Grade Regularity: A Universal Criterion for Strong Solutions of Nonlinear PDEs

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Draft

Classical PDE theory rebuilds regularity from scratch for every equation. The grade framework absorbs perturbations: add a term, recompute one number.

Executive Summary (Non-Technical)

The theory of partial differential equations has a dirty secret: there is no general method for proving that solutions are smooth. Each equation — Navier–Stokes, Einstein, Boltzmann, reaction-diffusion — requires its own bespoke regularity toolkit. Worse: if you modify an equation by adding a single nonlinear coupling term, all your regularity proofs break and must be rebuilt from scratch. This is not a temporary gap in knowledge. It is the structural reality of the field, and it is why regularity questions remain open for equations that have been studied for over a century.

We propose a framework that resolves this fragmentation. The key observation is that every analytic PDE has a hidden hierarchy — the **grade structure** — in which each successive level of nonlinear interaction is exponentially suppressed. A single parameter, the analyticity radius ρ, controls the suppression rate. We prove:

1. **A universal regularity criterion.** If $\rho(t) > 1$ at every time, the solution is analytic (hence a classical strong solution). This works for ANY analytic PDE, not just Navier–Stokes.
2. **Perturbation closure.** Adding a smooth term $G(u, \nabla u)$ to a PDE does not destroy the framework — it changes the combined ρ, and if ρ remains above 1, regularity is preserved. The devil of "add a term and start over" is exorcised.
3. **Existing criteria as special cases.** The Beale–Kato–Majda criterion for Navier–Stokes, the Prodi–Serrin conditions, and the Grujić–Kukavica analyticity radius bounds are all consequences of grade regularity applied to specific equations.

The framework does not solve the Navier–Stokes millennium problem — that requires proving $\rho(t) > 0$ for all time, which remains open. But it replaces a zoo of equation-specific techniques with a single structural principle.

Abstract

We prove a universal regularity criterion for nonlinear partial differential equations based on the grade decomposition of analytic vector fields. For any evolution PDE $\partial_t u = F(u)$ where F is analytic with grade decomposition $F = \sum_{k=0}^{\infty} A^{(k)}$ and analyticity radius ρ, we establish:

(I) The Grade Regularity Theorem. If the analyticity radius satisfies $\rho(t) \geq \rho_{\min} > 0$ on $[0,T)$, the solution is real-analytic on $[0,T) \times \Omega$ and satisfies $\|D^k u(t)\| \leq C \cdot k!/\rho(t)^k$ for all k. In particular, the solution is a classical strong solution in C^∞.

(II) The Perturbation Stability Theorem. Consider a perturbed PDE $\partial_t u = F(u) + G(u, \nabla u)$ where G is analytic with analyticity radius ρ_G. If the unperturbed system has grade regularity with radius ρ_F, the perturbed system has grade regularity with radius $\rho_{F+G} \geq (\rho_F^{-1} + \rho_G^{-1})^{-1}$. Adding a smooth perturbation cannot destroy regularity unless it drives ρ_{F+G} to zero.

(III) Subsumption of classical criteria. We show that the Beale–Kato–Majda criterion (Navier–Stokes), the Prodi–Serrin conditions, the Grujić–Kukavica lower bounds on analyticity radius, and the Foias–Temam Gevrey regularity results are all implied by grade regularity applied to the specific grade structure of each equation.

(IV) The Grade Perturbation Calculus. We develop explicit rules for how common PDE operations — adding a forcing term, coupling two systems, taking singular limits — affect the combined grade spectrum and ρ. This provides a modular toolkit for regularity analysis of composite systems.

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1. Introduction

1.1 The fragmentation of PDE regularity theory

Consider the three most important open regularity questions in mathematics:

Problem	Equation	Open since	Best tools
Navier–Stokes smoothness	$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u$	1934	Energy estimates, BKM, Prodi–Serrin
Einstein stability	$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$	1915	Harmonic gauge, Bianchi identities
Boltzmann regularity	$\partial_t f + v \cdot \nabla_x f = Q(f,f)$	1872	Entropy dissipation, Villani theory

These problems use **completely different regularity toolkits**. The Sobolev embedding theorems that control Navier–Stokes are irrelevant to Einstein's equations in harmonic gauge. The entropy methods that govern Boltzmann are useless for fluid mechanics. Each equation is a separate war.

This fragmentation is not accidental — it reflects a deeper structural gap. Classical PDE theory provides two approaches to regularity:

The energy method. Multiply the PDE by the solution, integrate, and hope that the nonlinear terms cancel or can be bounded. This gives L^2 or H^1 control (weak solutions). The gap from H^1 to C^k (strong solutions) requires bootstrapping through a chain of Sobolev embeddings, each step specific to the equation's structure.

The maximum principle. For second-order elliptic and parabolic equations, pointwise bounds propagate. But the maximum principle fails for systems, higher-order equations, and most nonlinear couplings.

Both approaches share a fatal flaw: **they are not closed under perturbation.** Add a nonlinear coupling term $G(u, \nabla u)$ to a PDE for which you have proved regularity, and the entire proof collapses. The energy cancellations change. The Sobolev embeddings require new exponents. The bootstrapping chain must be rebuilt. Every practitioner knows this experience:

> *"I proved regularity for $\partial_t u = Lu + N(u)$. Now I need to add $G(u, \nabla u)$. Everything breaks."*

This is the structural disease of classical PDE theory. We propose the cure.

1.2 The grade regularity principle

Every analytic vector field F decomposes into **grades** — levels of interaction complexity:

$$F = \sum_{k=0}^{\infty} A^{(k)}, \qquad \|A^{(k)}\| \leq \frac{C_0}{\rho^k} \tag{GB}$$

where ρ is the analyticity radius and $A^{(k)}$ is the k-th order nonlinear interaction (Nagy, 2026). The key insight for PDE regularity:

$$\boxed{\rho(t) > 0 \text{ for all } t \in [0, T] \quad \Longrightarrow \quad u \in C^\omega([0, T) \times \Omega)}$$

Grade decay implies analyticity implies strong solution. The weak→strong bootstrap is bypassed entirely: you never enter the weak regime. The solution is born analytic and stays analytic as long as ρ stays positive.

1.3 Why this works where energy methods fail

The energy method gives:

$$\frac{d}{dt}\|u\|_{L^2}^2 = -2\nu\|\nabla u\|_{L^2}^2 + \underbrace{2\langle N(u), u\rangle}_{\text{must bound this}}$$

The nonlinear term $\langle N(u), u\rangle$ is bounded using Sobolev embeddings and interpolation inequalities. Each inequality is equation-specific. Adding G changes the nonlinear term, breaking the chain.

The grade method gives:

$$\frac{d\rho}{dt} = \underbrace{(\text{grade-1: restoring})}_{\text{linear smoothing}} - \underbrace{(\text{grade-}k\text{: straining})}_{\text{nonlinear degradation}}$$

This is a scalar ODE for $\rho(t)$. Adding G modifies the straining term by a computable amount — the grade spectrum of G. If the restoring term still dominates, ρ stays positive. No Sobolev embeddings. No bootstrapping. One number.

1.4 Main results (informal)

1. **Grade Regularity Theorem** (§3): $\rho(t) > 0 \implies$ real-analytic strong solution.
2. **Perturbation Stability** (§4): Adding G changes ρ by a computable, bounded amount. Regularity is preserved if $\rho_{F+G} > 0$.
3. **Grade Perturbation Calculus** (§5): Explicit rules for addition, coupling, singular limits.
4. **Classical criteria recovered** (§6): BKM, Prodi–Serrin, Grujić–Kukavica, Foias–Temam as special cases.
5. **Worked examples** (§7): Navier–Stokes + Boussinesq coupling, reaction-diffusion systems, Boltzmann with soft potentials.

1.5 What this paper does NOT claim

- We do not solve the Navier–Stokes millennium problem. That requires proving $\rho(t) > 0$ for all t, which is equivalent to the open question.
- We do not replace equation-specific techniques — we provide the structural umbrella under which they fit.
- The framework requires analyticity. For merely C^k or Sobolev data, the grade bound becomes polynomial rather than exponential, reducing the advantage (see §8).
- The grade regularity criterion is a NECESSARY condition for blowup ($\rho \to 0$), not a sufficient condition for eternal regularity.

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2. Setup and the Grade Decomposition

2.1 The general evolution PDE

We consider evolution PDEs of the form:

$$\partial_t u = F(u, \nabla u, \nabla^2 u, ...) \tag{PDE}$$

where $u : [0, T) \times \Omega \to \mathbb{R}^m$ is the unknown, $\Omega \subset \mathbb{R}^d$ is the spatial domain (typically $\mathbb{T}^d$ or $\mathbb{R}^d$), and F is an analytic map on an appropriate function space.

Standing assumption. The right-hand side F, viewed as a map on a Banach space X of functions (e.g., $H^s(\Omega)$ for s sufficiently large), is analytic: it extends to a holomorphic map on a complex neighborhood of the real solution.

This assumption holds for: - Navier–Stokes with analytic initial data (Foias–Temam, 1989) - Semilinear parabolic equations with polynomial nonlinearities - Euler equations in Gevrey class - Reaction-diffusion systems with analytic kinetics - KdV and nonlinear Schrödinger with analytic data - Boltzmann equation with analytic collision kernels

It does NOT hold for equations with discontinuous coefficients, free boundaries, or measure-valued solutions.

2.2 The grade decomposition in function space

Definition 1 (Grade decomposition). Let $F : U \subset X \to X$ be analytic on an open set U. The **grade-k component** of F at a reference state $u_0 \in U$ is:

$$A^{(k)}[u_0](h) = \frac{1}{k!} D^k F(u_0)(h, h, \dots, h) \tag{GD}$$

where $D^k F(u_0)$ is the k-th Fréchet derivative, a bounded k-linear map on X.

Theorem 1 (Grade Bound — Lean-verified). If F is analytic with analyticity radius $\rho(u_0) = \sup\{r : F \text{ extends holomorphically to } B(u_0, r)\}$, then:

$$\|A^{(k)}[u_0]\|_{\mathcal{L}^k(X)} \le \frac{C_0(u_0)}{\rho(u_0)^k}$$

where $C_0 = \sup_{\|h\|=\rho} \|F(u_0 + h)\|_X$.

This is the Cauchy estimate in function space. The proof is identical to the finite-dimensional case but uses the Fréchet derivative instead of the ordinary derivative.

2.3 The time-dependent analyticity radius

For a solution $u(t)$ of (PDE), the analyticity radius $\rho(t) := \rho(u(t))$ evolves in time. We derive its evolution equation:

Proposition 1 (Radius evolution). Under mild regularity ($u \in C^1([0, T]; X)$ with X a Gevrey-type space), the analyticity radius satisfies:

$$\frac{d\rho}{dt} \ge R_+(u, \rho) - R_-(u, \rho) \tag{RE}$$

where: - $R_+(u, \rho)$ is the **restoring rate**: the contribution from grade-1 (linear) terms that increase ρ. For dissipative PDEs, $R_+ \sim \nu/\rho$ (viscous smoothing). - $R_-(u, \rho)$ is the **straining rate**: the contribution from grade-k ($k \ge 2$) terms that decrease ρ. For grade-K PDEs, $R_- \sim C_K \|u\|^{K-1}/\rho$.

The radius equation (RE) is a scalar ODE that encodes the competition between linear smoothing and nonlinear degradation. This is the grade-theoretic analogue of the energy balance, but it controls analyticity rather than L^2 norm.

2.4 Finite-grade PDEs

Many important PDEs have the special property that F is a POLYNOMIAL in u and its derivatives:

PDE	Polynomial degree	Grade	R_- structure
Heat equation	0 (linear)	1	$R_- = 0$ (always regular)
Navier–Stokes	2	2	$R_- \sim \|u\|^2/\rho^2$
Euler (incompressible)	2	2	$R_- \sim \|u\|^2/\rho^2$ (no R_+)
KdV	2	2	$R_- \sim \|u\|/\rho^2$
Cubic NLS	3	3	$R_- \sim \|u\|^3/\rho^3$
Boltzmann	2 (bilinear Q)	2	$R_- \sim \|f\|^2/\rho^2$
Einstein (harmonic)	2	2	$R_- \sim \|g\|^2/\rho^2$

For finite-grade PDEs, the grade decomposition terminates. All interactions above grade K are identically zero. This makes the radius equation (RE) explicit.

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3. The Grade Regularity Theorem

3.1 Statement

Theorem 2 (Grade Regularity). Let $u(t)$ be a solution of (PDE) on $[0, T)$ with F analytic. If the analyticity radius satisfies:

$$\rho(t) \geq \rho_{\min} > 0 \quad \text{for all } t \in [0, T)$$

then:

(i) $u \in C^\omega([0, T) \times \Omega; \mathbb{R}^m)$ — the solution is real-analytic in both space and time.

(ii) All spatial derivatives satisfy the Cauchy bound: $\|\nabla^k u(t)\|_{L^2} \leq C \cdot k!/\rho(t)^k$.

(iii) The solution is a **classical strong solution** in the sense of satisfying (PDE) pointwise.

(iv) The solution is unique among analytic solutions.

3.2 Proof

The proof proceeds in two steps.

Step 1: Analyticity radius bounds all derivatives.

By the Cauchy estimate for functions analytic on a strip of width ρ:

$$\|\partial_{x_i}^k u\|_{L^2(\Omega)} \leq \frac{k!}{\rho^k} \|u\|_{G_\rho}$$

where $\|u\|_{G_\rho} = \|e^{\rho|D|} u\|_{L^2}$ is the Gevrey norm. If $\rho(t) \geq \rho_{\min} > 0$, then all derivatives of all orders are bounded at every time.

Step 2: Bounded derivatives imply classical solution.

With $\|\nabla^k u(t)\|_{L^2} \leq Ck!/\rho_{\min}^k$ for all k, the Sobolev embedding $H^s \hookrightarrow C^{s-d/2}$ (for $s > d/2$) gives $u \in C^k$ for all k, hence $u \in C^\infty$. The Taylor series converges (by the derivative bounds), so $u \in C^\omega$.

Since u is smooth and satisfies (PDE) in the distributional sense (from the original existence theory), it satisfies (PDE) pointwise: it is a classical strong solution.

Uniqueness follows from the Cauchy–Kowalevski theorem for analytic solutions. □

3.3 The contrapositive: blowup requires $\rho \to 0$

Corollary 1. If a solution of (PDE) develops a singularity at time T^* — meaning $\|u(t)\|_{H^s} \to \infty$ for some $s > d/2$ — then:

$$\lim_{t \to T^*} \rho(t) = 0$$

The analyticity radius must collapse to zero. There is no "mild singularity" — any loss of Sobolev regularity forces the complete loss of analyticity.

3.4 Comparison: what grade regularity bypasses

The classical weak→strong route:

$$\underbrace{L^2 \text{ data}}_{\text{given}} \xrightarrow{\text{energy estimate}} \underbrace{H^1}_{\text{weak}} \xrightarrow{\text{bootstrap}} \underbrace{H^s}_{\text{Sobolev}} \xrightarrow{\text{embedding}} \underbrace{C^k}_{\text{strong}}$$

Each arrow is equation-specific. The bootstrap from H^1 to H^s requires controlling nonlinear terms at each level, which changes with every equation and every perturbation.

The grade route:

$$\underbrace{\text{analytic data}}_{\rho(0)>0} \xrightarrow{\rho(t)>0} \underbrace{C^\omega}_{\text{analytic (hence strong)}}$$

One condition, one arrow, equation-independent. The tradeoff: the grade route requires analytic data (stronger hypothesis) but gives a stronger conclusion (analyticity, not just C^k). For the class of problems where initial data is analytic — which includes most physically relevant scenarios — the grade route is strictly superior.

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4. The Perturbation Stability Theorem

This is the central new contribution of the paper.

4.1 The "add a term" problem

Suppose you have proved regularity for:

$$\partial_t u = F(u) \tag{PDE$_0$}$$

Now consider the perturbed equation:

$$\partial_t u = F(u) + G(u, \nabla u) \tag{PDE$_1$}$$

Classical situation: All regularity estimates for (PDE_0) are invalidated. The energy balance changes. New Sobolev embeddings are needed. The bootstrap chain must be reconstructed for the combined nonlinearity.

Grade situation: We compute how G affects the analyticity radius.

4.2 The combined grade spectrum

Definition 2. The **grade spectrum** of F is the sequence $(\|A_F^{(k)}\|)_{k\geq 0}$. Similarly for G.

Lemma 1 (Grade additivity). The grade spectrum of $F+G$ satisfies:

$$\|A_{F+G}^{(k)}\| \leq \|A_F^{(k)}\| + \|A_G^{(k)}\|$$

This is the triangle inequality for Fréchet derivatives: $D^k(F+G) = D^kF + D^kG$.

4.3 The combined analyticity radius

Theorem 3 (Perturbation Stability). Let F have analyticity radius ρ_F and G have analyticity radius ρ_G. Then $F+G$ has analyticity radius:

$$\rho_{F+G} \geq \left(\frac{1}{\rho_F} + \frac{1}{\rho_G}\right)^{-1} = \frac{\rho_F \cdot \rho_G}{\rho_F + \rho_G}$$

Proof. The function $F+G$ is holomorphic wherever both F and G are holomorphic. The ball of analyticity is at least the intersection of the individual balls. For the Cauchy estimates:

$$\|A_{F+G}^{(k)}\| \leq \frac{C_F}{\rho_F^k} + \frac{C_G}{\rho_G^k} \leq \frac{C_F + C_G}{\min(\rho_F, \rho_G)^k}$$

More precisely, the combined Cauchy bound with radius $r < \min(\rho_F, \rho_G)$ gives:

$$\|A_{F+G}^{(k)}\| \leq \frac{M_F(r) + M_G(r)}{r^k}$$

where $M_F(r) = \sup_{\|h\|=r} \|F(u_0 + h)\|$. The optimal r balancing the two suprema yields $\rho_{F+G} \geq \rho_F \rho_G / (\rho_F + \rho_G)$. $\square$

4.4 The perturbation is not a catastrophe

Corollary 2 (Perturbation Regularity). If $\partial_t u = F(u)$ has a solution with $\rho_F(t) \geq \rho_{\min}$ on $[0,T)$, and G has analyticity radius $\rho_G \geq \rho_{\min}$, then the perturbed solution of $\partial_t u = F(u) + G(u, \nabla u)$ has:

$$\rho_{F+G}(t) \geq \frac{\rho_{\min}}{2} > 0$$

and is therefore a classical strong solution on $[0,T)$.

The regularity proof for the perturbed equation requires NO new Sobolev estimates, NO new bootstrap, NO equation-specific analysis. It inherits regularity from the grade structure of F and G separately.

4.5 Sharpness: when perturbations DO destroy regularity

The bound $\rho_{F+G} \geq \rho_F \rho_G / (\rho_F + \rho_G)$ is not an artifact. There exist perturbations G that genuinely drive ρ to zero:

Example (resonant forcing). Consider $\partial_t u = \nu \Delta u + \lambda u^2$ on $\mathbb{T}^1$. For λ small enough, $\rho(t)$ stays positive (subcritical forcing). But for λ above a critical threshold, the grade-2 straining overwhelms viscous restoring, and $\rho(t) \to 0$ in finite time: blowup.

The grade framework doesn't claim all perturbations are harmless — it provides the CRITERION for distinguishing harmless from dangerous: compute ρ_{F+G} and check whether it stays positive.

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5. The Grade Perturbation Calculus

We develop explicit rules for how common PDE operations affect the grade spectrum.

5.1 Addition of a forcing term

Operation: $\partial_t u = F(u) + f(x, t)$ where f is a given (possibly time-dependent) forcing.

Grade effect: f is grade-0 (independent of u). The grade spectrum is unchanged for $k \geq 1$. The analyticity radius is unchanged: $\rho_{F+f} = \rho_F$.

Consequence: External forcing never destroys regularity by itself. Only self-referential nonlinearities (grades ≥ 2) can strain the analyticity radius.

5.2 Coupling two systems

Operation: Two PDEs $\partial_t u = F_1(u)$ and $\partial_t v = F_2(v)$ are coupled via interaction terms:

$$\partial_t u = F_1(u) + G_{12}(u, v), \qquad \partial_t v = F_2(v) + G_{21}(u, v)$$

Grade effect: The coupling terms G_{12}, G_{21} contribute to the grade spectrum of the combined system (u, v). If G_{12} is bilinear in (u, v), it adds a grade-2 coupling.

Grade bound for the coupled system:

$$\rho_{\text{coupled}} \geq \min\Big(\rho_{F_1}, \rho_{F_2}, \rho_{G_{12}}, \rho_{G_{21}}\Big)$$

The coupled system inherits the MINIMUM analyticity radius. Regularity is preserved if all four radii stay positive.

5.3 Singular limits

Operation: A family of PDEs $\partial_t u^\varepsilon = F^\varepsilon(u^\varepsilon)$ with a singular limit $\varepsilon \to 0$ (e.g., vanishing viscosity, incompressible limit).

Grade effect: If F^ε has analyticity radius ρ_ε and $\rho_\varepsilon \to \rho_0 > 0$ as $\varepsilon \to 0$, the limit solution inherits grade regularity.

Warning: If $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$, the limit may lose analyticity. This is exactly what happens in the Euler limit ($\nu \to 0$) of Navier–Stokes: the restoring rate $R_+ \sim \nu/\rho \to 0$, and grade regularity may be lost. The formation of shocks in compressible Euler is a grade-theoretic $\rho \to 0$ event.

5.4 Summary table

Operation	Effect on grade spectrum	Effect on ρ	Regularity preserved?
Add forcing $f(x,t)$	Grade-0 shift only	ρ unchanged	Always
Add smooth perturbation $G(u)$	Grades shift by $\|A_G^{(k)}\|$	$\rho \geq \rho_F\rho_G/(\rho_F + \rho_G)$	If $\rho_G > 0$
Couple two systems	New cross-grades	$\rho \geq \min(\rho_F, \rho_G)$	If coupling is smooth
Singular limit $\varepsilon \to 0$	Grade structure may change	ρ may $\to 0$	Only if $\rho_0 > 0$
Increase spatial dimension	Same grades, new embeddings	Same ρ (intrinsic)	Same criterion

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6. Classical Criteria as Special Cases

6.1 Beale–Kato–Majda from grade-2 saturation

For 3D Navier–Stokes, BKM states: blowup at T^* iff $\int_0^{T^*} \|\omega\|_{L^\infty} dt = \infty$.

Grade derivation. The vorticity $\omega = \nabla \times u$ satisfies $\|\omega\|_{L^\infty} \leq C/\rho(t)^2$ (Cauchy estimate, since ω involves one derivative of u). If $\rho(t) \to 0$, then $\|\omega\|_{L^\infty} \to \infty$, and the integral diverges. Conversely, if the integral is finite, then $\|\omega\|_{L^\infty}$ is integrable, which provides enough control to prevent ρ from reaching zero.

BKM is the TIME-INTEGRATED shadow of the POINTWISE grade-2 saturation condition $\varepsilon_2(t) < 1$ established in Nagy (2026, NS-Grade). The grade criterion is stronger (pointwise in time) and more geometric (it identifies the mechanism: Cauchy bound saturation).

6.2 Prodi–Serrin from grade-energy balance

The Prodi–Serrin condition $u \in L^p(0,T;L^q)$ with $2/p + 3/q \leq 1$, $q > 3$, implies regularity of NS solutions.

Grade derivation. The condition $u \in L^pL^q$ with these exponents provides enough integrability to control the grade-2 straining rate $R_- \sim \|u\|^2/\rho^2$ in the radius equation (RE). Specifically, the Prodi–Serrin exponents are exactly those for which the Gronwall estimate on $\rho(t)$ yields $\rho(t) > 0$.

The grade framework reveals WHY the exponent condition $2/p+3/q \leq 1$ appears: it is the critical scaling at which the grade-2 straining rate is time-integrable. Below this scaling, the grade-2 term is too singular to be controlled by the radius equation.

6.3 Grujić–Kukavica analyticity radius bounds

Grujić and Kukavica (1998, 2002) proved lower bounds on the analyticity radius of NS solutions: $\rho(t) \geq c/\|u(t)\|_{L^3}$ for smooth solutions.

Grade derivation. This is a direct consequence of the radius equation (RE) for grade-2 PDEs. The straining rate $R_- \sim \|u\|^2/\rho^2$ is balanced by the Cauchy normalization $C_0 \sim \|u\|$, giving $\rho \sim 1/\|u\|$ at grade balance.

6.4 Foias–Temam Gevrey regularity

Foias and Temam (1989) proved that NS solutions with analytic data are Gevrey regular: $\|e^{\sigma(t)|D|}u(t)\|_{L^2} < \infty$ for $\sigma(t) > 0$ as long as the solution is smooth.

Grade derivation. The Gevrey radius $\sigma(t)$ IS the analyticity radius $\rho(t)$ in the grade framework. The Foias–Temam result is exactly the statement that $\rho(t)$ stays positive for smooth solutions. Grade regularity subsumes this: it provides the evolution equation for $\rho(t)$ (equation RE) and the mechanism for its potential collapse (grade-2 saturation).

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7. Worked Examples

7.1 Navier–Stokes + Boussinesq thermal coupling

The Boussinesq equations couple velocity u and temperature θ:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + \alpha \theta \mathbf{e}_3$$
$$\partial_t \theta + (u \cdot \nabla)\theta = \kappa \Delta \theta$$

Classical approach: Separate energy estimates for u and θ, coupled through the buoyancy term $\alpha\theta\mathbf{e}_3$ and the advection $(u \cdot \nabla)\theta$. The coupling makes the bootstrap significantly harder than pure NS.

Grade approach: Both equations are grade-2 (the couplings $\theta\mathbf{e}_3$ and $(u \cdot \nabla)\theta$ are bilinear). The combined system (u, θ) is grade-2 with analyticity radius:

$$\rho_{\text{Boussinesq}} \geq \min(\rho_{\text{NS}}, \rho_{\text{heat}}, \rho_{\text{buoyancy}}, \rho_{\text{thermal advection}})$$

Since the buoyancy coupling $\alpha\theta\mathbf{e}_3$ is bilinear (grade-2) and the thermal advection $(u \cdot \nabla)\theta$ is also bilinear (grade-2), the coupling does not introduce higher grades. The regularity criterion is the SAME as NS: $\varepsilon_2(t) < 1$, where ε_2 now accounts for both velocity and thermal grade-2 terms.

Result: If pure NS is regular (which is open), then Boussinesq is regular with the same criterion. The thermal coupling CANNOT introduce a new blowup mechanism — it only modifies the saturation ratio within the existing grade-2 framework.

7.2 Reaction-diffusion with arbitrary kinetics

$$\partial_t u_i = d_i \Delta u_i + f_i(u_1, \dots, u_m), \qquad i = 1, \dots, m$$

where f_i are analytic reaction terms (polynomials, rational functions, etc.).

Classical approach: Global existence depends delicately on the structure of f. Mass conservation, entropy dissipation, Lyapunov functions — all equation-specific. Adding one more species or reaction term can change the regularity picture completely.

Grade approach: If f_i is a polynomial of degree K, the system is grade-K. The radius equation is:

$$\frac{d\rho}{dt} \geq \frac{d_{\min}}{\rho} - \frac{C_K \|u\|^{K-1}}{\rho^K}$$

Regularity criterion: $d_{\min} > C_K \|u\|^{K-1}/\rho^{K-1}$, i.e., diffusion dominates the K-th grade nonlinearity.

Adding a new reaction term $g_j(u)$ of degree K' modifies ρ by the perturbation theorem. If $K' \leq K$, the grade structure is unchanged. If $K' > K$, a new higher grade appears, and the criterion tightens — but the FRAMEWORK absorbs it without rebuilding.

7.3 Boltzmann equation with soft potentials

$$\partial_t f + v \cdot \nabla_x f = Q(f, f) = \int_{\mathbb{R}^3 \times S^2} B(|v - v_*|, \sigma)[f' f'_* - f f_*] \, dv_* d\sigma$$

The collision operator Q is bilinear: grade-2. The grade analysis mirrors NS. The straining rate depends on the collision kernel B:

- **Hard spheres** ($B = |v - v_*|$): $R_- \sim \|f\|^2/\rho^2$. Same structure as NS.
- **Soft potentials** ($B = |v - v_*|^\gamma$, $\gamma < 0$): The kernel is singular, but still analytic away from $v = v_*$. The grade framework applies with a modified ρ that accounts for the velocity-space singularity.

The regularity criterion: $\rho(t) > 0$ iff the collision frequency doesn't concentrate mass at a point in velocity space. This recovers the known result (Villani, 2002) that entropy dissipation prevents concentration for Boltzmann with a Grad cutoff.

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8. Limitations and the Analyticity Hypothesis

8.1 When the framework weakens: non-analytic data

For initial data $u_0 \in C^s$ (finite smoothness, not analytic), the grade bound becomes polynomial:

$$\|A^{(k)}\| \leq \frac{C_0}{k^s}$$

instead of exponential C_0/ρ^k. The effective number of grades is $k_{\text{eff}} \sim \varepsilon^{-1/s}$ instead of $\log(1/\varepsilon)/\log \rho$.

In this regime, the grade framework reduces to standard Sobolev theory. The advantage of grade regularity diminishes as the data becomes rougher. For L^2 data (no smoothness), the grade framework provides no compression — you recover the full Fourier expansion.

8.2 The analyticity hypothesis in context

The requirement of analytic data is a REAL restriction, not a technicality. However:

1. **Physical initial conditions are typically analytic.** Thermal fluctuations, smooth forcing, and diffusive processes all produce analytic fields. The non-analytic pathologies studied in PDE theory (C^k but not C^{k+1}, Sobolev but not Hölder) are mathematical constructions, not physical states.
2. **Gevrey class G_σ interpolates.** For $\sigma > 1$ (sub-analytic), the grade bound is $\|A^{(k)}\| \leq C_0 \exp(-ck^{1/\sigma})$. The framework works with super-algebraic but sub-exponential decay. This covers a wide class of smooth-but-not-analytic data.
3. **The framework is honest about its limitations.** For L^2 or H^1 data, energy methods remain the appropriate tool. Grade regularity is complementary, not competing: it covers the regime where the data is smooth enough for the question to be "does it STAY smooth?" rather than "is it smooth at all?"

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9. Discussion

9.1 The structural diagnosis

Classical PDE regularity theory is fragmented because its primary tool — the energy estimate — is a SCALAR projection of a VECTOR inequality. The L^2 norm $\|u\|^2$ loses all information about HOW the energy distributes across grades. Two solutions with the same L^2 norm but different grade spectra can have completely different regularity properties.

The grade framework replaces the scalar $\|u\|_{L^2}$ with the SPECTRAL object $(\|A^{(k)}\|)_{k\geq 0}$. This retains the structural information that energy estimates discard.

9.2 Connections to existing frameworks

Framework	What it controls	Grade-theoretic interpretation
Sobolev spaces H^s	Derivative regularity	Algebraic grade decay (k^{-s})
Gevrey classes G_σ	Sub-exponential smoothness	Grade decay $\exp(-ck^{1/\sigma})$
Analyticity	Exponential smoothness	Grade decay ρ^{-k} (the full framework)
BKM criterion	Vorticity integrability	Time-integrated grade-2 saturation
Prodi–Serrin	$L^p L^q$ integrability	Grade-2 scaling balance
Entropy methods	$L \log L$ bounds	Grade-0 (background) control

9.3 Open problems

1. **The ρ-persistence problem.** For which PDEs does $\rho(t) > 0$ persist for all time? This is the grade-theoretic formulation of the NS millennium problem.
2. **Optimal perturbation bounds.** The harmonic mean $\rho_{F+G} \geq \rho_F \rho_G/(\rho_F + \rho_G)$ is a lower bound. Can it be improved for structured perturbations?
3. **Data-driven ρ estimation.** Can $\rho(t)$ be estimated from numerical simulations or experimental data without resolving the full grade spectrum?
4. **Non-autonomous grade theory.** For PDEs with time-dependent coefficients, the grade structure itself evolves. What conditions on the time dependence preserve grade regularity?

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