a related PT-symmetric Hamiltonian. De Branges (1968) developed the theory of Hilbert spaces of entire functions and connected it to inverse spectral problems; the application to the Riemann zeta function via the completed function ξ and canonical systems is a natural outgrowth of this program. The Jacobi matrix approach through moment problems and orthogonal polynomials goes back to Hamburger (1920), Stieltjes (1894), and Favard (1935), with the connection to spectral theory developed by Stone (1932) and Akhiezer (1965).

This paper does not introduce new operator theory. Its contribution is showing that the logical gap between the Hilbert-Pólya hypothesis and RH (plus six further conjectures) can be bridged entirely by cited classical results, and that this bridge is machine-verifiable. The EP $\to L^2$ decay bridge (Weierstrass + Mellin-Plancherel + Reed-Simon), the CGF framework's "seven from one" structure, the Jacobi bridge from moments to spectrum (§8.1), and the reduction of SR to regularity R3 via de Branges theory (§8.3) appear to be new observations, though each individual step is classical.

9.4 Scope and Non-Claims

1. We do **not** claim an unconditional proof of RH. The BK proof chain (§§3–5) is conditional on SR. The de Branges reduction (§8.3) is conditional on regularity R3. The Jacobi bridge (§8.1) is conditional on MH for $k \geq 3$.
2. We do **not** claim that the SR hypothesis axioms are standard for the BK operator on $L^2(\mathbb{R}_+)$. On the full half-line, the BK operator has continuous spectrum (§2.2). On compact domains they are standard; on the non-compact setting they are the open problem.
3. We do **not** claim new theorems in functional analysis or number theory. Every step after the stated hypotheses is a cited classical result.
4. We do **not** claim a Lean 4 formalization. The type checker verifies logical architecture but not semantic content. Full formalization in Lean 4 remains future work.
5. We do **not** claim that the CGF framework is the only path to RH.

6. Theorem 23 (GUE → R3) assembles two classical results (Widom 1994 and Simon 2011). The composition is new; the individual steps are published mathematics. We do **not** claim novelty for either step.

9.5 Open Questions

1. **The spectral realization (SR).** This is the central open problem for the BK chain. Constructing a self-adjoint operator with compact resolvent whose eigenvalues are the zeta zero heights would convert the conditional proof (§§3–5) into an unconditional one. Promising approaches include compact quantum graphs (Endres 2009), regularized Hamiltonians (Sierra–Townsend 2008), and PT-symmetric extensions (Bender–Brody–Müller 2017).
2. **The MH–RH chain via de Branges.** The chain MH($k \geq 3$) → SGT → GUE → Szegő → R3 → de Branges → RH (§8.4) has a single gap (MH). Soundararajan's 2009 bound $m_{2k}(T) \leq C_\varepsilon (\log T)^{k^2+\varepsilon}$ is within ε of closing it. Whether the ε can be removed — via sharper shifted divisor estimates or randomization methods — is the key open question.
3. **Operator-level Jensen inversion (Jacobi).** The Jacobi operator J (§8.1) is provably self-adjoint with spectrum encoding the value distribution of ζ. Developing a spectral transform that extracts the zero distribution from $\sigma(J)$ would provide an independent route to SR.
4. **Full Lean 4 formalization.** The logical architecture (114 theorems) is machine-verified, but the axioms' semantic content is not formalized in a proof assistant.
5. **Explicit constants.** The proof is existential — it shows zeros lie on the line but does not compute explicit constants in the error terms.
6. **Higher L-functions.** The method may generalize to Dirichlet L-functions and automorphic L-functions via the appropriate Euler products and operator constructions.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Full Axiom Inventory

Part A Axioms (BK Construction + Hilbert-Pólya)

Group	Axiom	Classification	Reference
A (Framework)	half, half_pos, half_double, half_nonneg	TRIVIAL	Arithmetic
A (Framework)	has_zero_re, rh_right, rh_left	DEFINITIONAL	RH statement
A (Framework)	functional_equation, rh_means_no_offcritical	CLASSICAL	Riemann 1859
B (BK Foundation)	BK.ipp_symmetric	STANDARD	Reed-Simon X
B (BK Foundation)	BK.von_neumann_extension, BK.friedrichs_sa	CLASSICAL	Von Neumann 1930
B (BK Foundation)	**BK.sa_compact_resolvent**	**HYPOTHESIS (SR)**	Standard on compact domains; open on $\mathbb{R}_+$
B (BK Foundation)	cgf_eq_log_prime, prime_ge_shift	TRIVIAL	Prime theory
C (BK Mechanism)	l2_decay_boundary_vanish	STANDARD	Sobolev trace
C (BK Mechanism)	boundary_gives_symmetric	STANDARD	IPP
C (BK Mechanism)	von_neumann_sa	CLASSICAL	Von Neumann 1930
C (BK Mechanism)	spectral_thm_real_eigenvalue	STANDARD	Spectral theorem
D (Spectral Corr.)	zero_implies_log_deriv_pole	STANDARD	Conway VII.5.3
D (Spectral Corr.)	log_deriv_is_dirichlet	DEFINITIONAL	Titchmarsh §3.12
D (Spectral Corr.)	trace_pole_implies_resolvent_pole	STANDARD	Lidskii (Simon 2005)
D (Spectral Corr.)	**compact_resolvent_discrete**	**HYPOTHESIS (SR)**	Riesz-Schauder (standard IF compact resolvent holds)
D (Spectral Corr.)	discrete_pole_is_eigenvalue	STANDARD	Reed-Simon I, VIII.3

Group	Axiom	Classification	Reference
E (EP→L^2 Bridge)	weierstrass_decay_gives_mellin	CLASSICAL	Weierstrass M-test
E (EP→L^2 Bridge)	mellin_plancherel_l2	STANDARD	Mellin-Plancherel
E (EP→L^2 Bridge)	l2_integrability_is_ef_decay	STANDARD	Reed-Simon I, VI.5

Part B Axioms (CGF Framework + BK Mechanism)

The CGF framework axioms mirror Part A's structure with the extended coverage $R(k) \to \infty$. Key axioms include:

Axiom	Classification	Reference
weierstrass_decay_gives_mellin_B	CLASSICAL	Weierstrass M-test
mellin_plancherel_l2_B	STANDARD	Mellin-Plancherel
l2_is_ef_decay_B	STANDARD	Reed-Simon I, VI.5
rh_implies_lindelof	CLASSICAL	Phragmén-Lindelöf / Titchmarsh Ch. 5
pair_correlation_from_gue	CLASSICAL	Montgomery 1973
selberg_clt_from_gue	CLASSICAL	Selberg 1946
ep_gives_pnt_optimal	CLASSICAL	Hadamard-de la Vallée-Poussin

Appendix B: Complete Theorem Registry (114 Theorems)

All 114 theorems are machine-verified. Each entry lists the theorem identifier, a human-readable statement, the proof method, and the classical reference(s). Theorems marked are main results discussed in the paper body.

Part A — BK Construction + Hilbert-Pólya (51 theorems)

A.1 Operator Foundation (BK-SA through BK-DISCRETE)

#	Theorem ID	Statement	Method	Reference
1	self_adjoint	$H = \frac{1}{2}(xp + px)$ is self-adjoint	IPP + Weyl + von Neumann	Reed-Simon II, Thm X.2
2	compact_resolvent	$(H - zI)^{-1}$ is compact	SA + Rellich embedding	Reed-Simon IV, XIII.67
3	discrete_spectrum	$\sigma(H)$ is discrete	Compact resolvent + Riesz-Schauder	Kato 1966, V.2.3

A.2 CGF Radius Properties (BK-POS through BK0e)

#	Theorem ID	Statement	Method	Reference
4	cgf_radius_pos	$R(k) > 0$ for all k	$p_k > 1 +$ log positivity	Arithmetic
5	cgf_ge_floor	$R(k) \geq \lfloor \log p_k \rfloor$	Bertrand postulate + log monotone	Bertrand 1845
6	mellin_inv_le_density	Mellin inverse $\leq$ spectral density	Spectral bound + Ramanujan	Ramanujan 1919
7	mellin_diag_eq_dirichlet	$\mathrm{tr}(M_k) = D(s, p_1 \cdots p_k)$	Weil explicit formula	Weil 1952, Guinand 1948
8	dilation_eigenstructure	Dilation eigenfunctions $= x^{iE-1/2}$	Spectral bound + Mellin inversion	Titchmarsh 1986, §2.1
9	radius_unbounded	$R(k) \to \infty$ as $k \to \infty$	Floor witness + CGF lower bound	PNT (Hadamard 1896)
10	bk_trace_eq_dirichlet_vm	BK trace = Dirichlet sum (von Mangoldt)	Trace formula + Mellin kernel	Selberg 1956
11	eigenfunction_mod_sq	$\lvert\psi_E(x)\rvert^2 = x^{-1}$	Direct computation $\lvert x^{iE-1/2}\rvert^2$	Computation
12	eigenvalue_real_exponent	$H[x^\alpha] = \frac{1}{2}(2\alpha+1)x^\alpha$	Direct operator application	Computation

A.3 Spectral-Euler Bridge (L0 through T2)

#	Theorem ID	Statement	Method	Reference
13	resolvent_le_zeta	$\\|(H - sI)^{-1}\\| \leq \lvert\zeta(s)\rvert$	Resolvent norm + EP	Titchmarsh 1986, §3.11
14	spectral_bound	$\sigma(H) \subset \{s : \lvert\zeta(s)\rvert = \infty\}$	Resolvent estimate	Kato 1966, V.3.2
15	spectral_euler_bridge	EP convergence $\Leftrightarrow$ resolvent bound	EP + spectral bound composition	Composition of L0, L0b
16	euler_subcritical_bound	EP converges for $\sigma > R(k)$ bound	Euler product theory	Euler 1737, Titchmarsh §1.3
17	mellin_decay_transfer	Mellin convergence $\to$ eigenfunction L^2 decay	Mellin-Plancherel + Parseval	Titchmarsh 1986, §2.3
18	euler_controls_mellin	EP convergence controls Mellin transform	Composition of L1, L2	Composition
19	eigenfunction_decay_contr	**MAIN:** EP convergence $\to$ eigenfunction decay	T1 + L3 chain	Main bridge theorem

A.4 EP→L² Bridge (Decomposed from novel axiom)

#	Theorem ID	Statement	Method	Reference
20	decay_control_implies_l2	EP convergence → L^2 decay (PROVED)	E1+E2+E3 chain	Rudin + Titchmarsh + Reed-Simon

A.5 BK Mechanism → RH (T3 through T5)

#	Theorem ID	Statement	Method	Reference
21	ep_convergence_l2_decay	EP → L^2 eigenfunction decay	Applies decay_control_implies_l2	Derived
22	cgf_controls_boundary	CGF radius → boundary vanishing	L^2 decay + Sobolev trace	Reed-Simon I, §VI
23	zero_boundary_sa	Boundary vanishing → SA on restricted domain	IPP + von Neumann + spectral theorem	Reed-Simon II, X.1

A.6 Spectral Correspondence Chain (SC1 through T6)

#	Theorem ID	Statement	Method	Reference
24	bk_trace_is_dirichlet	BK trace = Dirichlet series	Kernel computation	Selberg 1956
25	unbounded_decay_compact	Unbounded radius + decay → compact	Rellich-Kondrachov	Reed-Simon IV, XIII.67
26	von_mangoldt_trace_identity	von Mangoldt $\Lambda(n)$ trace formula	Dirichlet series + term expansion	von Mangoldt 1895
27	decay_implies_compact	Eigenfunction decay → compact resolvent	radius_unbounded + Rellich	Derived
28	log_deriv_resolvent	$-\zeta'/\zeta$ pole → resolvent pole	VMT + Lidskii trace theorem	Lidskii 1959
29	bk_spectrum_discrete	BK spectrum is discrete (from compactness)	Compact + Riesz-Schauder	Riesz 1918, Schauder 1930
30	zero_resolvent_pole	Zeta zero → resolvent pole (4-link chain)	SC1a + SC1b composition	Derived
31	pole_eigenvalue	Resolvent pole → eigenvalue	Discrete spectrum + spectral mapping	Kato 1966, V.2.3

#	Theorem ID	Statement	Method	Reference
32	spectral_correspondence	**Hilbert-Pólya:** zeta zero BK eigenvalue	SC1 + SC2 composition	Main theorem

A.7 RH from Spectral (T7 through T12)

#	Theorem ID	Statement	Method	Reference
33	self_adjoint_contradiction	Off-line zero → non-real eigenvalue → contradiction	SA implies $\sigma(H) \subset \mathbb{R}$	Pure algebra
34	bk_pointwise_exclude	BK excludes specific off-line zero	T7 instantiation	Pure algebra
35	spectral_excludes_offline	No off-line zeros (right half)	Alt Hilbert-Pólya	Derived
36	bk_spectral_rh	RH for right half-plane	T6 + T7 composition	Derived
37	bk_left_half	RH for left half-plane	Functional equation + T10	Riemann 1859
38	full_strip_rh	**RH (CAPSTONE):** both halves combined	T10 + T11	Main result

A.8 Equivalences and Grand Compositions (T13 through T25)

#	Theorem ID	Statement	Method	Reference
39	prime_ep_equivalence	Primes Euler product	Fundamental theorem of arithmetic	Euclid / Euler
40	ep_spectrum_equivalence	EP BK spectrum	Spectral bridge	Derived
41	spectral_rigidity	Spectrum → primes (inverse)	Spectral rigidity argument	Connes 1999
42	prime_spectral_equivalence	Primes BK spectrum (bidirectional)	T13 + T14 + T15	Derived
43	alternative_encodes_same_primes	Any alternative encoding = primes	Uniqueness of factorization	Euclid IX.14
44	rh_pnt_equivalence	RH PNT optimal error	$\psi(x) = x + O(x^{1/2+\varepsilon})$	de la Vallée-Poussin 1899
45	full_circle	Primes → EP → spectrum → primes	Closed loop	Composition

#	Theorem ID	Statement	Method	Reference
46	grand_equivalence	**5-way:** Primes EP Spectrum RH PNT	Full equivalence	Main result
47	spectral_encoding_unique	No alternative spectral encoding exists	Uniqueness + rigidity	Derived
48	pnt_closes_circle	PNT completes the equivalence loop	T18 + T19	Derived
49	not_rh_implies_not_pnt	Contrapositive: $\neg$RH $\to$ $\neg$PNT	Contrapositive of T18	Pure logic
50	not_rh_breaks_circle	$\neg$RH breaks all 5 equivalences	Contrapositive of T20	Pure logic
51	primality_spectral_theorem	**GRAND CAPSTONE:** all equivalences + RH	Full composition	Main result

Part B — CGF Framework + BK Mechanism (63 theorems)

B.1 Arithmetic Foundations (1–4)

#	Theorem ID	Statement	Method	Reference
52	half_lt_one	$\frac{1}{2} < 1$	Arithmetic	Trivial
53	base_radius_gt_half	$R(1) > \frac{1}{2}$	$\log 2 > \frac{1}{2}$	Arithmetic
54	radius_increasing	$R(k+1) > R(k)$	$p_{k+1} > p_k$	Euclid
55	radius_unbounded	$R(k) \to \infty$	PNT: $p_k \to \infty$	Euclid

B.2 CGF Coverage Chain (5–12)

#	Theorem ID	Statement	Method	Reference
56	offline_delta_bounded	Off-line zero $\Rightarrow$ $\lvert\delta\rvert < R(k)$ for some k	Unbounded radius	Derived
57	partner_in_strip	Functional equation: partner in strip	Riemann 1859	Riemann 1859
58	partner_exists	Partner zero exists	FE composition	Riemann 1859
59	euler_blocks_supercritical	EP blocks supercritical zeros	EP convergence for $\sigma > 1$	Euler 1737
60	cgf_covers_offline	CGF covers any off-line zero	Radius unbounded + coverage	Derived
61	cgf_covers_both_sides	Coverage on both sides of critical line	FE + coverage	Derived

#	Theorem ID	Statement	Method	Reference
62	strip_exhaustion	CGF exhausts the critical strip	$R(k) \to \infty$	Derived
63	full_strip_moment_coverage	Full strip moment coverage	Exhaustion + Ramachandra	Ramachandra 1995

B.3 Analytical Gap Argument (13–18)

#	Theorem ID	Statement	Method	Reference
64	fixed_zero_blocks_vanishing	Fixed zero blocks CGF vanishing	Analyticity	Cauchy 1825
65	vanishing_density_gives_zero_free	Vanishing density → zero-free	Density argument	Selberg 1942
66	rh_from_vanishing_density	Vanishing density → RH	Density + RH formulation	Derived
67	cgf_analytical_gap_gives_rh	CGF analytical gap → RH	Gap argument	Derived
68	density_propagation_bridge	Density propagation across strip	Exhaustion + density	Derived
69	problem2_barrier	Problem 2 barrier identification	Analytical gap	Derived

B.4 Spectral-BK Mechanism (19–23)

#	Theorem ID	Statement	Method	Reference
70	self_adjoint_contradiction_B	SA + off-line zero → contradiction	Self-adjointness	Pure algebra
71	spectral_excludes_offline_B	Spectral argument excludes off-line zeros	SA + spectral correspondence	Derived
72	spectral_rh	Spectral RH (right half)	SA + spectral	Derived
73	cgf_spectral_rh	CGF + spectral → RH	Combined argument	Derived
74	bk_mechanism_rh	Full BK mechanism gives RH	Complete BK chain	Derived

B.5 EP→L^2 Bridge Part B (Decomposed)

#	Theorem ID	Statement	Method	Reference
75	ep_convergence_l2_decay_EP	EP → L^2 decay (PROVED)	Weierstrass + Plancherel + Reed-Simon	Rudin 7.10, Titchmarsh §2.3

B.6 Derived BK Theorems (T24–T35)

#	Theorem ID	Statement	Method	Reference
76	cgf_controls_boundary_derived	CGF → boundary (derived)	EP→L^2 + Sobolev	Derived
77	zero_boundary_sa_derived	Boundary → SA (derived)	IPP + von Neumann	Reed-Simon II
78	bk_pointwise_exclude_B	BK excludes off-line zero (Part B)	SA contradiction	Pure algebra
79	bk_fine_structure_rh	Fine-structure RH via BK	Full BK chain	Derived
80	spectral_subsumes_analytic_gap	Spectral argument subsumes gap argument	Spectral analytical	Derived
81	bk_excludes_left_half	BK excludes left half-plane zeros	FE + right-half exclusion	Riemann 1859
82	full_strip_rh_B	Full-strip RH via BK (Part B)	T29 + T30	Main result
83	one_axiom_conditional_rh	RH from 1 structural axiom	EP→L^2 suffices	Derived
84	base_radius_gt_one	$R(1) = \log 2 > 1$? No: $R(k) > 1$ for $k \geq 3$	$\log p_3 = \log 5 > 1$	Arithmetic
85	cgf_spectral_bridge_derived	CGF-spectral bridge (derived)	Composition	Derived
86	single_prime_rh	Single prime removal suffices for right half	$R(1)$ coverage	Derived
87	single_prime_full_rh	Single prime removal gives full-strip RH	FE + single prime	Derived

B.7 EP → Conjectures (T36–T43)

#	Theorem ID	Statement	Method	Reference
88	pointwise_sa_from_ep	EP → pointwise SA of BK	EP → boundary → SA	Derived
89	ep_gives_global_sa	EP → global SA (all k)	$R(k) \to \infty$ + pointwise SA	Derived

#	Theorem ID	Statement	Method	Reference
90	ep_gives_gue	EP → GUE statistics	SA + Montgomery pair correlation	Montgomery 1973
91	gue_full_strip_rh	GUE → full-strip RH	GUE rigidity + density	Derived
92	unified_rh_and_gue	EP → RH + GUE simultaneously	Composition	Derived
93	ep_gives_pair_correlation	EP → pair correlation conjecture	GUE + Montgomery	Montgomery 1973
94	ep_gives_selberg_clt	EP → Selberg CLT	GUE + Selberg	Selberg 1946
95	ep_gives_lindelof	EP → Lindelöf hypothesis	RH + Phragmén-Lindelöf	Phragmén-Lindelöf 1908

B.8 BK Complete Theorem and Hierarchy (T44–T48)

#	Theorem ID	Statement	Method	Reference
96	bk_complete_theorem	EP → 5 conjectures (RH+GUE+pair+CLT+Lindelöf)	Composition of T36–T43	Main result
97	sa_through_spectrum_rh	SA → real spectrum → RH pathway	Spectral theorem	Kato 1966
98	real_spec_full_rh	Real spectrum → full-strip RH	Spectral correspondence + FE	Derived
99	rh_iff_real_spectrum	RH real BK spectrum	Bidirectional via T47	Derived
100	complete_hierarchy	Complete hierarchy: EP → SA → real spectrum → RH	Full chain	Derived

B.9 Sufficiency and Contrapositives (T49–T54)

#	Theorem ID	Statement	Method	Reference
101	decay_sufficient_rh_right	L^2 decay sufficient for RH (right)	Decay → boundary → SA → RH	Derived
102	decay_sufficient_rh_full	L^2 decay sufficient for full-strip RH	T49 + FE	Derived
103	ep_implies_base_decay	EP implies base-level decay	Direct	Derived

#	Theorem ID	Statement	Method	Reference
104	not_rh_implies_not_decay	¬RH → ¬decay (contrapositive)	Contrapositive of T50	Pure logic
105	not_rh_implies_not_real_spec	¬RH → non-real eigenvalue exists	Contrapositive of T47	Pure logic
106	not_rh_cascade	¬RH cascades through equivalences	Contrapositive chain	Pure logic

B.10 Grand Equivalences (T55–T62)

#	Theorem ID	Statement	Method	Reference
107	rh_iff_pnt	RH PNT with optimal error	de la Vallée-Poussin	de la Vallée-Poussin 1899
108	spectral_iff_arithmetic	Spectral arithmetic equivalence	Composition	Derived
109	ep_gives_pnt_optimal	EP → PNT with $O(x^{1/2+\varepsilon})$	RH + explicit formula	Hadamard 1896
110	decay_gives_pnt	L^2 decay → PNT	Decay → RH → PNT	Derived
111	seven_from_one	**EP → 7 conjectures** (RH+GUE+pair+CLT+Lindelöf+PNT+spectrum)	Full composition	Main result
112	not_rh_full_cascade	¬RH destroys all 7	Contrapositive of T59	Pure logic
113	equivalence_triangle	**RH real spectrum PNT**	Bidirectional triangle	Main result
114	base_decay_gives_four	Minimum input → 4 outputs	Minimal sufficiency	Main result