An Analytic Proof of $\{}\{}kappa_+ < \{}\{}kappa_-$ for Non-Degenerate Rotating Kerr-Newman-de Sitter Black Holes via Vieta Root Relations

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Abstract

For every non-degenerate rotating Kerr-Newman-de Sitter (KNdS) black hole — any configuration whose radial polynomial Δ_r admits four simple real roots $r_n < 0 < r_- < r_+ < r_c$ — we prove analytically that the outer-horizon surface gravity is strictly smaller than the Cauchy-horizon surface gravity,

$$\kappa_+ < \kappa_-.$$

The proof uses a single Vieta root-sum identity on the Δ_r quartic and contains no numerical computation. Combined with the standard leading-order identification $\alpha_{\text{NH}} = \kappa_+/2$ of the near-horizon QNM damping rate, this gives the strong-cosmic-censorship (SCC) ratio $\beta_{\text{NH}} = \kappa_+/(2\kappa_-) < 1/2$ uniformly over the KNdS family — strictly below the Christodoulou C^0-extension threshold. To our knowledge, this is the first analytic proof of the surface-gravity ordering across the full rotating KNdS family, as opposed to numerical scans or near-extremal asymptotic expansions.

The core algebraic steps are machine-verified independently in two formal systems: the Platonic proof kernel (Python, four theorems) and Lean 4 under Mathlib v4.28 (six theorems, zero axioms beyond the standard library). A $7 \times 6 \times 5 = 210$-point numerical scan over $(a^*, q^*, \Lambda) \in [0.01, 0.99] \times [0, 0.99] \times [0.001, 0.15]$ yields 131 non-degenerate configurations; $\max(\kappa_+/\kappa_-) = 0.856$, giving $\max \beta_{\text{NH}} = 0.428$ and a uniform safety margin of at least 0.072 below the Christodoulou threshold.

1. Introduction

1.1 Motivation: surface gravities and the SCC threshold

Drop an electromagnetic perturbation onto a Reissner-Nordström-de Sitter black hole near extremality. The perturbation decays outside at a rate α set by the quasinormal-mode (QNM) spectral gap; the same perturbation is blueshifted as it falls into the Cauchy horizon at a rate κ_-, the surface gravity of the inner horizon. Whether the field stays bounded or gets amplified enough to reshape the geometry beyond the Cauchy horizon depends on the ratio $\beta := \alpha/\kappa_-$. That single dimensionless number decides whether general relativity predicts its own future or loses control of it.

For any asymptotically de Sitter black hole, the surface gravity κ_H at a non-degenerate Killing horizon controls two physical quantities: the thermodynamic temperature $T_H = \kappa_H/(2\pi)$ [1-3] and exactly the blueshift rate described above. The blueshift is central to Penrose's strong cosmic censorship (SCC) conjecture [4]: when α outruns κ_-, fields stay regular enough to admit a C^0 extension through the Cauchy horizon, and SCC fails [5-7].

Hintz [6], following Dias-Eperon-Reall-Santos [8] and Cardoso-Costa-Destounis-Hintz-Jansen [9], formalized this as the *Christodoulou threshold* [5]:

$$\beta := \frac{\alpha}{\kappa_-} > \frac{1}{2} \quad \Longrightarrow \quad C^0 \text{ extension exists.}$$

Numerical work on rotating Kerr-de Sitter [8, 9] showed that β does cross 1/2 in a thin near-extremal window, so SCC can fail — putting the question of where, precisely, the threshold sits squarely in the geometry of the horizons.

1.2 The near-horizon branch and the surface-gravity ratio

In the near-extremal regime the KNdS QNM spectrum splits into three families — near-horizon (NH), photon-sphere (PS), and de Sitter (dS) modes [6, 9, 11]. The NH branch, a Matsubara-quantized thermal mode in the $\mathrm{AdS}_2 \times S^2$ throat, has the standard leading-order damping rate $\alpha_{\mathrm{NH}} = \kappa_+/2$ [9, 11], so the NH contribution to the SCC ratio becomes

$$\beta_{\mathrm{NH}} = \frac{\kappa_+}{2\kappa_-}.$$

Whenever the NH branch dominates the spectral gap, the Christodoulou safety condition $\beta < 1/2$ collapses to a purely geometric statement:

$$\boxed{\kappa_+ < \kappa_- \ .}$$

Numerical scans verify this inequality across wide regions of KNdS [8, 9] and its Reissner-Nordström-de Sitter (RNdS, $a = 0$) subcase [14]; Hintz [6] derives it asymptotically as $r_+ \to r_-$. What has been missing is a *general* analytic proof — one holding uniformly across the full non-degenerate rotating KNdS family, for every admissible (a, Q, Λ). That gap is what this note closes.

1.3 Contributions of this note

1. Using the Vieta root-sum relation of the Δ_r quartic, a short algebraic argument proves $\kappa_+ < \kappa_-$ **for every non-degenerate rotating KNdS black hole** (Theorem 3.3).
2. As an immediate corollary, the NH contribution to the SCC ratio stays *uniformly* below 1/2 over the full non-degenerate rotating KNdS family (Corollary 4.1). When the NH mode dominates the QNM spectral gap, Christodoulou C^0-extension safety follows with no assumption of near-extremality, no restriction on Λ, and no numerical input.
3. The two algebraic lemmas and the division step are machine-verified independently in two formal systems: the Platonic proof kernel (four theorems) and Lean 4 under Mathlib v4.28 (six theorems). Both formalizations are closed: no `sorry`, no added axioms. The Lean module is reproduced in Section 5.
4. A $7 \times 6 \times 5 = 210$-point scan over $(a^*, q^*, \Lambda) \in [0.01, 0.99] \times [0, 0.99] \times [0.001, 0.15]$ contains 131 non-degenerate configurations; at every one of them $\kappa_+ < \kappa_-$, with $\max(\kappa_+/\kappa_-) = 0.856$ (Section 6). Quartile data and saturation trends appear in Appendix B.

1.4 Relation to existing work

The ordering $\kappa_+ < \kappa_-$ shows up throughout the SCC literature, but only as an observed numerical fact used to bound β in chosen subregions of parameter space [8, 9, 15]. Hintz [6] recovers it asymptotically as $r_+ \to r_-$ via local expansion — a statement about one endpoint of the parameter space, not the whole. Casals and Ottewill [13] handle the Schwarzschild case by direct calculation, and Mo et al. [14] verify aspects of the ordering in the RNdS ($a = 0$) subcase along specific one-parameter families. What has been missing is an **analytic, uniform** theorem across the full rotating KNdS family — and, separately, a formal-verification certificate establishing it. This note supplies both.

2. Setup

2.1 The Kerr-Newman-de Sitter metric

In Boyer-Lindquist-type coordinates (t, r, θ, ϕ), the KNdS metric is

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,dt - \frac{r^2+a^2}{\Xi}d\phi\right)^2,$$

with

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Xi = 1 + \frac{\Lambda a^2}{3},$$

$$\Delta_\theta = 1 + \frac{\Lambda a^2}{3}\cos^2\theta,$$

and the radial polynomial

$$\Delta_r(r) = \left(1 - \frac{\Lambda r^2}{3}\right)(r^2 + a^2) - 2Mr + Q^2. \tag{2.1}$$

Here $M > 0$ is the mass parameter, $a \in \mathbb{R}$ the spin-per-mass, $Q \in \mathbb{R}$ the electric charge, and $\Lambda > 0$ the cosmological constant. Expanding (2.1):

$$\Delta_r(r) = -\frac{\Lambda}{3}r^4 + \left(1 - \frac{\Lambda a^2}{3}\right)r^2 - 2Mr + (a^2 + Q^2). \tag{2.2}$$

2.2 Non-degenerate rotating case

We consider the **non-degenerate rotating regime**, defined by

$$a \neq 0, \qquad \Delta_r(r) = 0 \text{ has four distinct real roots, in increasing order } r_1 < r_2 < r_3 < r_4. \tag{A}$$

Under (A), the four roots must split as *one negative and three positive.* Three observations from (2.2) pin down the signs:

1. The leading coefficient $-\Lambda/3 < 0$, so $\Delta_r(r) \to -\infty$ as $r \to \pm\infty$.
2. $\Delta_r(0) = a^2 + Q^2 > 0$ (strictly, because $a \neq 0$).
3. Applying Descartes' rule of signs to $\Delta_r(-s)$ for $s > 0$ gives at most one negative root.

Items 1 and 2, combined with the intermediate-value theorem, force at least one negative root when four real roots exist; item 3 caps that at exactly one. So the sign pattern is fixed: $r_1 < 0 < r_2 < r_3 < r_4$.

The three positive roots are, in increasing order, the Cauchy (inner) horizon, the event (outer) horizon, and the cosmological horizon. The negative root has no physical interpretation. Throughout the paper we use the physical names

$$r_n := r_1 < 0, \qquad r_- := r_2, \qquad r_+ := r_3, \qquad r_c := r_4,$$

so the ordering reads $r_n < 0 < r_- < r_+ < r_c$, and the physical exterior is $r_+ < r < r_c$.

2.3 Vieta's relations

Writing (2.2) as

$$\Delta_r(r) = -\frac{\Lambda}{3}\,(r - r_n)\,(r - r_-)(r - r_+)(r - r_c),$$

matching coefficients yields the Vieta relations:

$$r_n + r_- + r_+ + r_c = 0, \tag{V1}$$

$$r_n r_- + r_n r_+ + r_n r_c + r_- r_+ + r_- r_c + r_+ r_c = -\frac{3}{\Lambda}\left(1 - \frac{\Lambda a^2}{3}\right) = a^2 - \frac{3}{\Lambda}, \tag{V2}$$

$$r_n r_- r_+ + r_n r_- r_c + r_n r_+ r_c + r_- r_+ r_c = -\frac{6M}{\Lambda}, \tag{V3}$$

$$r_n r_- r_+ r_c = -\frac{3}{\Lambda}(a^2 + Q^2). \tag{V4}$$

Only (V1) is used in the proof below. The crucial fact from (V1) is

$$r_n = -(r_- + r_+ + r_c) < 0. \tag{V1'}$$

2.4 Surface gravities

At each (non-degenerate) horizon $r_H \in \{r_-, r_+, r_c\}$, the surface gravity is

$$\kappa_H = \frac{|\Delta_r'(r_H)|}{2(r_H^2 + a^2)}. \tag{2.3}$$

Differentiating the factorized form at $r = r_+$:

$$\Delta_r'(r_+) = -\frac{\Lambda}{3}(r_+ - r_n)(r_+ - r_-)(r_+ - r_c).$$

The signs satisfy $r_+ - r_n > 0$, $r_+ - r_- > 0$, $r_+ - r_c < 0$; the leading factor is $-\Lambda/3 < 0$, so $\Delta_r'(r_+) > 0$ and

$$\kappa_+ = \frac{\Lambda}{6} \cdot \frac{(r_+ - r_n)(r_+ - r_-)(r_c - r_+)}{r_+^2 + a^2}. \tag{2.4}$$

Similarly at $r = r_-$:

$$\Delta_r'(r_-) = -\frac{\Lambda}{3}(r_- - r_n)(r_- - r_+)(r_- - r_c),$$

whose signs are $+, -, -$, giving $\Delta_r'(r_-) < 0$ and

$$\kappa_- = \frac{\Lambda}{6} \cdot \frac{(r_- - r_n)(r_+ - r_-)(r_c - r_-)}{r_-^2 + a^2}. \tag{2.5}$$

The factor $r_+ - r_-$ is common to both numerators.

2.5 The ratio κ_+/κ_-

From (2.4), (2.5), the common factor $(\Lambda/6)(r_+ - r_-)$ cancels:

$$\frac{\kappa_+}{\kappa_-} = \frac{(r_+ - r_n)(r_c - r_+)}{(r_- - r_n)(r_c - r_-)} \cdot \frac{r_-^2 + a^2}{r_+^2 + a^2}. \tag{2.6}$$

Denoting the two factors as

$$F_1 := \frac{(r_c - r_+)(r_+ - r_n)}{(r_c - r_-)(r_- - r_n)}, \qquad F_2 := \frac{r_-^2 + a^2}{r_+^2 + a^2}, \tag{2.7}$$

equation (2.6) reads $\kappa_+/\kappa_- = F_1 \cdot F_2$.

Strategy. We prove each factor is a positive number strictly less than 1: Lemma 3.1 handles F_1 using only Vieta relation (V1), and the elementary Lemma 3.2 handles F_2. Their product is then strictly less than 1, giving $\kappa_+ < \kappa_-$.

3. Main Theorem

3.1 The Vieta root-ratio lemma

Lemma 3.1 (Vieta root-ratio lemma). *Let r_n, r_-, r_+, r_c be four distinct real numbers satisfying*

$$r_n < 0 < r_- < r_+ < r_c \qquad \text{and} \qquad r_n + r_- + r_+ + r_c = 0. \tag{3.1}$$

Then

$$(r_c - r_+)(r_+ - r_n) < (r_c - r_-)(r_- - r_n). \tag{3.2}$$

Proof. Let $L := (r_c - r_+)(r_+ - r_n)$ and $R := (r_c - r_-)(r_- - r_n)$. Using the Vieta constraint (3.1) we substitute $r_n = -(r_- + r_+ + r_c)$ into both products and expand. Writing $p := r_+, m := r_-, c := r_c$ for brevity (with $n = -(m + p + c)$):

$$L = (c - p)(p - n) = (c - p)(p + m + p + c) = (c - p)(2p + m + c),$$

$$R = (c - m)(m - n) = (c - m)(m + m + p + c) = (c - m)(2m + p + c).$$

Computing $R - L$ directly:

$$\begin{aligned}
R - L &= (c - m)(2m + p + c) - (c - p)(2p + m + c) && (1)\\
&= \left[(2mc + pc + c^2) - (2m^2 + mp + mc)\right] - \left[(2pc + mc + c^2) - (2p^2 + mp + pc)\right] && (2)\\
&= 2mc + pc + c^2 - 2m^2 - mp - mc - 2pc - mc - c^2 + 2p^2 + mp + pc && (3)\\
&= (2mc - 2mc) + (pc - 2pc + pc) + (c^2 - c^2) - 2m^2 + (-mp + mp) + 2p^2 && (4)\\
&= 2p^2 - 2m^2 = 2(p - m)(p + m). && (5)
\end{aligned}$$

Since $0 < m < p$ by (3.1), both $p - m > 0$ and $p + m > 0$, so $R - L = 2(p - m)(p + m) > 0$, i.e. $L < R$, which is (3.2). □

3.2 Radius-squared ordering

Lemma 3.2. *For any $0 < r_- < r_+$ and any $a \in \mathbb{R}$,*

$$r_-^2 + a^2 < r_+^2 + a^2. \tag{3.3}$$

Proof. $r_+^2 - r_-^2 = (r_+ - r_-)(r_+ + r_-) > 0$, so $r_-^2 < r_+^2$, and adding a^2 to both sides gives (3.3). □

3.3 The surface gravity ordering

Theorem 3.3 (KNdS Surface-Gravity Ordering). *For every non-degenerate rotating KNdS black hole satisfying assumption (A) of Section 2.2,*

$$\kappa_+ < \kappa_-. \tag{3.4}$$

Proof. With F_1, F_2 defined in (2.7), Lemma 3.1 gives $F_1 < 1$ (both numerator and denominator are positive by the horizon ordering of Section 2.2, so the inequality is preserved after division). Lemma 3.2 gives $F_2 < 1$; both F_1 and F_2 are strictly positive for the same reason. The surface gravities κ_+ and κ_- are strictly positive by the sign computation in (2.4)-(2.5), so from (2.6),

$$0 < \frac{\kappa_+}{\kappa_-} = F_1 F_2 < 1,$$

which gives $\kappa_+ < \kappa_-$. □

3.4 What Theorem 3.3 does NOT say

Two clarifications are in order.

(i) **Degenerate cases are excluded.** When two horizons coincide — extremal cases with $r_+ = r_-$ (extremal Kerr-Newman limit in the appropriate Λ regime), $r_+ = r_c$ (Nariai-type limits), or $r_- = r_+ = r_c$ (ultracold corner) — assumption (A) fails and the proof does not apply. The surface gravity vanishes ($\kappa = 0$) on each degenerate horizon, so the ordering $\kappa_+ < \kappa_-$ is trivially undefined in those limits. The non-degenerate rotating regime is open and dense in the physically admissible parameter space.

(ii) **The non-rotating case $a = 0$ is formally outside (A), but the same proof chain covers it.** For Reissner-Nordström-de Sitter (RNdS, $a = 0$, $Q \neq 0$), the radial polynomial $\Delta_r^{\mathrm{RNdS}}(r) = -(\Lambda/3)r^4 + r^2 - 2Mr + Q^2$ is still quartic in r with vanishing r^3 coefficient, so (V1) holds. Theorem 3.3 as stated requires $a \neq 0$ and so does not apply, but the algebraic chain does: Lemma 3.1 never mentions a, and Lemma 3.2 with $a = 0$ simply gives $F_2 = r_-^2/r_+^2 < 1$. The conclusion $\kappa_+ < \kappa_-$ follows whenever Δ_r^{RNdS} has four distinct real roots. The RNdS ordering is already known from direct computation [14]; we emphasize the rotating case because it is astrophysically relevant and, until now, had no analytic proof.

4. Application: Unconditional SCC Safety

4.1 The near-horizon SCC ratio

Combining Theorem 3.3 with the standard leading-order identification of the near-horizon damping rate $\alpha_{\mathrm{NH}} = \kappa_+/2$ for the NH branch of the KNdS QNM spectrum [9, 11], we obtain:

Corollary 4.1 (Unconditional NH SCC safety). *For every non-degenerate rotating KNdS black hole, the near-horizon contribution to the SCC ratio satisfies*

$$\beta_{\mathrm{NH}} = \frac{\alpha_{\mathrm{NH}}}{\kappa_-} = \frac{\kappa_+}{2\kappa_-} < \frac{1}{2}. \tag{4.1}$$

Equivalently, whenever the near-horizon mode dominates the QNM spectral gap, the Christodoulou C^0-extension threshold is never crossed.

Proof. By Theorem 3.3, $\kappa_+ < \kappa_-$ and both are positive. Dividing by $2\kappa_- > 0$:

$$\frac{\kappa_+}{2\kappa_-} < \frac{\kappa_-}{2\kappa_-} = \frac{1}{2}.\square$$

4.2 Dominance of the NH branch

Corollary 4.1 protects SCC safety *whenever the NH mode dominates the spectral gap.* Two independent sources establish this dominance:

1. **Spectrum bifurcation in nearly extremal Kerr** [10, 11], computed via the Leaver continued-fraction method [16], places the crossing between the damped-mode (DM) and zero-damped-mode (ZDM = NH) branches of Kerr at $a/M \approx 0.886$. For $a/M > 0.886$, the NH branch gives the smaller damping rate and so controls α_{min}.
2. **Coupled-spectrum numerics** [9, 12] in KNdS, together with [14] in the RNdS subcase, confirm NH dominance in the near-extremal regime $\rho^* := 1 - a^{*2} - q^{*2} \leq 0.02$ — precisely the region where SCC had previously been feared to fail.

Far from extremality ($\rho^* \geq 0.1$) the PS branch typically dominates and β is much smaller than 1/2. In the transition regime $0.02 \leq \rho^* \leq 0.1$, NH and PS modes compete; both stay below 1/2 by a comfortable margin.

4.3 The safety margin

Corollary 4.1 gives $\beta_{\text{NH}} < 1/2$ strictly, but not how far below. The numerical scan of Section 6 quantifies the margin:

Numerical upper bound (Section 6). *Over the parameter cube* $(a^*, q^*, \Lambda) \in [0.01, 0.99] \times [0, 0.99] \times [0.001, 0.15]$ *the* $7 \times 6 \times 5 = 210$ *grid points yield* 131 *non-degenerate KNdS configurations (the remaining* 79 *are super-extremal or degenerate and fail assumption (A)). Across all* 131 *valid points,* $\max(\kappa_+/\kappa_-) = 0.856$*, so* $\max \beta_{\text{NH}} = 0.428$*. The NH safety margin* $1/2 - \beta_{\text{NH}}$ *stays uniformly at or above* 0.072*.*

The analytic bound of Corollary 4.1 guarantees $\beta_{\text{NH}} < 1/2$ for every non-degenerate rotating KNdS configuration; the numerical upper bound quantifies how far below the threshold the ratio sits on the 131 admissible points of the scan cube. Together they give a *quantitative* SCC safety statement for the NH branch: strict in theory, margin ≥ 0.072 in practice.

5. Formal Verification

5.1 Scope of the formalization

Two independent machine-verified certificates accompany the paper: a Platonic-kernel formalization (Python) and a Lean 4 formalization (Mathlib v4.28). Both encode the same mathematical content — the two algebraic lemmas (Lemma 3.1, Lemma 3.2), the abstract contraction step synthesizing Theorem 3.3 from the factorization (2.6), and the division step yielding Corollary 4.1.

The identification of the formal real variables $r_n, r_-, r_+, r_c, a, \kappa_\pm$ with KNdS horizon radii and surface gravities is *not* part of either formal proof. That identification lives in equations (2.4)-(2.6) of Section 2 — GR calculations outside the Lean/Platonic systems. What the certificates verify is the algebraic skeleton on which Theorem 3.3 rests.

5.2 Platonic kernel

The analytic core of Section 3 is formalized as four theorems in the Platonic proof kernel (elysium/fields/vorticity_causality_bridge/vorticity_causality_bridge_proof.py, Part 59):

#	Theorem name	Encoded content
281	vieta_root_ratio_lemma	Lemma 3.1: for four reals with $r_n < 0 < r_- < r_+ < r_c$ and $r_n + r_- + r_+ + r_c = 0$, $(r_c - r_+)(r_+ - r_n) < (r_c - r_-)(r_- - r_n)$.
282	radius_squared_ordering	Lemma 3.2: for $0 < u < v$ and any a, $u^2 + a^2 < v^2 + a^2$.
283	kappa_plus_lt_kappa_minus	Synthesis step of Theorem 3.3: given $\kappa_- > 0$ and two contraction factors $F_1, F_2 \in (0, 1)$, $F_1 F_2 \kappa_- < \kappa_-$.
284	unconditional_christodoulou_safety	Division step of Corollary 4.1: given $0 < \kappa_+ < \kappa_-$, $\kappa_+/(2\kappa_-) < 1/2$.

The full proof file contains 288 verified theorems; running

```
python3 elysium/fields/vorticity_causality_bridge/vorticity_causality_bridge_proof.
```

produces Proved: 288/288, Errors: 0. The four theorems above close entirely through the Platonic nonlinear-arithmetic tactic (ts.nlinarith()); no ts.sorry() is invoked, and no p.axiom(...) call is introduced.

5.3 Lean 4 module

The same content is independently formalized in Lean 4 as kernel/LeanProofs/SCC.lean, self-contained modulo Mathlib v4.28. The complete module is reproduced below verbatim. In the Lean code, r_m, r_p, kappa_p, kappa_m stand for the paper's $r_-, r_+, \kappa_+, \kappa_-$; the subscripted forms are avoided because Unicode minus and plus are ambiguous parser targets in Mathlib.

```
/-
# Strong Cosmic Censorship — Kerr-Newman-de Sitter

## Key analytic theorem

For any non-degenerate KNdS black hole with horizons `r_n < 0 < r_m < r_p < r_c`
on the Δ_r quartic (whose sum of roots equals zero by Vieta), the outer
surface gravity is strictly smaller than the Cauchy horizon surface gravity:

   _+ <  _-.

As a corollary, the near-horizon SCC ratio  _NH =  _+ / (2  _-) is
unconditionally bounded by 1/2, establishing Christodoulou-SCC safety
```

```
throughout the NH-dominated regime.

This file formalizes:

  1. the purely algebraic heart of the result
     (`vieta_root_ratio_lemma`) — a strict inequality on the roots of the
     quartic that holds whenever their sum is zero and the ordering is
     `r_n < 0 < r_m < r_p < r_c`;

  2. the radius-squared ordering (`inner_over_outer_radius_squared_lt_one`)
     which contributes the second < 1 factor in _+/_-;

  3. the abstract contraction step (`kappa_contraction_lemma`):
     given `_- > 0` and two contraction factors `F_1, F_2  (0, 1)`,
     `F_1 · F_2 ·  _- < _-`. This synthesizes Theorem 3.3 once the
     geometric identification `_+/_- = F_1 F_2` (equation (2.6) of
     the paper) is made;

  4. the abstract universal bound
     (`beta_NH_universal_bound`): if `_+ < _-` (with `_- > 0`)
     then `_+ / (2 _-) < 1/2`.

Taken together, these establish — at the level of the Platonic kernel and
exported to Lean 4 — that the SCC near-horizon spectral ratio is strictly
below the Christodoulou threshold, unconditionally, for every KNdS black
hole with non-degenerate horizons in the rotating regime.

Physical: combining this with the near-horizon damping ansatz
`_NH = _+/2` gives `_NH = _+/(2_-) < 1/2` whenever the near-horizon
mode dominates. NH dominance is established numerically in the coupled
KNdS QNM spectrum by Cardoso-Costa-Destounis-Hintz-Jansen (paper §4.2).
-/

import Mathlib.Data.Real.Basic
import Mathlib.Tactic.Linarith
import Mathlib.Tactic.Positivity
import Mathlib.Tactic.FieldSimp

set_option linter.unusedVariables false

noncomputable section

namespace SCC.KNdS

/--
**Core algebraic lemma** (`vieta_root_ratio_lemma`).

Given four real numbers with `r_n < 0 < r_m < r_p < r_c`
```

```
and the Vieta constraint `r_n + r_m + r_p + r_c = 0`, we have

    (r_c − r_p)·(r_p − r_n)  <  (r_c − r_m)·(r_m − r_n).

This is the combinatorial heart of the surface−gravity inequality
_+ <  _− for any KNdS black hole.

**Proof sketch**:
Expanding LHS − RHS and substituting r_n = −(r_m + r_p + r_c) via Vieta gives
    LHS − RHS = −2 (r_p − r_m) (r_p + r_m),
which is strictly negative because r_p > r_m > 0.
−/
theorem vieta_root_ratio_lemma
    (r_n r_m r_p r_c :  )
    (h_n_neg : r_n < 0)
    (h_nm : 0 < r_m)
    (h_mp : r_m < r_p)
    (h_pc : r_p < r_c)
    (h_vieta : r_n + r_m + r_p + r_c = 0) :
    (r_c − r_p) * (r_p − r_n) < (r_c − r_m) * (r_m − r_n) := by
  have h_rn : r_n = −(r_m + r_p + r_c) := by linarith
  have h_sum_pos : 0 < r_p + r_m := by linarith
  have h_diff_pos : 0 < r_p − r_m := by linarith
  have h_prod_pos : 0 < (r_p − r_m) * (r_p + r_m) := mul_pos h_diff_pos h_sum_pos
  nlinarith [h_prod_pos, sq_nonneg (r_p − r_m), sq_nonneg (r_p + r_m)]

/−−
**Second factor** (`inner_over_outer_radius_squared_lt_one`).

With the radial ordering `0 < r_m < r_p`, we have
    r_m² + a²  <  r_p² + a²
for any `a   `. This is the contraction factor appearing in the
`r_+^2 + a^2` denominator of the surface−gravity formula.
−/
theorem inner_over_outer_radius_squared_lt_one
    (r_m r_p a :  )
    (h_m_pos : 0 < r_m)
    (h_mp : r_m < r_p) :
    r_m^2 + a^2 < r_p^2 + a^2 := by
  have : r_m^2 < r_p^2 := by nlinarith
  linarith

/−−
**Abstract contraction step** (`kappa_contraction_lemma`).

For any `_− > 0` and two contraction factors `F_1, F_2   (0, 1)`,

    F_1 · F_2 ·  _−  <   _−.
```

```
This is the synthesis step of Theorem 3.3 of the paper. Once the
geometric identification `_+ / _− = F_1 · F_2` from §2.5 is made,
`vieta_root_ratio_lemma` supplies `F_1 < 1`, and
`inner_over_outer_radius_squared_lt_one` supplies `F_2 < 1`.
This lemma then gives `_+ = F_1 · F_2 · _− < _−`.
−/
theorem kappa_contraction_lemma
    (kappa_m F_1 F_2 : )
    (h_kappa_pos : 0 < kappa_m)
    (h_F1_pos : 0 < F_1)
    (h_F1_lt : F_1 < 1)
    (h_F2_pos : 0 < F_2)
    (h_F2_lt : F_2 < 1) :
    F_1 * F_2 * kappa_m < kappa_m := by
  nlinarith [mul_pos h_F1_pos h_F2_pos,
             mul_pos h_kappa_pos h_F1_pos,
             mul_pos h_kappa_pos h_F2_pos]

/−−
**Abstract universal bound** (`beta_NH_universal_bound`).

For any two positive real surface gravities `_+ < _−` with `_− > 0`,

    _NH := _+ / (2 _−) < 1/2.

This is the final step converting the KNdS surface−gravity ordering
(established via `vieta_root_ratio_lemma` + `inner_over_outer_radius_squared_lt_one
+ `kappa_contraction_lemma`) into the Christodoulou−SCC safety bound.

**Physical content**: whenever the near−horizon mode dominates the
QNM spectral gap (so the gap is `_NH = _+/2`), we have ` < 1/2`
unconditionally — the Christodoulou regularity threshold is never violated.
−/
theorem beta_NH_universal_bound
    (kappa_p kappa_m : )
    (h_ord : kappa_p < kappa_m)
    (h_pos : 0 < kappa_m) :
    kappa_p / (2 * kappa_m) < 1 / 2 := by
  have h_two_km_pos : (0 : ) < 2 * kappa_m := by positivity
  have h_two_km_ne : (2 * kappa_m)  0 := ne_of_gt h_two_km_pos
  have h_half_eq : (1 : ) / 2 = kappa_m / (2 * kappa_m) := by
    field_simp
  rw [h_half_eq]
  apply div_lt_div_of_pos_right h_ord h_two_km_pos

/−−
**Chained corollary** (`kapp_ord_implies_beta_safety`).
```

```
Given the Vieta hypotheses on KNdS horizon radii and the positivity of −_,
if _+ < −_ is established from the upstream factorization , then _NH < 1/2.
−/
theorem kappa_ord_implies_beta_safety
    (kappa_p kappa_m :  )
    (h_ord : kappa_p < kappa_m)
    (h_pos : 0 < kappa_m) :
    kappa_p / (2 ∗ kappa_m) < 1 / 2 :=
  beta_NH_universal_bound kappa_p kappa_m h_ord h_pos

/−−
∗∗Integrated safety theorem∗∗ (`scc_near_horizon_safety`).

Packages the near−horizon SCC safety bound as a single Lean 4 theorem
for external citation:

Given positive reals `kappa_p`, `kappa_m` with `kappa_p < kappa_m`, the
quantity `kappa_p / (2 ∗ kappa_m)` is strictly less than `1/2` and strictly
positive.

∗∗Physical reading∗∗ (not part of the Lean proof term): under the identifications
`kappa_p = _+`, `kappa_m = _−` for a non−degenerate KNdS black hole and
the near−horizon QNM identification `_NH = _+/2`, this is the SCC ratio
bound `_NH < 1/2`. The consequences for ` C`−extension obstructions across
the Cauchy horizon are discussed in the paper (§§1, 4) and in the cited
SCC literature; they are not claims of this Lean theorem.
−/
theorem scc_near_horizon_safety
    (kappa_p kappa_m :  )
    (h_kp_pos : 0 < kappa_p)
    (h_km_pos : 0 < kappa_m)
    (h_ord : kappa_p < kappa_m) :
    −− (1) ratio is strictly less than threshold
    kappa_p / (2 ∗ kappa_m) < 1 / 2
    −− (2) ratio is strictly positive (non−vacuous)
     0 < kappa_p / (2 ∗ kappa_m) := by
  refine ?_, ?_
  · exact beta_NH_universal_bound kappa_p kappa_m h_ord h_km_pos
  · have h_two_km_pos : (0 :  ) < 2 ∗ kappa_m := by positivity
    exact div_pos h_kp_pos h_two_km_pos

end SCC.KNdS
```

The module compiles with

```
lake env lean kernel/LeanProofs/SCC.lean
```

and exits with code 0, zero warnings, and zero sorry. Running #print axioms on each of the six theorems returns only Lean 4's three core axioms

$$\{\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}\}$$

— the foundational axioms of Lean's type theory, on which all of Mathlib's real-analysis development rests. No additional axioms, sorry, or unverified assumptions enter the proof.

5.4 Verification summary

System	Theorems	File	Status
Platonic kernel	4	elysium/ fields /vorticity_causality_bridge/vorticity_causality_ (Part 59)	288/288 pass
Lean 4	6	kernel/LeanProofs/SCC.lean	lake env lean exit 0
Numerical scan	131/210 grid points satisfy (A)	NumPy quartic root finder, grid per Appendix B	0 violations

Together, the certificates establish the algebraic core: the two lemmas behind Theorem 3.3 (Lemma 3.1, Lemma 3.2), the abstract contraction $F_1 F_2 \kappa_- < \kappa_-$ for $\kappa_- > 0$ and $F_1, F_2 \in (0,1)$, and the division $\kappa_+/(2\kappa_-) < 1/2$ for $0 < \kappa_+ < \kappa_-$.

What remains pen-and-paper is the geometric input from Section 2: identifying F_1, F_2 with the specific ratios in (2.6), and the positivity of the KNdS surface gravities. This division of labor — algebra in Lean/Platonic, geometry on paper — is standard for theorems resting on differential-geometric computation.

6. Numerical Confirmation

Theorem 3.3 gives strict inequality; a numerical scan quantifies the margin. We sampled a $7\times6\times5 = 210$-point tensor grid on

$$(a^*, q^*, \Lambda) \in [0.01, 0.99] \times [0, 0.99] \times [0.001, 0.15],$$

with $a^* = a/M$ and $q^* = Q/M$, using 5 approximately log-spaced Λ-values, 7 near-linear a^*-values, and 6 near-linear q^*-values (Appendix B). At each point we compute the four roots of Δ_r numerically, check assumption (A) (four distinct real roots with $r_n < 0 < r_- < r_+ < r_c$), evaluate κ_+ and κ_- from (2.4) and (2.5), and record the ratio. Of the 210 grid points, 131 yield a non-degenerate KNdS configuration satisfying (A); the remaining 79 are super-extremal or degenerate and are excluded. The statistics below are over the 131 admissible points.

Statistic	Value
Total grid points	210
Points satisfying non-degeneracy (A)	131
Violations of $\kappa_+ < \kappa_-$	0
$\max(\kappa_+/\kappa_-)$	0.856
$\min(\kappa_+/\kappa_-)$	≈ 0

Statistic	Value
Median κ_+/κ_-	0.068
Max β_{NH} in scan	0.428
Min NH safety margin $1/2 - \beta_{\mathrm{NH}}$	0.072

The maximum 0.856 is attained at $(a^*, q^*, \Lambda) = (0.3, 0.99, 0.15)$ — a configuration that is super-extremal in flat space ($a^2 + Q^2 > M^2$) but stabilized by the large cosmological constant, with $r_+ - r_- \approx 0.077$ indicating strong near-extremality. This is consistent with Theorem 3.3 approaching (but never crossing) $\kappa_+/\kappa_- = 1$ in the extremal limit $r_- \to r_+$. For reference, the near-extremal Kerr corner ($a^* = 0.99, q^* = 0, \Lambda = 0.001$) gives $\kappa_+/\kappa_- \approx 0.749$.

7. Discussion

7.1 Why the Vieta argument works

The proof reduces to a single identity: after Vieta substitution, $R - L = 2(r_+ - r_-)(r_+ + r_-)$. Both factors are strictly positive by the horizon ordering $0 < r_- < r_+$. Every quartic coefficient involving M, a, Q, Λ cancels in the Δ_r' ratio. The proof is minimal: it uses one of the four Vieta relations (V1), the horizon ordering, and nothing else.

Read structurally, the Vieta relation (V1) is a *conservation law on horizon radii*: the four roots — including the unphysical r_n — distribute around 0 so that their sum vanishes. The unphysical root plays the role of a balancing weight, the way an invisible counterweight sets the equilibrium of a lever. Asymmetries in the physical sector (r_- versus r_+) are then forced by that balance. The inequality $\kappa_+ < \kappa_-$ is the geometric shadow of this conservation law, refracted through the factorization of Δ_r'. Take (V1) out of the argument — for instance, by bounding κ_+ and κ_- separately as functions of (a, Q, Λ, M) — and the sign of $\kappa_+ - \kappa_-$ becomes a question about the interplay of four dimensionful parameters rather than a statement about the shape of the root distribution. (V1) is what converts the inequality from an analytic-estimate problem to a combinatorial one.

7.2 Why previous derivations were less general

The near-extremal analysis [6] expands around $r_+ = r_- + \epsilon$ with $\epsilon \to 0^+$ and recovers $\kappa_+/\kappa_- \to 1^-$. That is a local limit: it does not give the strict inequality uniformly across the parameter space. Numerical scans [8, 9] in KNdS and [14] in the RNdS subcase observe $\kappa_+ < \kappa_-$ but do not prove it. A direct expansion of κ_+/κ_- as a function of (a, Q, Λ, M) is possible in principle, but produces a ratio of quartic-coefficient-dependent expressions whose sign is opaque. The Vieta reformulation is the simplification that makes the sign visible.

7.3 Implications for the SCC program

Corollary 4.1 upgrades the existing SCC safety results from *numerical* to *analytic*: whenever the NH mode dominates the spectral gap, SCC holds across the full non-degenerate rotating KNdS family — a single statement replacing the point-by-point parameter-space scan. The only regime where SCC could still fail is when the photon-sphere (PS) or de Sitter (dS) branches dominate with $\beta > 1/2$. Recent work [9] isolates that risk to a narrow near-extremal window in KdS; an analytic treatment of those branches remains open.

7.4 Formal verification: methodological remark

The Lean 4 module in Section 5.3 drops directly into a Mathlib extension or a stand-alone general-relativity library. It is, to our knowledge, the first Lean 4 certificate for an analytic theorem on KNdS surface gravities. The take-away is methodological: once a GR statement reduces to elementary algebraic inequalities, the Mathlib ecosystem verifies it in a handful of lines — provided the differential geometry is first reduced to algebra by hand and the algebraic core is exposed cleanly.

7.5 Future work

1. **Analytic bounds on α_{PS} and α_{dS}.** The PS and dS branches of KNdS QNMs still require numerical continued-fraction computation. An analytic bound $\alpha_{\mathrm{PS}} \leq \kappa_+ \cdot g(a^*, q^*, \Lambda)$ with $g < 1$ in a regime of interest would close the gap to a full-spectrum analytic SCC theorem.
2. **Higher-dimensional black holes.** The Vieta argument uses only the quartic structure of Δ_r. For Myers-Perry-de Sitter in $d > 4$, the radial polynomial has higher degree and more horizons; whether an analogous surface-gravity ordering survives is open.
3. **Coupled NH-PS resonances.** In the near-extremal window, NH and PS branches mix. Controlling the spectral gap there — the very regime where Theorem 3.3 is tightest — is the natural next step.

8. Acknowledgments

The formal proofs were developed and verified in the Platonic proof system and in the Lean 4 / Mathlib ecosystem. The numerical scans of Section 6 use standard NumPy quartic root-finding routines. The author thanks the maintainers of Mathlib for the underlying real-analysis infrastructure that made the Lean formalization a few-line exercise.

Disclosure on AI tooling

This work was produced with substantial assistance from AI coding and proof-engineering tools (Claude, Cursor). The tools were used to develop, refactor, and machine-verify the Platonic-kernel theorems (Part 59 of vorticity_causality_bridge_proof.py) and the Lean 4 module (kernel/LeanProofs/SCC.lean). The same mathematical content — the two algebraic lemmas, the abstract contraction step, and the division step — was encoded and independently verified in each kernel. The mathematical substance of the paper — the Vieta identity, the surface-gravity derivation in Section 2, and the SCC application in Section 4 — was authored and reviewed by the author, and every claim in the text was validated against the machine-verified artifacts.

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Appendix A: Surface-Gravity Computation Details

We record here the explicit calculation of $\Delta_r'(r_\pm)$ introduced in Section 2.4 and used in Section 2.5.

Factorizing $\Delta_r(r) = -(\Lambda/3)(r-r_n)(r-r_-)(r-r_+)(r-r_c)$ and applying the product rule,

$$\Delta_r'(r) = -\frac{\Lambda}{3}\Big[(r-r_-)(r-r_+)(r-r_c)+(r-r_n)(r-r_+)(r-r_c)+(r-r_n)(r-r_-)(r-r_c)+(r-r_n)(r-r_-)(r-r_+)\Big].$$

Each of the four terms contains $(r-r_H)$ for one of $r_H \in \{r_n, r_-, r_+, r_c\}$. At $r = r_+$ three of them vanish (the three that include a factor $(r-r_+)$, i.e. the 1st, 2nd, and 4th in the sum above), leaving only the term without $(r - r_+)$:

$$\Delta_r'(r_+) = -\frac{\Lambda}{3}(r_+ - r_n)(r_+ - r_-)(r_+ - r_c).$$

With the sign pattern noted in Section 2.4, this is > 0, confirming $\kappa_+ = +\Delta_r'(r_+)/[2(r_+^2 + a^2)]$. The analogous computation at $r = r_-$ yields (2.5). $\square$

Appendix B: Numerical Scan Details

The scan of Section 6 uses the following procedure.

Sampling. Approximately log-spaced in $\Lambda \in \{0.001, 0.01, 0.05, 0.1, 0.15\}$ (5 values). The a^* grid $\{0.01, 0.1, 0.3, 0.5, 0.7, 0.9, 0.99\}$ (7 values) and the q^* grid $\{0.0, 0.2, 0.4, 0.6, 0.8, 0.99\}$ (6 values) are near-linear with endpoint adjustments: $a^* = 0.01$ and $a^* = 0.99$ replace the naive 0 and 1 endpoints to keep the sample strictly inside the non-degenerate rotating regime (assumption (A) requires $a \neq 0$, and a^*, q^* strictly less than their extremal values). The tensor product has $5 \times 7 \times 6 = 210$ points.

Admissibility filter. Of the 210 grid points, 131 yield four distinct real roots of Δ_r with $r_n < 0 < r_- < r_+ < r_c$, hence satisfy (A). The remaining 79 points are super-extremal in flat space ($a^{*2} + q^{*2} > 1$) and remain so for the Λ value on the grid; these have fewer than four real roots or lose the $0 < r_- < r_+ < r_c$ ordering and are excluded. The statistics below are conditional on (A).

Quartile distribution of κ_+/κ_- (over the 131 admissible points).

Quartile	Value
Q1 (25th percentile)	0.012
Q2 (median)	0.068
Q3 (75th percentile)	0.214
Max	0.856

Where the ratio is largest. κ_+/κ_- approaches 1 as $r_+ \to r_-$ (extremality). Over the admissible subset, the maximum 0.856 is attained at $(a^*, q^*, \Lambda) = (0.3, 0.99, 0.15)$, a configuration that is super-extremal in flat space ($a^{*2} + q^{*2} = 1.070 > 1$) but stabilized by a large cosmological constant, with horizon gap $r_+ - r_- \approx 0.077$. The near-extremal Kerr corner $(a^*, q^*, \Lambda) = (0.99, 0.0, 0.001)$, by contrast, gives $\kappa_+/\kappa_- \approx 0.749$. Both are consistent with Theorem 3.3 being tightest as $r_- \to r_+$.

Where $\beta_{\rm NH}$ is largest. $\beta_{\rm NH} = \kappa_+/(2\kappa_-)$ inherits its maximum from the same configuration: $\max \beta_{\rm NH} = 0.856/2 = 0.428$. The NH safety margin (gap to Christodoulou threshold 1/2) is $0.500 - 0.428 = 0.072$ uniformly across the admissible subset.

Reproducibility. The scan script topics/phy_knds_kappa_ordering_vieta/scan.py computes the coefficients of Δ_r, uses NumPy's roots for quartic root-finding, applies the admissibility filter, and prints the full statistics. All values in this appendix reproduce by running it.

Appendix C: Independent Tensor-Level Cross-Check of the Geometric Setup

The scan of Appendix B writes the quartic Δ_r directly from the analytic formula (2.2). Section 2 and Appendix A likewise quote this formula without re-deriving it from the Boyer-Lindquist metric. To close this methodological gap and to confirm that the geometry underlying the ordering theorem is the full Einstein-Maxwell solution rather than an algebraic artifact, we ran an independent tensor-level audit (elysium/ fields /vorticity_causality_bridge/knds_tensor_validation.py). The script is the third consumer of the reusable SymPy pipeline in tools/physics/gr_tensor.py (after the Gödel universe and Kerr-de Sitter validations).

Three checks are performed:

(C1) Horizon polynomial from the metric. With (M, a, Λ, Q) symbolic, the Boyer-Lindquist metric (2.1) is built, and $\Delta_r(r)$ is extracted from $g_{rr} = \rho^2/\Delta_r$. After sp.expand and sp. simplify, the result is identically $-(\Lambda/3)r^4 + (1 - \Lambda a^2/3)r^2 - 2Mr + a^2 + Q^2$, matching (2.2) and scan.py symbolically. The Vieta relation $\sum_i r_i = 0$ (V1) is re-read off from the vanishing r^3-coefficient.

(C2) Crown-max numerical match against Appendix B. At the specific crown-max point $(a^*, q^*, \Lambda) = (0.3, 0.99, 0.15)$ the tensor-derived coefficients

$$c_4 = -0.0500000000,\ c_3 = 0,\ c_2 = +0.9955000000,\ c_1 = -2,\ c_0 = +1.0701000000$$

agree with those produced by scan.py's delta_r_coeffs to within 10^{-12}, and the NumPy roots

$$r_n \approx -5.310,\ r_- \approx 1.1237,\ r_+ \approx 1.2014,\ r_c \approx 2.985$$

yield $\kappa_+ \approx 0.01472$, $\kappa_- \approx 0.01720$, $\kappa_+/\kappa_- \approx 0.8556$, matching the 0.856 headline of Section 6 to three decimals. This anchors the chain metric $\to \Delta_r \to$ scan output at the paper's tightest observational point.

(C3) Full Einstein-Maxwell equation at the tensor level. With (M, a, Λ, Q) fixed to rational values $(1, \frac{1}{2}, \frac{1}{10}, \frac{1}{4})$ and $(1, \frac{3}{10}, \frac{1}{20}, \frac{4}{10})$ and (r, θ) kept symbolic, the Christoffel symbols $\Gamma^{\mu}{}_{\nu\rho}$, the Ricci tensor $R_{\mu\nu}$, the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the electromagnetic stress-energy

$$T^{\text{EM}}_{\mu\nu} = \frac{1}{4\pi}\Big[F_{\mu\alpha}F_{\nu}{}^{\alpha} - \tfrac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\Big]$$

are computed symbolically from $g_{\mu\nu}$ and the Carter potential $A_\mu = -(Qr/(\Xi\rho^2))(dt - a\sin^2\theta\, d\phi)$. The residual of the Einstein-Maxwell equation with cosmological constant,

$$R_{\mu\nu} - \Lambda\, g_{\mu\nu} - 8\pi\, T^{\text{EM}}_{\mu\nu},$$

is evaluated componentwise at several physical-region (r, θ) points for each preset. All 16 components vanish to within $\approx 10^{-166}$ (rational arithmetic precision), for each preset and each point. Thus the metric of Section 2 is genuinely an Einstein-Maxwell solution with the stated cosmological constant, and the quartic Δ_r that the theorem factorizes is the horizon polynomial of the full solution rather than a truncated or linearized approximation.

This appendix does not enter any proof — Theorems 3.1–3.3 are self-contained algebraic statements. Its purpose is methodological: it verifies from the metric up that the object on which the Vieta machinery acts is the correct one, using a pipeline that is reused without modification across three distinct GR spacetimes (Gödel dust, Kerr-de Sitter vacuum, Kerr-Newman-de Sitter electrovacuum).