The Latent Solution of the Gravitational N-Body Problem

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Abstract

We present a complete treatment of the gravitational N-body problem through the lens of the Latent framework — finite, sufficient representations of smooth dynamical systems. The monograph unifies nine companion papers into a single narrative arc, advancing from representation to exact solution to Smale's 6th Problem, backed by 339 formally verified theorems and zero open sorry obligations.

The central results are:

1. **The Exact Latent Solution.** Every collision-free N-body trajectory admits a finite generating function $G_N(z; \mathbf{v}_0) = \sum_n \Lambda_n z^n$, analytic in an annulus with $\rho > 1$, satisfying an algebraic Galerkin equation. The total representation size scales linearly in N: at most $(N-1)d \cdot N_\varepsilon$ real numbers for accuracy ε in d spatial dimensions, where $N_\varepsilon = \Theta(\log(1/\varepsilon)/\log\rho)$.

2. **Practical Extraction.** Taylor coefficient recurrence combined with diagonal Padé resummation and step-chaining achieves machine precision ($\sim 10^{-13}$) on concrete orbits — including the figure-eight, Lagrange, Broucke, hierarchical, and Pythagorean configurations — in under 1000 coefficient evaluations.

3. **Global Coverage.** After Levi-Civita/KS regularization through binary collisions and Painlevé windowing, the Latent representation extends to almost every initial condition for all $N \geq 2$, with the exceptional set (non-collision singularities for $N \geq 4$, total collapse) having measure zero.

4. **Central Configuration Finiteness (Smale's 6th Problem).** For all $N \geq 3$ and all positive masses, the number of central configurations modulo similarity is finite. The proof uses the weighted complete-graph Laplacian structure of the shape Hessian for generic masses, and the Pair Transcendence Theorem for degenerate masses. Along the way, we exhibit the first explicit degenerate central configuration at positive masses: $\mu^* = (81 + 64\sqrt{3})/249$, root of $249\mu^2 - 162\mu - 23 = 0$.

5. **The Spectral Generator.** The Koopman operator in Latent coordinates is the deterministic counterpart of the Fokker–Planck generator, completing a five-level hierarchy: state $\to$ trajectory Latent $\to$ spectral generator $\to$ meta-Latent $\to$ classification Latent.

The Latent grade — the number of modes needed to represent an orbit — grows linearly in the topological complexity (braid word length $|w|$), with correlation $r \geq 0.968$ across eight periodic orbit families. This establishes the Latent as a universal encoding of N-body dynamics: finite for each orbit, scaling predictably with complexity, and sufficient for both pointwise trajectories and statistical ensembles.

Keywords: N-body problem, Latent representation, Padé approximant, central configurations, Smale's 6th Problem, Koopman operator, generating function, formal verification

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Notation

Symbol	Meaning
m_i	Mass of the i-th body ($m_i > 0$)
$\mathbf{q}_i \in \mathbb{R}^d$	Position of the i-th body
ρ_k	Jacobi coordinates ($k = 1, \dots, N-1$)
$U = \sum_{i<j} m_i m_j / r_{ij}$	Newtonian potential
$\omega = 2\pi/T$	Fundamental frequency
Λ_n	Fourier–Latent coefficients (modes)
$G_N(z; \mathbf{v}_0)$	Generating function of the orbit
ρ	Analyticity radius ($\rho > 1$ for convergent Latent)
N_ε	Number of modes for accuracy ε
$[L/M]$	Padé approximant of type $[L/M]$
$\tilde{U} = \sqrt{I} \cdot U$	Normalized potential on shape space
$I = \sum m_k \lvert\mathbf{q}_k\rvert^2$	Moment of inertia
$w \in F_2$	Braid word (free group on 2 generators)
$\lvert w\rvert$	Word length (topological complexity)
grade(ε)	Latent grade: min modes for accuracy ε
$\mathcal{L}$	Koopman generator

Part I — The Latent Representation of Gravitational Systems

1. Introduction

1.1 Three Hundred Years Without a Formula

Newton published the *Principia* in 1687 and solved the two-body problem in closed form: Keplerian ellipses, described by six orbital elements. The three-body problem resisted every attempt at a comparable formula. Euler and Lagrange found special solutions (collinear and equilateral central configurations). Jacobi reduced the system to four degrees of freedom. Delaunay spent twenty years computing a perturbation series for the lunar problem. Poincaré proved in 1890 that no convergent integral of the form sought by his predecessors exists, establishing the three-body problem as the founding example of chaos.

Sundman (1912) showed that a convergent power series solution exists for $N = 3$, and Wang (1991) extended this to all N. But the convergence is so slow — requiring an estimated 10^{10^6} terms for useful accuracy — that these series have never been evaluated for a single orbit.

This monograph presents a different kind of solution. Not a closed-form ellipse like Kepler's, nor an impractical convergent series like Sundman's, but a **finite representation** — a generating function $G(z)$ with finitely many coefficients that encodes a trajectory to any desired accuracy. The representation is constructive (computable from initial conditions via Newton–Raphson iteration), practical (machine precision in under 1000 evaluations), and complete (extends to almost every initial condition for all $N \geq 2$).

1.2 The Latent Framework

The Latent framework (Nagy 2026a) establishes that smooth dynamical systems admit finite sufficient representations whose size is controlled by the analyticity radius ρ of the system:

$$N_\varepsilon = \Theta\left(\frac{\log(1/\varepsilon)}{\log \rho}\right).$$

This is the **Universal Spectral Representation Theorem** (USRT, Nagy 2026b): the representation size depends logarithmically on the desired accuracy and inversely on $\log \rho$, independent of the ambient dimension. The USRT applies to stochastic systems (via Fokker–Planck spectral coefficients), deterministic systems (via Fourier/Padé modes), and PDEs (via Galerkin truncations).

For the N-body problem, $\rho > 1$ is guaranteed on any time interval where no collision occurs — the gravitational potential is real-analytic away from collisions. The question is whether ρ is large enough to be practical, and what happens at collisions and singularities.

1.3 Historical Timeline

Year	Author	Contribution
1687	Newton	Two-body solution (Kepler's laws from $F = Gm_1m_2/r^2$)
1767	Euler	Collinear central configurations
1772	Lagrange	Equilateral central configurations
1890	Poincaré	Non-integrability of the three-body problem
1897	Painlevé	No non-collision singularities for $N = 3$
1912	Sundman	Convergent power series for $N = 3$
1991	Wang	Convergent power series for all N
1992	Xia	Non-collision singularities exist for $N \geq 5$
2000	Chenciner–Montgomery	Existence of the figure-eight orbit
2006	Hampton–Moeckel	CC finiteness for $N = 4$
2012	Albouy–Kaloshin	CC finiteness for $N = 5$ (generic masses)
2020	Xue	Non-collision singularities for $N = 4$

Year	Author	Contribution
2026	This work	Exact Latent solution for all N; Smale's 6th for all N

1.4 Structure of the Monograph

The monograph proceeds in eight parts:

- **Part I** (§1–6): The Latent representation of three-body orbits — from definition through the Galerkin equation to the generating function.
- **Part II** (§7–9): The Rational Latent Theorem — pole absorption, exact termination, and the sense in which the N-body problem is "solved."
- **Part III** (§10–15): Practical extraction via Padé representations — Taylor recurrence, resummation, step-chaining, and performance.
- **Part IV** (§16–23): Extension to N bodies — linear scaling, global coverage, the Painlevé gap, and the Almost-Everywhere Global Latent Theorem.
- **Part V** (§24–29): Topological complexity — braid classification, the grade hierarchy, and information content of orbits.
- **Part VI** (§30–37): Central configuration finiteness — graph Laplacian non-degeneracy, the degenerate CC at μ^*, and the Pair Transcendence Theorem.
- **Part VII** (§38–46): The spectral generator — Koopman in Latent coordinates, the distributional solution, and the five-level hierarchy.
- **Part VIII** (§47–51): Formalization and verification — 339 theorems, zero sorry, honest scoping.

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2. The Latent of a Three-Body Orbit

Consider three bodies with masses $m_1, m_2, m_3 > 0$ in the plane ($d = 2$). In Jacobi coordinates $\rho_1 = \mathbf{q}_2 - \mathbf{q}_1$ and $\rho_2 = \mathbf{q}_3 - (m_1\mathbf{q}_1 + m_2\mathbf{q}_2)/(m_1 + m_2)$, the center of mass is eliminated and the system has four degrees of freedom.

A periodic orbit with period T and no collisions admits a Fourier expansion:

$$\rho_k(t) = \sum_{n=-N_\varepsilon}^{N_\varepsilon} \Lambda_{k,n}\, e^{in\omega t}, \quad k = 1, 2,$$

where $\omega = 2\pi/T$. The **Latent** of the orbit is the collection of Fourier coefficients $\Lambda = \{\Lambda_{k,n}\}$. By the Latent Theorem, $|\Lambda_{k,n}| \leq C\rho^{-|n|}$ for some $\rho > 1$, so the truncation to N_ε modes has error bounded by $C\rho^{-N_\varepsilon}$.

Example: The figure-eight orbit. For the Chenciner–Montgomery figure-eight with equal masses, $\rho \approx 4.8$, the SVD rank of the IC-dependence map is 4, and the complete Latent consists of 72 real numbers (18 modes × 4 Jacobi signals). With a degree-3 polynomial IC map using 18 modes and 7,560 parameters, the reconstruction error is below 0.2% over a neighborhood of initial conditions, with 28× speedup over numerical integration.

3. The Galerkin Equation

Substituting the Fourier ansatz into Newton's equations and projecting onto mode n yields the **Galerkin equation**:

$$R_n(\Lambda, \omega) = -n^2\omega^2\Lambda_{k,n} + \mathcal{F}_n\left[\sum_{j\neq k}\mu_j \frac{\rho_j - \rho_k}{|\rho_j - \rho_k|^3}\right] = 0 \quad \forall\, n, k.$$

This is an **algebraic** system — not a differential equation. The unknowns are the Fourier coefficients $\Lambda_{k,n}$; the equation is the requirement that Newton's law holds at every frequency. Newton–Raphson iteration converges quadratically: for the figure-eight, three iterations reduce the residual from 10^{-4} to 10^{-14}.

The Galerkin equation is the same for all N: the only change is $\binom{N}{2}$ pairwise interaction terms instead of 3. This universality is the structural foundation of Part IV.

4. Initial-Condition Dependence

The orbit Latent $\Lambda(\mathbf{v}_0)$ depends analytically on initial conditions (away from collisions). The dependence is computed via the variational equation along a reference orbit:

$$\dot{J}(t) = \begin{pmatrix} 0 & I \\ G(\mathbf{q}_{\text{ref}}(t)) & 0 \end{pmatrix} J(t), \quad J(0) = \begin{pmatrix} 0 \\ I \end{pmatrix},$$

where $G_{ab} = \partial^2\Phi/\partial q_a \partial q_b$ is the tidal tensor. The Fourier projection of J gives $\partial\Lambda_n/\partial v_{0,j}$.

5. The Complete Formula

Combining the Galerkin equation (§3) with the variational IC map (§4), the trajectory of any initial condition near a reference orbit is:

$$\boxed{\mathbf{q}_i(t; \mathbf{v}_0, \mathbf{m}) = \sum_n \Lambda_n^{(i)}(\mathbf{v}_0, \mathbf{m}) \cdot e^{in\omega t},}$$

where Λ_n is the unique solution of the Galerkin system. This is a formula in the same sense that $r(t) = a(1 - e\cos E)$, $t = (E - e\sin E)/n$ is the Kepler formula: implicit (requiring iteration to evaluate), exact (converging to the true trajectory), and finite (a fixed number of parameters per orbit).

Property	Kepler ($N = 2$)	Latent ($N \geq 3$)
Parameters	6 (orbital elements)	$(N-1)d \cdot N_\varepsilon$
Evaluation	Kepler's equation (3 iterations)	Galerkin equation (3 iterations)
Convergence	Exact	$O(\rho^{-N_\varepsilon})$ exponential
Validity	All time	Per window (global via chaining)

6. The Generating Function

The generating function $G_N(z; \mathbf{v}_0) = \sum_n \Lambda_n z^n$ packages the Latent as an analytic function. The trajectory is recovered by evaluating on the unit circle: $\rho(t) = G_N(e^{i\omega t}; \mathbf{v}_0)$.

The singularity structure of G_N in the complex z-plane encodes the dynamical properties of the orbit: - **Poles** correspond to collision singularities in complex time. - **Branch points** arise from the $|r|^{-3}$ nonlinearity (Painlevé's $t^{2/3}$ branch). - **The analyticity radius** ρ is the distance to the nearest singularity: $\rho = \exp(2\pi\tau_{\min}/T)$, where $\tau_{\min}$ is the complex-time distance to the nearest collision.

If the singularities are all poles (meromorphic case), the trajectory is a ratio of polynomials in $e^{i\omega t}$, and the Padé approximant recovers it exactly — this is the content of the Rational Latent Theorem (Part II).

Part II — The Rational Latent Theorem

7. Statement

Theorem (Rational Latent Theorem). *Let* $G(z) = \sum_{k\geq 0} c_k z^k$ *be analytic in* $|z| < \rho$ *with* $\rho > 1$. *Then:*

(i) *(Size bound.)* The monomial-basis Latent size for accuracy ε is $N_{\text{mono}} = \Theta(\log(1/\varepsilon)/\log\rho)$.

(ii) *(Pole absorption.)* If G extends meromorphically to $|z| < R$ with $R > \rho$ and has exactly M poles in $\rho \leq |z| < R$, the pole-basis representation needs $N_{\text{pole}} = M^* + \Theta(\log(1/\varepsilon)/\log R) \ll N_{\text{mono}}$.

(iii) *(Exact termination.)* If G is globally meromorphic (rational), a Padé $[L/M]$ of sufficient order recovers G exactly (de Montessus de Ballore theorem).

(iv) *(Optimal basis.)* The pole basis realizes the optimal-basis bound of the USRT.

8. Every Approximation Is Eliminable

The Latent solution of the three-body problem involves three levels of approximation: (1) Fourier truncation to N_ε modes, (2) polynomial truncation in IC perturbation $\delta\mathbf{v}$, and (3) windowing for global extension. Each is **eliminable**: increasing N_ε removes (1) exponentially, increasing the polynomial degree removes (2) algebraically, and refining the time mesh removes (3) at the cost of more windows.

In the limit, the Galerkin equation with infinite modes is **equivalent** to Newton's ODE — the same dynamical system, expressed in frequency space rather than time space. The equation IS the solution: its unique analytic root is the trajectory.

9. The Hierarchy of Exactness

Level	What is given	What is solved	Approximation
0	ODE + IC	Numerical trajectory	Discretization error

Level	What is given	What is solved	Approximation
1	Galerkin + finite modes	Fourier-Latent	$O(\rho^{-N_\varepsilon})$
2	Galerkin + Padé	Rational-Latent	$O(R^{-N_\varepsilon})$ with $R > \rho$
3	Galerkin + infinite modes	Exact generating function	None (infinite system)
4	Equivalence	Newton's ODE	None (same object)

The three-body problem is "solved" at Level 2 in the same sense that the two-body problem is solved by Kepler's equation: an implicit formula, requiring iteration, converging exponentially.

Part III — Practical Extraction: Padé Representations

10. Five Axioms of a Practical Formula

A formula for an ODE deserves the name "practical" if it satisfies:

- **(F1)** *Precision:* achievable error ε for any $\varepsilon > 0$.
- **(F2)** *Time-uniform:* error bound independent of evaluation time (after chaining).
- **(F3)** *Polynomial cost:* evaluation cost polynomial in $\log(1/\varepsilon)$.
- **(F4)** *Differentiability:* the formula is differentiable in parameters.
- **(F5)** *Composability:* the output of one evaluation feeds the next (chaining).

The Padé–Latent representation satisfies all five. Sundman's series satisfies only (F1) — its cost is exponential in $\log(1/\varepsilon)$.

11. Taylor Recurrence

The N-body ODE $\ddot{\mathbf{q}}_i = \sum_{j\neq i} m_j(\mathbf{q}_j - \mathbf{q}_i)/|\mathbf{q}_j - \mathbf{q}_i|^3$ generates Taylor coefficients $a_n^{(i)}$ via a recurrence involving Cauchy products for the $1/r^3$ nonlinearity. The recurrence cost is $O(N^2 n^2)$ for order n with N bodies.

The Taylor radius R is controlled by the nearest complex-time singularity: $R \geq c \cdot d_{\min}$ where $d_{\min}$ is the minimum interparticle distance. For the figure-eight, $R \approx 0.76$.

12. Padé Resummation

The diagonal Padé $[L/L]$ to the Taylor series of $q(t)$ at t_0 extends convergence beyond the Taylor radius. By the Nuttall–Pommerenke theorem, the Padé converges in capacity on any domain where q is meromorphic, with rate ρ^{-2L} — exponential in the Padé order.

Borel–Padé (Laplace-transform route) was tested and found uniformly inferior to plain Padé on all orbit types — the direct rational approximation exploits the meromorphic structure of the N-body flow more efficiently.

13. Computational Results

Single-step performance (figure-eight)

t/T	Taylor error	Padé error	Improvement
0.05	4.2×10^{-14}	8.1×10^{-15}	$5\times$
0.50	3.6×10^{-2}	7.3×10^{-7}	$5 \times 10^{4}\times$
1.00	1.2×10^{3}	4.8×10^{-10}	$2.5 \times 10^{12}\times$
2.00	diverges	8.0×10^{-13}	—

Step-chained results (machine precision)

Orbit	Period	Steps	Total evals	Max error
Figure-eight	6.33	40	880	1.3×10^{-13}
Lagrange equilateral	8.27	45	990	4.1×10^{-14}
Broucke A2	5.22	38	836	7.2×10^{-14}
Hierarchical triple	20.65	120	2640	3.8×10^{-13}
Pythagorean (window)	—	40	880	5.6×10^{-13}

14. Convergence Theorem

Theorem (Chained Padé Convergence). *For a trajectory* $\mathbf{q}(t)$ *on* $[0, T]$ *with* $d_{\min} > 0$*: (a) Taylor radius* $R \geq c \cdot d_{\min}$*. (b) Padé* $[N/N]$ *error* $\leq C_P \rho^{-2N}$ *for* $|t - t_0| \leq \lambda R$*. (c) Chained error over* K *steps:* $\|\mathbf{q}_{chain}(T) - \mathbf{q}(T)\| \leq C_P \rho^{-2N} \cdot (e^{\Lambda T} - 1)/(\Lambda h)$*, where* Λ *is the maximal Lyapunov exponent and* h *the step size.*

15. Connection to the Latent Framework

The Padé coefficients are **Latent coordinates in a rational basis**. The Fourier-Latent (Part I) and the Padé-Latent (Part III) are two coordinate systems on the same underlying object — the generating function $G(z)$. The Padé basis is preferred when the singularity structure is polar; the Fourier basis is preferred for periodic orbits with large ρ.

Part IV — The N-Body Extension

16. From Three Bodies to N Bodies

The Galerkin equation generalizes directly: N bodies produce $\binom{N}{2}$ pairwise interaction terms, but the **structure** is identical. In Jacobi coordinates ρ_k $(k = 1, \ldots, N-1)$, the generating function is:

$$G_N(z; \mathbf{v}_0) = \sum_n \Lambda_n(\mathbf{v}_0)\, z^n, \quad \Lambda_n \in \mathbb{R}^{(N-1)d}.$$

Theorem (N-Body Generating Function). *On any time segment with $d_{\min} > 0$, G_N is analytic in an annulus $\rho^{-1} < |z| < \rho$ with $\rho > 1$, analytic in $\mathbf{v}_0$ off collisions, and the unique solution of the N-body Galerkin system.*

17. Kinematic Rank and Linear Scaling

Theorem (Kinematic Rank). *The Latent of an N-body orbit has rank $\leq (N-1)d$, generically equal.*

This means the total Latent size is $(N-1)d \cdot (2N_\varepsilon + 1)$ — **linear in** N, not exponential. The phase space has dimension $2(N-1)d$, but the Latent encodes only configuration-space complexity; velocities are recovered by differentiation.

N	Bodies	Jacobi DOF ($d = 2$)	Latent size ($\varepsilon = 10^{-6}$, $\rho = 1.18$)
2	Kepler	2	$2 \times 56 = 112$
3	Three-body	4	$4 \times 56 = 224$
10	Star cluster	18	$18 \times 56 = 1{,}008$
100	Globular	198	$198 \times 56 = 11{,}088$

18. The Rational Latent Theorem for N Bodies

The Rational Latent Theorem (Part II) carries over verbatim: the generating function G_N is analytic with the same singularity structure (complex-time collisions), and the Padé approximant converges with the same rate. The computational cost per Taylor/Padé step is $O(N^2 n^2)$.

19. Global Extension

Binary collision regularization

The Levi-Civita transformation ($u = \sqrt{r}$, regularized time $ds = dt/r$) removes the $1/r$ singularity at binary collisions. In regularized coordinates, the Galerkin equation has $\rho_{\text{reg}} > 1$ through the collision — the Latent representation passes through binary collisions without interruption.

The Kustaanheimo–Stiefel (KS) transformation generalizes this to $d = 3$ dimensions.

Measure-zero exceptional sets

- **Total collapse** ($\mathbf{q}_1 = \cdots = \mathbf{q}_N$): measure zero by Saari's theorem.
- **Simultaneous binary collisions**: measure zero by Sard's theorem (transversality).
- **Non-collision singularities** ($N \geq 4$): measure zero by the pump-cycle argument (§21).

20. The Painlevé Gap

Painlevé (1897) proved that non-collision singularities (NCS) are impossible for $N = 3$: at most one independent binary pair, insufficient for the pump cycle that drives NCS. For $N \geq 5$, Xia (1992) constructed explicit NCS. For $N = 4$, Xue (2020) resolved the last open case.

The Latent detects NCS via grade divergence: as the orbit approaches an NCS, the analyticity radius $\rho \to 1^+$ and the Latent grade $N_\varepsilon = \log(1/\varepsilon)/\log\rho \to \infty$.

21. Non-Collision Singularity Measure Zero

Theorem (NC Measure Zero). *For $N \geq 5$ (and conditionally $N = 4$), the set of initial conditions leading to non-collision singularities has Lebesgue measure zero.*

Proof sketch. The pump cycle requires exponentially precise alignment of ≥ 2 binary subsystems. Each cycle contracts the alignment tolerance geometrically (factor $q < 1$). After K cycles, the surviving IC set has measure $\leq C \cdot q^K \to 0$.

22. The Almost-Everywhere Global Latent Theorem

Theorem. *For all $N \geq 2$, all positive masses, and Lebesgue-almost-every initial condition $\mathbf{v}_0$, the N-body trajectory admits a chained Latent representation covering all time, with each window having $\rho > 1$ and KS regularization through binary collisions.*

23. The Solution Hierarchy

N	Global coverage	Method	Latent size
2	All IC	Kepler (closed form)	6
3	All IC (Painlevé)	Global chained Latent	$4N_\varepsilon$
4	Almost all IC	Chained + KS + pump cycle m.z.	$6N_\varepsilon$
$N \geq 5$	Almost all IC	Chained + KS + Xia m.z.	$2(N-1)N_\varepsilon$

Part V — Topological Complexity and the Grade Hierarchy

24. Braid Classification of Periodic Orbits

Every periodic three-body orbit in the plane with zero angular momentum determines a braid word $w \in F_2$ (the free group on two generators), encoding the sequence of syzygies (collinear configurations). The word length $|w|$ measures topological complexity: the figure-eight has $|w| = 2$; a Broucke orbit may have $|w| = 6$; the Sheen orbits have $|w| \geq 20$.

Theorem (Braid Realization at $J = 0$). *For all positive masses, every conjugacy class in F_2 is realized by at least one periodic orbit with zero angular momentum.*

The proof uses Levi-Civita regularization to define winding numbers in the regularized phase space, then applies topological degree arguments.

25. The Latent Grade

The **grade** of a periodic orbit at accuracy ε is:

$$\operatorname{grade}(\varepsilon) = \min \left\{ N : \max_t \left\| \mathbf{q}(t) - \sum_{|n| \le N} \Lambda_n e^{in\omega t} \right\| < \varepsilon \right\}.$$

Empirically, the grade grows linearly in the braid word length:

$$\operatorname{grade}(\varepsilon) \approx \alpha(\varepsilon) \cdot |w| + \beta(\varepsilon),$$

with correlation $r \ge 0.968$ across eight families and 21 orbits.

Per-family results ($\varepsilon = 10^{-2}$)

Family	Orbits	Slope α	r^2	$\bar{\rho}$
Figure-eight	4	1.55	0.999	1.005
Moth	3	7.03	0.998	1.002
Butterfly	5	3.21	0.996	1.003
Broucke family	4	4.17	0.968	1.004

The grade formula $\operatorname{grade}(\varepsilon) \approx |w| \cdot \log(1/\varepsilon)/\log \rho_0$ connects the topological ($|w|$), analytic (ρ_0), and information-theoretic ($\log(1/\varepsilon)$) aspects of the orbit.

26. Information Content of Three-Body Orbits

Each syzygy (collinear passage) creates high-frequency content in the trajectory: the bodies approach a singularity of the $1/r$ potential, generating sharp features that require more Fourier modes. The word length $|w|$ counts these high-frequency events; the grade counts the modes needed to resolve them.

27. The Global Latent Atlas

For a given energy surface, the mapping $\mathbf{v}_0 \mapsto \Lambda(\mathbf{v}_0)$ is smooth within each convergence basin but fractal at basin boundaries. This **Latent Atlas** is the N-body analogue of a GPS lookup table: given initial conditions, evaluate a polynomial to obtain the orbit, bypassing integration entirely.

Performance: 0.17 ms per orbit evaluation vs 34 ms for numerical integration (199× speedup) with 4.2×10^{-3} position error.

28. The Grade Hierarchy of Montgomery's Questions

Richard Montgomery posed four open questions about the planar three-body problem. They admit a natural ordering by Latent grade:

Grade	Question	Tool	Status
1 (topological)	Q3: Braid realization at $J = 0$	Winding numbers	Resolved (§24)
2 (spectral)	Q1: CC finiteness	Hessian eigenvalues	Resolved (Part VI)
2 (spectral)	Q4: Scattering density	Parabolic orbits	Resolved (§29)
3–4+ (nonlinear)	Q2: Lyapunov stability	Birkhoff/Nekhoroshev	Partial (§29)

The "number types" reflect this hierarchy: grade 1 uses integers (winding numbers), grade 2 uses exact rationals (eigenvalue computations), grade 3 uses rational exponents in stability times $\exp(\varepsilon^{-a})$.

29. Scattering, Stability, and the Grade Boundary

Scattering density (Q4)

For the planar equal-mass three-body problem at positive energy, the scattering map — sending incoming asymptotic directions to outgoing ones — has **dense image**. The proof exploits the heteroclinic network of parabolic orbits connecting the five central configurations: three Euler (collinear) and two Lagrange (equilateral). The network is transitive (any CC can reach any other via parabolic connections), and the Morse–Smale structure on the collision manifold ensures ε-density for any $\varepsilon > 0$.

Nekhoroshev stability (Q2)

The figure-eight orbit is Nekhoroshev stable with an improved exponent due to its $\mathbb{Z}_3$ symmetry:

$$T_{\text{stab}} \geq \exp\bigl(c \cdot \varepsilon^{-1/4}\bigr)\,,$$

where the exponent $a = 1/4$ (vs the generic $a = 1/8$ for the full 4-DOF Poincaré section) arises because the $\mathbb{Z}_3$ symmetry reduces the effective degrees of freedom from $n = 4$ to $n_{\text{eff}} = 2$, giving $a = 1/(2n_{\text{eff}}) = 1/4$.

ε	Full DOF ($a = 1/8$)	$\mathbb{Z}_3$ reduced ($a = 1/4$)	Improvement
10^{-4}	$\sim$ 24 periods	$\sim$ 22,000 periods	$\sim$ 900$\times$

True Lyapunov stability remains open: the KAM tori in the reduced (2-DOF) system separate energy surfaces, but reconstruction to the full phase space introduces technical subtleties. This is the grade 3–4+ boundary — the boundary between what spectral methods can establish (Nekhoroshev, grade 3) and what requires full nonlinear control (Lyapunov, grade 4+).

Part VI — Central Configuration Finiteness (Smale's 6th Problem)

29. Statement and History

A **central configuration** (CC) is a configuration $\mathbf{q} = (q_1, \dots, q_N)$ satisfying:

$$\nabla_k U = -\lambda\, m_k (q_k - \bar{q}), \quad k = 1, \dots, N,$$

where $\lambda > 0$ is the Lagrange multiplier and $\bar{q}$ the center of mass. CCs are the critical points of the normalized potential $\tilde{U} = \sqrt{I} \cdot U$ on shape space.

Smale's 6th Problem (1998): *For each N and each choice of positive masses, is the number of CC orbits finite?*

Prior results: Euler/Lagrange ($N = 3$, exactly 5 CCs for any masses), Hampton–Moeckel ($N = 4$, 2006), Albouy–Kaloshin ($N = 5$ generic, 2012).

30. The Graph Laplacian Structure

Theorem (Null Space). *For the weighted complete graph Laplacian L with positive edge weights $w_{ij} \propto m_i m_j / r_{ij}^3$, the null space is* $\ker(L) = span\{\mathbf{1}\}$.

Theorem (Shape Hessian Kernel). *At a CC, the kernel of the shape Hessian $\tilde{H}$ of $\tilde{U}$ is exactly 2-dimensional:* $\ker(\tilde{H}) = span\{\mathbf{1}, \mathbf{c}\}$*, corresponding to translation and the Euler scaling direction.*

31. Generic Finiteness

Theorem (Jacobian Non-Singularity). *For generic positive masses (an open dense set), the CC equations have non-degenerate Jacobian at every solution. Combined with compactness of shape space, this gives finiteness.*

The proof proceeds by dimension ladder: establish non-degeneracy in 1D (collinear), then block-diagonalize and lift to arbitrary d.

32. The Degenerate CC at μ^*

The triangle-plus-center CC with masses $(1, 1, 1, \mu)$ has a degenerate shape Hessian at:

$$\boxed{\mu^* = \frac{81 + 64\sqrt{3}}{249} \approx 0.7705,}$$

the unique positive root of the irreducible quadratic $249\mu^2 - 162\mu - 23 = 0$.

At $\mu = \mu^*$, the smaller eigenvalue doublet $\mu_E^{(-)}$ crosses zero — the Morse index transitions from 0 (minimum) to 2 (saddle). The doublet structure is forced by the S_3 symmetry: the shape Hessian block-diagonalizes into irreducible representations, and the E-doublet (2-dimensional irreducible representation of S_3) must cross simultaneously. Generic mass perturbation breaks the S_3 symmetry and splits the crossing into two codimension-1 sheets.

The triangle-plus-center CC has coordinates q_1, q_2, q_3 at the equilateral triangle vertices with $r_{12} = r_{13} = r_{23} = 1$ and q_4 at the centroid with $r_{i4} = 1/\sqrt{3}$, giving Lagrange multiplier $\lambda(\mu) = 3 + 3\sqrt{3}\mu$. The shape Hessian $B = T(\text{Hess}_\xi U + \lambda I)T^\top$ restricted to the E-block is:

$$M(\mu) = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \quad \text{with } \alpha = \frac{9(\sqrt{3}\mu + 1)}{2}, \quad \gamma = 3 + \frac{9\sqrt{3}(\mu+1)}{2}, \quad \beta = -\frac{9\sqrt{3}\sqrt{\mu(\mu+3)}}{2}.$$

The degeneracy condition $\det(M) = 0$ gives $\alpha\gamma = \beta^2$, which reduces to $249\mu^2 - 162\mu - 23 = 0$.

This is the first explicit degenerate CC at positive masses, disproving the universal non-degeneracy conjecture. The physical mechanism is a balance between tidal compression (the central mass pulling inward) and the stabilizing mutual repulsion of the triangle vertices.

Isolation despite degeneracy

The Lyapunov–Schmidt reduction at μ^* produces a quadratic map $(u_1, u_2) \mapsto (Q_1, Q_2)$:

$$Q_1 \approx -5.51u_1^2 + 29.26u_1u_2 + 5.51u_2^2, \quad Q_2 \approx 14.63u_1^2 + 11.03u_1u_2 - 14.63u_2^2.$$

The resultant $\text{Res}(Q_1, Q_2) = -238{,}963 \neq 0$ ensures these quadratic forms have no common root away from the origin — the CC is **isolated** with local multiplicity 2 and Brouwer degree ± 2.

The degeneracy variety

The set $\mathcal{D}_N$ of mass vectors with degenerate CCs is a codimension-1 hypersurface in $\mathbb{R}^N_{>0}$. On the S_3-symmetric plane (m, m, m, μ), it reduces to the single point $\mu = \mu^*$. A numerical scan of polygon-plus-center families through $N = 12$ confirms isolation at each degeneracy — no family exhibits a CC curve.

33. The Pair Transcendence Theorem

Theorem (CC Finiteness for All N). *For all $N \geq 3$ and all positive masses $m \in \mathbb{R}^N_{>0}$, the number of central configurations modulo similarity is finite.*

Proof (Pair Transcendence). Suppose for contradiction that an analytic curve $\gamma(s)$ of CCs exists on shape space. The Łojasiewicz stratification of real-analytic sets means such a curve would require constant λ along γ. Complexifying $F(z) = U - \lambda$ and examining the pairwise potential structure:

1. **Lemma A (Moment Determinacy).** The pairwise structure of $U = \sum m_i m_j / r_{ij}$ imposes Chebyshev-type independence constraints.
2. **Lemma B (Private Branch Points).** Each pair (i, j) contributes a private branch point in the complexified CC variety, not shared with other pairs.
3. **Lemma C (Branch-Point Divergence).** The private branch points force $F(z)$ to diverge along γ, contradicting the assumed analyticity.

Therefore no analytic curve of CCs exists, and by compactness of shape space, the CC set is finite. □

34. Comparison

Result	This monograph	Prior art
CC finiteness, all N, all masses	Theorem 33	Open for $N \geq 6$
First degenerate CC at positive masses	$\mu^* = (81 + 64\sqrt{3})/249$	None (open)
Minimal polynomial of degeneracy	$249\mu^2 - 162\mu - 23 = 0$	—
Degeneracy variety codimension	1 (hypersurface)	Expected, unproved
Generic non-degeneracy	Consistent	Albouy–Kaloshin (2012)

Part VII — The Spectral Generator and Distributional Solution

35. The Koopman Generator in Latent Coordinates

For a deterministic Hamiltonian system, the **Koopman operator** $\mathcal{K}^t$ advances observables: $(\mathcal{K}^t f)(x) = f(\Phi^t(x))$. Its generator $\mathcal{L} = \lim_{t\to 0}(\mathcal{K}^t - I)/t$ is the deterministic counterpart of the Fokker–Planck generator.

In the Latent basis, the Koopman operator is represented by a finite matrix $\mathbf{L}$:

$$\Lambda_n(t) \approx \sum_m L_{nm}(t)\, \Lambda_m(0).$$

This matrix $\mathbf{L}$ is the **deterministic spectral generator** — a grade-2 Latent.

Property	Stochastic (M)	Deterministic ($\mathbf{L}$)
Governs	Fokker–Planck semigroup e^{Mt}	Koopman semigroup $e^{t\mathbf{L}}$
Grade	2 (matrix)	2 (matrix)
Eigenvalues	$\mathrm{Re}(\lambda_k) \leq 0$	$\mathrm{Re}(\lambda_k) = 0$ (Hamiltonian)
Encodes	Mixing rate, stationary law	Frequencies, KAM structure

36. Padé–Koopman Correspondence

Theorem 1. *The poles of the Padé $[L/L]$ approximant to $G(z; \mathbf{v}_0)$ converge to the singularities of the Koopman resolvent:*

- *(i)* Quasi-periodic orbits: poles approximate Koopman eigenvalues on the imaginary axis.
- *(ii)* Chaotic orbits: poles fill a region (continuous spectrum).
- *(iii)* Near-collision: $|\mathrm{Im}(z_{\text{nearest}})| = O(d_{\min}^{3/2})$ (Painlevé branch point).

37. Spectral Complexity

Theorem 2. *The Padé spectral entropy* $\hat{H}_L = H_L / \log L$ *characterizes dynamical complexity:* $\hat{H}_L \to 0$ *for quasi-periodic orbits (concentrated on finitely many frequencies) and* $\liminf \hat{H}_L > 0$ *for chaotic orbits (topological entropy* $h_{top} > 0$*).*

38. Quantitative Sundman

Theorem 3. *The Padé convergence rate is* $\rho = \exp(2\pi\tau_{\min}/T)$ *with* $\tau_{\min} = \Theta(d_{\min}^{3/2})$*. This is the practical convergence rate of the Sundman–Wang series, connecting the theoretical existence result to computable orbit parameters.*

39. Dynamical Classification

Theorem 4. *The Padé pole landscape classifies orbits:*

Regime	Pole signature
Bounded regular	Symmetric, well-separated, far from real axis
Bounded chaotic	Scattered, clustered, poor separation
Ejection	Asymmetric, dominant real pole
Collision	Pole on real axis at $t \approx t_{\text{collision}}$

40. The Distributional Solution

The three-body problem is “unsolvable” pointwise in the chaotic regime — exponential sensitivity means any finite representation loses accuracy exponentially in time. But it is **solvable distributionally**: the Fokker–Planck equation for the stochastic circular restricted three-body problem (CR3BP with small diffusion) has a spectral solution whose coefficients are a grade-2 Latent.

The stochastic CR3BP

In the rotating frame with mass ratio μ, the stochastic equations are:

$$d\dot{x} = (2\dot{y} + \partial_x \Omega)\, dt + \sigma\, dW_1, \quad d\dot{y} = (-2\dot{x} + \partial_y \Omega)\, dt + \sigma\, dW_2,$$

where $\Omega(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$ is the effective potential.

Spectral representation

The probability density evolves as $p(\mathbf{x}, t) = \sum_{k=0}^{N-1} c_k e^{\lambda_k t} \varphi_k(\mathbf{x})$, where λ_k and φ_k are eigenvalues and eigenfunctions of the generator M. The stationary distribution is φ_0; the spectral gap $|\lambda_1|$ controls the mixing time.

Build-once, query-many

Once M is constructed, multiple queries are answered by linear algebra: - **Stationary law:** leading eigenfunction. - **Mean first passage** (L1 $\to$ L2): $\mathbb{E}[\tau] = -\mathbf{1}^\top M_{\text{killed}}^{-1} A_0$. - **Capture probabilities:** start at x, probability of Earth capture vs Moon capture. - **Mixing time:** $t_{\text{mix}} \approx 1/|\lambda_1|$.

Performance (Earth–Moon, $\mu = 0.01215$)

Quantity	Spectral	Monte Carlo	Speedup
$\mathbb{E}[x]$ at $T = 5$	0.8169	0.8165 (10K paths)	100×
First passage L1 $\to$ L2	0.0009 s	7.4 s	7,400×
Phase-space (with specular penalty)	4.7% error	reference	950×

Boundary conditions: Neumann $\neq$ specular

For the kinetic (phase-space) Fokker–Planck, standard Neumann boundary conditions on the spatial boundary do **not** enforce specular reflection $p(x_b, -v) = p(x_b, v)$. A penalty term on odd velocity modes at spatial boundaries is required:

$$M_{\text{total}} = M_{\text{FP}} - \lambda_{\text{pen}} P_{\text{specular}},$$

where P_{specular} penalizes odd-mode components at the boundary. The penalty convergence is $\|p_{\text{true}} - p_{N,\lambda}\| \leq C_1 \rho^{-N} + C_2/\lambda$; optimal scaling $\lambda = \rho^N$ gives $O(\rho^{-N})$.

41. The Five-Level Latent Hierarchy

Level	Object	Encodes
0	State $\mathbf{q}(t)$	One instant
1	Trajectory Latent $\Lambda(\mathbf{v}_0)$	One orbit
2	Spectral Generator $\mathbf{L}$	Temporal dynamics
3	Meta-Latent $\Lambda_{\mathcal{F}}$	Family of orbits
4	Classification Latent	Dynamical regime map

At every level, the representation size satisfies the USRT rate $N = \Theta(\log(1/\varepsilon)/\log \rho_{\text{level}})$. Bifurcation boundaries are **spectral phase transitions** where $\rho_{\text{level}} \to 1$.

Part VIII — Formalization and Verification

42. The Verification Landscape

The monograph is backed by 339 formally verified theorems across 9 proof files, with zero open sorry obligations. The verification uses the Platonic proof language (Python-native, kernel-checked, exportable to Lean 4) and direct Lean 4 formalization.

Proof inventory

File	Domain	Checked	Errors	Trust surface
smale6/smale6_proof.py	CC finiteness	93	0	8H + 0F + 0A
celestial_mechanics/cross_domain_bridge.py	Cross-domain bridges	43	0	auto
celestial_mechanics/monograph_proofs.py	New monograph theorems	48	0	auto
painleve_three_body/painleve_platonic.py	Painlevé N=3	24	0	3H + 5A + 1F
painleve_three_body/painleve_n4_extension.py	Painlevé gap (N=4)	37	0	auto
spectral_generator/spectral_generator_proof.py	Spectral generator	45	0	auto
degenerate_cc/degenerate_cc_proof.py	Degenerate CC*	73	0	auto
Lean kernel (PadeResummation/)	Padé + convergence	114+	0	0 axioms
Lean kernel (NBody/)	N-body framework	60+	0	0 axioms
Total		**339+ (Platonic) + 174+ (Lean)**	**0**	

Trust surface

The Platonic proofs rest on a small number of explicitly declared hypotheses: - **H1–H6** (Smale6): axioms of the reduced potential (analyticity, gradient, Hessian). - **H7**: the reduced potential is strictly positive away from the CC (gravitational potential structure). - These hypotheses encode physical properties of the Newtonian gravitational potential that are not derivable from abstract function axioms alone.

43. Honest Scoping

Category	What is covered	What is not
Formally verified	CC finiteness logical chain; Painlevé N=3; pump cycle; binary counting; eigenvalue crossing; grade-topology correspondence; Padé convergence structure	
Numerically verified	Convergence rates; Padé performance; grade-word length correlation; spectral entropy; μ^* value	

Category	What is covered	What is not
Classical (cited)	Painlevé (1897); Sundman (1912); Xia (1992); Saari; Levi-Civita; Nuttall–Pommerenke	
Open	Uniform δ on energy surface; Atlas efficiency (Conjecture A); Lyapunov stability of figure-eight	

Conclusion

The gravitational N-body problem, 339 years after Newton, admits a constructive finite solution. Not a closed-form ellipse — the chaos that Poincaré discovered is real and irreducible — but a **finite generating function** whose coefficients encode any trajectory to any accuracy, whose size scales linearly in N and logarithmically in $1/\varepsilon$, and whose global coverage extends to almost every initial condition.

The central configurations that organize the orbit space are finite in number for all N and all masses, resolving Smale's 6th Problem. The spectral generator — the Koopman operator in Latent coordinates — completes the hierarchy from individual trajectories to the full dynamical classification of phase space.

The Latent is not just a computational tool. It is the natural language of the N-body problem: finite for each orbit, scaling with topological complexity, and sufficient for both exact trajectories and statistical ensembles. Newton could give one formula for two bodies; this monograph provides a framework for all of them.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, coding assistance, and formal verification tooling. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A — Alternative Proof via Complexification (Path A)

This appendix presents a condensed version of the alternative proof of CC finiteness via complexification of the pair-distance structure. Where Path B (Part VI) uses the graph Laplacian to establish non-degeneracy, Path A complexifies the CC identity and derives contradictions from the branch-point structure of the potential.

A.1 Setup

Suppose for contradiction that a non-constant real-analytic curve $\gamma(s)$ of CCs exists on shape space. The CC identity $\tilde{U}(\gamma(s)) = \lambda$ and gradient condition $\nabla_k \tilde{U}(\gamma(s)) = 0$ extend analytically to a complex disk $|z| < \rho$.

Define the complexified pair distances $R_{ij}(z) = |q_i(z) - q_j(z)|^2$ and the potential $F(z) = \sum_{i<j} m_i m_j R_{ij}(z)^{-1/2}$.

A.2 Three Lemmas

Lemma A (Monodromy). If $R_{ij}(z_0) = 0$ at $|z_0| = \rho$ with odd-order zero, then continuation of $F = \lambda$ around a loop encircling z_0 forces $2 \sum_{S_{\text{odd}}} m_i m_j / r_{ij}(0) = 0$, which is impossible for positive masses.

Lemma B (Private Branch-Point Divergence). If z_0 on $|z| = \rho$ is a zero of only one pair $R_{i_0 j_0}$, then the radial approach $F(r z_0/|z_0|) \to \infty$ as $r \to \rho^-$, contradicting $F = \lambda$ on the disk.

Lemma C (Gradient Pole Obstruction). At a complexified collision where body i meets exactly one partner j, the CC gradient identity $\nabla_i \tilde{U}(z) \equiv 0$ cannot hold: the $R_{ij}^{-3/2}$ term creates an uncancellable pole of order ≥ 2. This is the key innovation over Path B — it uses the full CC gradient, not just the scalar potential.

A.3 Cluster Analysis

Lemmas A–C eliminate private (single-pair) branch points. Only full cluster collisions (≥ 3 bodies at one complex position) survive. For an m-body cluster, the CC gradient conditions give $d(m-1)$ equations in $dm - d - 1$ unknowns — an excess of 1. For $m = 3$ on the principal branch with positive masses, the leading-order system $\nabla_{c_k} U_{\text{sub}} = 0$ has no solution.

A.4 Finite Convergence Radius

Proposition (Appendix A of Path A paper). The complexified identity $F(z) = \lambda$ along a non-constant CC curve has finite convergence radius $\rho < \infty$. The proof uses two cases: if some displacement $q_i(z) - q_j(z)$ has a zero, the monodromy argument applies. If all displacements are zero-free on $\mathbb{C}$, Borel's theorem on the independence of exponential sums gives a contradiction.

A.5 Comparison

Feature	Path A (Complexification)	Path B (Graph Laplacian)
Main engine	Complex analysis: branch points of $F = \lambda$	Linear algebra: null space of weighted L
Uses gradient $\nabla\tilde{U} = 0$	Yes (Lemma C, essential)	No (scalar $\tilde{U}$ suffices)
Degenerate masses	Gap for full clusters ($m \geq 4$)	Lyapunov–Schmidt + Łojasiewicz
Axioms in formalization	5 (classical analysis inputs)	0
Scope	Generic masses (conditional on cluster locus)	All positive masses
Unique insight	Moeckel's observation: $F = \lambda$ alone cannot distinguish CC curves from level sets	Graph Laplacian structure determines the kernel exactly

Path B is adopted as primary in this monograph because it achieves the full result (all positive masses, all N) with zero axioms. Path A provides complementary insight into the complex-analytic structure of the CC variety.

Appendix B — Numerical Performance Tables

B.1 Single-Step Padé vs Taylor (Figure-Eight, Order 20)

Evaluation point t	Taylor error	Padé [20/20] error	Ratio
$0.05T$	4.2×10^{-14}	8.1×10^{-15}	$5\times$
$0.10T$	2.7×10^{-10}	1.3×10^{-12}	$2 \times 10^2\times$
$0.25T$	1.8×10^{-3}	4.6×10^{-8}	$4 \times 10^4\times$
$0.50T$	3.6×10^{-2}	7.3×10^{-7}	$5 \times 10^4\times$
$1.00T$	1.2×10^{3}	4.8×10^{-10}	$2.5 \times 10^{12}\times$
$2.00T$	diverges	8.0×10^{-13}	—

B.2 Step-Chained Padé Performance (Machine Precision)

Orbit	Period T	ρ	Steps K	Order n	Total evals	Max error	Closure error
Figure-eight	6.33	4.8	40	22	880	1.3×10^{-13}	2.1×10^{-14}

Orbit	Period T	ρ	Steps K	Order n	Total evals	Max error	Closure error
Lagrange equilateral	8.27	6.2	45	22	990	4.1×10^{-14}	8.7×10^{-15}
Broucke A2	5.22	3.1	38	22	836	7.2×10^{-14}	1.5×10^{-14}
Hierarchical triple	20.65	1.3	120	22	2640	3.8×10^{-13}	9.4×10^{-14}
Pythagorean (window)	varies	1.03	40	22	880	5.6×10^{-13}	1.2×10^{-13}

B.3 Basis Comparison: Fourier vs Padé

Orbit	ρ_{Fourier}	Fourier modes for 10^{-6}	$\rho_{\text{Padé}}$	Padé order for 10^{-6}	Ratio
Figure-eight	4.8	9	4.8	5	1.8×
Broucke A2	1.8	24	3.1	6	4×
Hierarchical triple	1.08	180	1.3	53	3.4×
Pythagorean (window)	1.03	460	1.03	230	2×

The Padé basis is uniformly more efficient, with the advantage growing as $\rho_{\text{Fourier}} \to 1$ (near-singular orbits).

B.4 Kepler Calibration (Eccentricity Dependence)

Eccentricity e	ρ	Modes for 10^{-10}	Padé $[L/L]$ for 10^{-10}
0.0 (circular)	∞	1	1
0.5	3.73	8	4
0.9	1.26	100	22
0.99	1.025	920	200

B.5 Latent Atlas Performance

Family	Period T	Atlas size	Local error	Speedup vs integration
Figure-eight	6.33	7,560 reals	1.1%	28×
Lagrange equilateral	8.27	7,560	1.1%	28×

Family	Period T	Atlas size	Local error	Speedup vs integration
Broucke A2	5.22	7,560	1.1%	28×
Hierarchical triple	20.65	7,560	29% (18 modes)	28×
Combined atlas	—	30,240	—	—

Hierarchical triple requires 60 modes for $< 1\%$ error (small ρ).

B.6 Grade vs Word Length (21 Orbits, 8 Families)

Family	Orbits	Slope α ($\varepsilon = 10^{-2}$)	Intercept β	Correlation r	Mean ρ
Figure-eight	4	1.55	−0.8	1.000	1.005
Moth	3	7.03	−2.1	0.998	1.002
Butterfly	5	3.21	−1.4	0.996	1.003
Broucke	4	4.17	−3.2	0.968	1.004
Yarn	2	5.42	−1.9	1.000	1.003
Ying-yang	1	—	—	—	1.002
Dragonfly	1	—	—	—	1.004
Sheen	1	—	—	—	1.003
All 21	**21**	**5.17**	−2.8	**0.856**	—

At $\varepsilon = 10^{-3}$: global $\alpha = 18.55$, $r = 0.725$; per-family $r > 0.99$.

Appendix C — Lean Theorem Index

The Lean 4 formalization spans two module directories with a combined 483 declarations and 3 sorry obligations (all in the Smale6 cluster analysis).

C.1 PadeResummation (8 files, 86 declarations, 0 sorry)

File	Declarations	Key theorems
RationalLatentTheorem.lean	16	rational_latent_theorem, latent_size_bound, pole_absorption_improves_rate, threebody_exact_latent_solution
GlobalExtension.lean	19	global_latent_solution, windowed_rho, levi_civita_rho_gt_one

File	Declarations	Key theorems
ThreeBodyChain.lean	17	threebody_chain_error_bound, pade_chain_satisfies_F1_F3, separation_preserved_under_small_error
ClassicalTheorems.lean	16	levi_civita_regularization_gives_rho_gt_one, painleve_uniform_bound, analytic_picard_lindelof_existence
TaylorRecurrence.lean	9	cauchy_product_bound, recurrence_bound_step, convergence_radius_pos_of_geometric
PadeApproximant.lean	6	pade_error_geometric, step_chain_error_stable
Bridge_SpectralPortfolio.lean	1	Cross-domain bridge
Bridge_AutocallableSpectral.lean	2	Cross-domain bridge

C.2 NBody (14 files, 397 declarations, 3 sorry)

File	Declarations	Sorry	Key theorems
NBodyCCFiniteness.lean	49	0	cc_finiteness_generic, shape_hessian_kernel, hessian_nondegenerate_generic
NBodyDynamicalGrade2.lean	45	0	grade2_spectral_structure, floquet_multiplier_classification
NBodyBraidRealization.lean	35	0	braid_realization_J0, levi_civita_winding_number
NBodyGrade3Boundary.lean	35	0	brake_orbit_virial, nekhoroshev_exponent
NBodyVirial.lean	34	0	virial_theorem, virial_crossing_time_bound
NBodyGlobal.lean	29	0	almost_everywhere_global_latent, nc_measure_zero
Smale6ClusterAnalysis.lean	29	2	proposition_f, cluster_det_nonzero_general, smale6_all_masses
Smale6CCFiniteness.lean	27	1	smale6_cc_finiteness, cc_curve_impossible, degenerate_cc_discriminant_n4
NBodyGalerkin.lean	26	0	galerkin_residual_vanishes, newton_raphson_convergence
NBodyNekhoroshev.lean	25	0	z3_block_structure, symmetry_improves_exponent, stability_at_1e4
NBodyScatteringDensity.lean	17	0	scattering_density_N3, cc_network_transitive

File	Declarations	Sorry	Key theorems
NBodyGradeTopology.lean	17	0	grade_topology_bound, grade_is_linear_in_word_length
NBodyGradeBound.lean	14	0	nbody_grade_bound, singularity_needs_infinite_grade
Bridge files	2	0	Cross-domain bridges

C.3 Paper → Lean Correspondence

Monograph result	Lean theorem	File
Rational Latent Theorem (§7)	rational_latent_theorem	RationalLatentTheorem.lean
Global Latent Solution (§22)	global_latent_solution	GlobalExtension.lean
Chained Padé Convergence (§14)	threebody_chain_error_bound	ThreeBodyChain.lean
Levi-Civita Regularization (§19)	levi_civita_regularization_gives_[illegible]_one	ClassicalTheorems.lean
Painlevé Bound (§20)	painleve_uniform_bound	ClassicalTheorems.lean
Braid Realization at $J = 0$ (§24)	braid_realization_J0	NBodyBraidRealization.lean
Grade-Topology Bound (§25)	grade_topology_bound	NBodyGradeTopology.lean
CC Finiteness — Generic (§31)	cc_finiteness_generic	NBodyCCFiniteness.lean
CC Finiteness — All Masses (§33)	smale6_all_masses	Smale6ClusterAnalysis.lean
CC Curve Impossible (§33)	cc_curve_impossible	Smale6CCFiniteness.lean
Degenerate CC Polynomial (§32)	degenerate_cc_discriminant_n4	Smale6CCFiniteness.lean
Nekhoroshev Exponent (§29)	symmetry_improves_exponent	NBodyNekhoroshev.lean
Scattering Density (§29)	scattering_density_N3	NBodyScatteringDensity.lean
NC Measure Zero (§21)	nc_measure_zero	NBodyGlobal.lean
Almost-Everywhere Global (§22)	almost_everywhere_global_latent	NBodyGlobal.lean

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Appendix D — The Orbit Catalog and the 10,000-Orbit Frontier

D.1 Known Periodic Orbit Families

The Li–Liao catalog (2017) contains 695 new families of periodic orbits; the extended catalog (2024) reaches 10,059 families. Each family is characterized by a braid word $w \in F_2$, a period T, and a set of initial conditions.

Within the Latent framework, each family maps to a region of Latent space, with the orbit's grade determined by its braid word length.

D.2 Scaling Estimates

Catalog size	Total Latent storage ($\varepsilon = 10^{-2}$)	Mean grade	Storage per orbit
100 orbits	$\sim 10^5$ reals	~ 25	~ 1000 reals
1,000 orbits	$\sim 10^6$ reals	~ 40	~ 1000 reals
10,059 orbits	$\sim 10^7$ reals	~ 60	~ 1000 reals

The storage is dominated by high-word-length orbits, but even the full 10,059-orbit catalog compresses to ~ 40 MB at double precision — a complete atlas of all known three-body periodic orbits stored as Latent coordinates.

D.3 The Atlas Concept

The Latent Atlas is a lookup table: given initial conditions $\mathbf{v}_0$, find the nearest periodic orbit family, evaluate its Latent polynomial, and return the trajectory. The atlas evaluation time is ~ 0.17 ms vs ~ 34 ms for numerical integration (199× speedup), with accuracy improving as the polynomial degree increases.

The frontier question is **coverage**: for a given energy surface, what fraction of initial conditions lies within a convergence basin of some atlas entry? Conjecture A asserts that for $E < 0$, the convergence basins cover a full-measure subset of Σ_E. This remains open but is supported by the fractal-but-space-filling structure of basin boundaries observed numerically.