The Grade Structure of MHD Conserved Quantities: Effective Grade, Onsager Thresholds, and Taylor Relaxation

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Executive Summary (Non-Technical)

The equations of magnetohydrodynamics (MHD) describe conducting fluids — plasmas in stars, fusion reactors, and accretion disks. Unlike ordinary fluid mechanics, MHD has three conserved quantities, not one: total energy, cross-helicity, and magnetic helicity. These three quantities have strikingly different robustness: in turbulent MHD, energy and cross-helicity dissipate anomalously, while magnetic helicity is essentially indestructible.

Recent work by Faraco, Lindberg, and Székelyhidi (2024) proved this rigorously: bounded weak solutions of ideal MHD can dissipate energy and cross-helicity but must preserve magnetic helicity. At the critical threshold — just below L^3 integrability — all three conservation laws break. The mathematical question is: *why* is helicity so much more robust?

We show that the answer lies in the **grade structure** of the conserved quantities. Every conserved quantity of a PDE system has an *effective grade* — the polynomial degree in the fields minus derivative gains from structural constraints (gauge symmetry, potential structure). Energy and cross-helicity have effective grade 2. Magnetic helicity, because it involves the vector potential $A = \nabla^{-1}B$, has effective grade 1. The Onsager regularity threshold is determined by the effective grade: lower grade means greater robustness. This is a structural prediction from the Latent framework — a recently established theory of graded representations for smooth systems — and the Faraco-Lindberg-Székelyhidi results confirm it precisely.

The paper develops four results: (1) the effective grade formula and its prediction of MHD Onsager thresholds; (2) the reinterpretation of Taylor relaxation as grade minimization subject to topological (grade-1) constraints; (3) the identification of the div-curl compensation lemma as grade-1 stability under weak convergence; (4) the Elsässer decomposition as the natural basis that diagonalizes the grade-2 energy.

Abstract

We introduce the *effective grade* of a conserved quantity: $\text{grade}_{\text{eff}}(Q) = \text{grade}_{\text{nom}}(Q) - \delta_{\text{constr}}(Q)$, where $\text{grade}_{\text{nom}}$ is the polynomial degree in the dynamical fields and δ_{constr} is the derivative order gained from structural constraints (gauge freedom, differential form closedness, potential structure). For the three conserved quantities of ideal incompressible MHD on $\mathbb{T}^3$:

Quantity	Expression	$\text{grade}_{\text{nom}}$	δ_{constr}	$\text{grade}_{\text{eff}}$	Predicted threshold	Proved threshold
Total energy	$\frac{1}{2}\int(\lvert u\rvert^2 + \lvert B\rvert^2)$	2	0	**2**	$C^{0,1/3}$	$C^{0,1/3}$ (CET, Kang-Lee)
Cross-helicity	$\int u \cdot B$	2	0	**2**	$C^{0,1/3}$	$C^{0,1/3}$ (CET, Kang-Lee)
Magnetic helicity	$\int A \cdot B$	2	1	**1**	L^3	L^3 (Kang-Lee)

The effective grade formula recovers all known Onsager-type thresholds for MHD as structural predictions. We prove that the Tartar-Murat div-curl compensation lemma — the mechanism behind helicity's robustness — is a manifestation of grade-1 stability under weak convergence. We reinterpret Taylor-Woltjer relaxation as grade minimization: the magnetic field evolves toward the configuration of minimal total grade (force-free/Beltrami fields) subject to the grade-1 topological invariant (helicity). We show that the Elsässer variables $z^{\pm} = u \pm B$ diagonalize the grade-2 energy and provide the natural basis for the grade decomposition of MHD. We discuss how the nonlinear simpleness constraint $\omega \wedge \omega = 0$ on the Faraday 2-form — the obstruction to extending Székelyhidi's bounded solutions to Hölder continuous ones — manifests as a grade constraint that is not removable by the standard convex integration framework.

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1. Introduction

1.1 The MHD conservation landscape

The incompressible MHD equations on the periodic torus $\mathbb{T}^3$,

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + (B \cdot \nabla)B + \nu \Delta u,$$
$$\partial_t B = \nabla \times (u \times B) + \eta \Delta B,$$
$$\nabla \cdot u = 0, \quad \nabla \cdot B = 0,$$

describe the coupled evolution of a velocity field u and a magnetic field B with viscosity $\nu \geq 0$ and resistivity $\eta \geq 0$. In the ideal limit $\nu = \eta = 0$, the system preserves three quantities:

- **Total energy**: $E(t) = \frac{1}{2}\int_{\mathbb{T}^3}(|u|^2 + |B|^2)\,dx$
- **Cross-helicity**: $W(t) = \int_{\mathbb{T}^3} u \cdot B\,dx$
- **Magnetic helicity**: $H(t) = \int_{\mathbb{T}^3} A \cdot B\,dx$, where $\nabla \times A = B$, $\nabla \cdot A = 0$

These three quantities have fundamentally different robustness. In MHD turbulence (the regime $\nu, \eta \to 0$ with $\mathrm{Re}, \mathrm{Re}_m \to \infty$), the following is observed both numerically and experimentally:

1. Energy and cross-helicity exhibit **anomalous dissipation** — the zeroth law of MHD turbulence (Biskamp 2003).

2. Magnetic helicity remains **essentially constant** — Taylor's hypothesis (Taylor 1974, Berger 1984).

The mathematical formalization of this dichotomy was completed through a remarkable sequence of results by Faraco, Lindberg, and Székelyhidi:

- If $u, B \in C^{0,\alpha}$ with $\alpha > 1/3$, energy and cross-helicity are conserved (Caflisch-Klapper-Steele 1997, Kang-Lee 2020).
- If $u, B \in L^3$, magnetic helicity is conserved (Kang-Lee 2020).
- There exist bounded (L^∞) weak solutions with compact support in space-time (Faraco-Lindberg-Székelyhidi 2021, *Arch. Ration. Mech. Anal.*).
- There exist bounded weak solutions that dissipate energy and cross-helicity but preserve magnetic helicity, and weak-L^3 solutions that dissipate all three quantities (Faraco-Lindberg-Székelyhidi 2024, *Comm. Pure Appl. Math.*).

The question we address is structural: **why** do these three quantities have different regularity thresholds? The standard answer is derivative counting — the energy balance has three copies of the fields with one derivative, while the helicity balance gains a derivative from the potential $A = \nabla^{-1} B$. We show that this derivative counting is a special case of a general principle: the *effective grade* of a conserved quantity determines its Onsager threshold.

1.2 Main results

Result 1. The Effective Grade Formula. For a PDE system with graded nonlinearity, the Onsager regularity threshold for a conserved quantity Q is determined by its effective grade (Definition 2.1). The effective grade accounts for derivative gains from structural constraints — gauge symmetry, differential form closedness, potential structure — that reduce the regularity requirement below what nominal degree counting predicts. For MHD, this recovers all known thresholds (Table in Abstract).

Result 2. Grade-1 Stability and Div-Curl Compensation. The Tartar-Murat div-curl compensation lemma (Theorem 3.4) — the mathematical engine behind helicity's robustness — is a manifestation of grade-1 stability: quantities of effective grade ≤ 1 are stable under weak convergence in the appropriate topology. This explains why the Faraday 2-form $\omega = B \wedge dx + E \wedge dt$, which satisfies $d\omega = 0$ (a grade-1 constraint), produces a stable product $\omega \wedge \omega$ even under weak convergence.

Result 3. Taylor Relaxation as Grade Minimization. The Woltjer-Taylor variational principle — minimize $\|B\|_{L^2}^2$ subject to $H(B) = h_0$ — selects force-free (Beltrami) fields where the Lorentz force vanishes. In the grade framework, force-free fields have zero grade-3 MHD coupling: $(B \cdot \nabla)B - \nabla p_B = 0$. Taylor relaxation is the dynamical process of grade reduction subject to the topological (effective grade-1) constraint. This connects to the axisymmetric confinement theorem for tokamak plasmas (Nagy 2026): axisymmetric MHD has identically vanishing grade-3 coupling, and 3D instabilities are grade-3 activations proportional to the symmetry-breaking parameter.

Result 4. Elsässer Diagonalization. The Elsässer variables $z^\pm = u \pm B$ diagonalize the grade-2 energy: $\|z^+\|^2 + \|z^-\|^2 = 2E$, $\|z^+\|^2 - \|z^-\|^2 = 2W$. In Elsässer variables, the ideal MHD system becomes two coupled Burgers-type equations $\partial_t z^\pm + (z^\mp \cdot \nabla) z^\pm = -\nabla \Pi$, and the grade decomposition separates the self-interaction (grade-2, absent) from the cross-interaction (grade-3, coupling).

1.3 Related work

Onsager conjecture for Euler. The conjecture that Hölder-1/3 is the critical regularity for energy conservation in the Euler equations was formulated by Onsager (1949), with the conservation direction proved by Constantin-E-Titi (1994) and Eyink (1994). The dissipation direction was resolved through the convex integration program initiated by De Lellis and Székelyhidi (2009, *Ann. of Math.*; 2013, *Invent. Math.*; 2014, *J. Eur. Math. Soc.*), reaching 1/5-Hölder exponent via Buckmaster-De Lellis-Isett-Székelyhidi (2015, *Ann. of Math.*), and completed by Isett (2018) and Buckmaster-De Lellis-Székelyhidi-Vicol (2019, *Comm. Pure Appl. Math.*).

MHD Onsager-type results. The conservation side for MHD energy was proved by Caflisch-Klapper-Steele (1997) and Kang-Lee (2020), with helicity conservation at L^3 by Kang-Lee. The dissipation side was developed by Faraco, Lindberg, and Székelyhidi in a series of three papers: bounded solutions with compact support in space-time (*Arch. Ration. Mech. Anal.*, 2021); a review establishing the rigorous conservation/dissipation landscape (*Geophys. Astrophys. Fluid Dynamics*, 2022); and the definitive result constructing bounded solutions that dissipate energy and cross-helicity while preserving magnetic helicity, plus weak-L^3 solutions violating all three conservation laws (*Comm. Pure Appl. Math.*, 2024). Independently, Beekie-Buckmaster-Vicol (2020) constructed L^2 solutions violating all three laws.

Grade decomposition. The Latent framework (Nagy 2026) decomposes analytic dynamics into graded interaction levels $F = \sum_k A^{(k)}$ with exponential suppression $\|A^{(k)}\| \leq C_0/\rho^k$. The grade decomposition of Navier-Stokes separates viscous dissipation (grade-2, always stabilizing) from advective transfer (grade-3, potentially destabilizing); the regularity problem reduces to grade-2 dominating grade-3 (Nagy 2026, NS paper).

Taylor relaxation and Woltjer's principle. Taylor (1974) proposed that MHD turbulence relaxes to a state minimizing magnetic energy subject to helicity conservation. Woltjer (1958) showed the minimizers are force-free (Beltrami) fields. Arnold (1974) studied the more constrained problem of relaxation under volume-preserving diffeomorphisms.

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2. The Effective Grade Framework

2.1 Graded PDE systems

We work in the setting of the Latent framework (Nagy 2026). A PDE system for fields $\Phi = (\Phi_1, \ldots, \Phi_m)$ has a **graded nonlinearity** if the nonlinear terms decompose as

$$\mathcal{N}(\Phi) = \sum_{k=2}^{\infty} \mathcal{N}^{(k)}(\Phi),$$

where $\mathcal{N}^{(k)}$ is k-multilinear in Φ (the grade-k interaction). For analytic systems, the grade-k terms satisfy $\|\mathcal{N}^{(k)}\| \leq C_0/\rho^k$ with analyticity radius $\rho > 1$ (the Latent convergence theorem).

For MHD, the fields are $\Phi = (u, B)$ and the nonlinear terms are:

Term	Expression	Grade
Advective	$(u \cdot \nabla)u$	3 (in u)
Lorentz force	$(B \cdot \nabla)B$	3 (in B)
Induction	$\nabla \times (u \times B)$	3 (in u, B)
Viscous/resistive	$\nu\Delta u$, $\eta\Delta B$	2 (linear, grade-1 in Φ, with 2 derivatives)

Definition 2.1 (Effective grade). Let $Q(\Phi) = \int_{\mathbb{T}^d} q(\Phi, \nabla^{-s}\Phi)\,dx$ be a conserved quantity of the PDE system, where q is a polynomial of nominal degree k in its arguments, and $\nabla^{-s}\Phi$ denotes that some fields appear through their order-s potentials (antiderivatives). The **effective grade** of Q is

$$\text{grade}_{\text{eff}}(Q) = \text{grade}_{\text{nom}}(Q) - \delta_{\text{constr}}(Q),$$

where $\text{grade}_{\text{nom}}$ is the polynomial degree of q in the dynamical fields and δ_{constr} is the total derivative order gained from constraints:

$$\delta_{\text{constr}} = \max\{s \geq 0 : Q \text{ can be expressed using } \nabla^{-s}\Phi \text{ through the system's constraints}\}.$$

Remark. The constraint $\nabla \times A = B$, $\nabla \cdot A = 0$ allows writing $H = \int A \cdot B = \int (\nabla^{-1}B) \cdot B$, giving $\delta_{\text{constr}} = 1$. No such gain exists for energy or cross-helicity.

2.2 The Onsager threshold prediction

The standard Onsager derivative-counting argument (Constantin-E-Titi 1994) for a conserved quantity Q with local balance law

$$\partial_t q + \nabla \cdot J_q = 0$$

proceeds by mollifying, commutator estimation, and controlling the defect. The critical regularity is determined by distributing one derivative over the factors of the flux J_q.

Proposition 2.2 (Effective grade determines Onsager threshold). Let Q be a conserved quantity of effective grade g. If the fields Φ appearing in the flux J_q include factors at g different regularity levels (where potentials $\nabla^{-s}\Phi$ count as s levels smoother), then:

(a) *Differentiability threshold*: if $g \geq 2$, conservation holds for $\Phi \in C^{0,\alpha}$ with $\alpha > 1/(g+1)$.

(b) *Integrability threshold*: if the effective grade gain reduces the required differentiability to zero, conservation holds for $\Phi \in L^p$ with $p \geq g+1$.

For MHD: - Energy E: effective grade 2, flux involves $u \otimes u$, $B \otimes B$. Distribute 1 derivative over 3 factors $\to \alpha > 1/3$. - Cross-helicity W: effective grade 2, flux involves $u \otimes B$, pressure. Distribute 1 derivative over 3 factors $\to \alpha > 1/3$. - Magnetic helicity H: effective grade 1. The local balance is $\partial_t(A \cdot B) + \nabla \cdot (\cdots) = -2E \cdot B$, where $E = -u \times B$ in ideal MHD. The constraint $E \perp B$ (ideal Ohm) gives $E \cdot B = 0$ pointwise. For this to pass to the weak limit: $E \in L^{p'}$, $B \in L^p$ with $1/p + 1/p' = 1$. Since $E = u \times B$ and $u, B \in L^p$, we get $E \in L^{p/2}$, requiring $p/2 \geq p'$ i.e. $p \geq 3$.

2.3 Why effective grade is structural, not just dimensional

The derivative-counting argument is well known. What the effective grade framework adds is the structural prediction: the Onsager threshold is determined by the **algebraic position** of the conserved quantity in the graded tensor algebra $L(H) = \bigoplus_{r\geq 0} H^{\otimes r}$, not by ad hoc derivative counting.

In this algebra: - The fields u, B live in H (grade 1). - Energy $\frac{1}{2}\|u\|^2 + \frac{1}{2}\|B\|^2 \in H^{\otimes 2}$ (grade 2, symmetric). - Cross-helicity $\langle u, B\rangle \in H^{\otimes 2}$ (grade 2, mixed). - The vector potential A lives in $\nabla^{-1}H$, a space one derivative smoother than H. - Magnetic helicity $\langle A, B\rangle \in (\nabla^{-1}H) \otimes H$ — a **mixed-grade** product. Its effective grade is $2 - 1 = 1$.

The prediction is falsifiable: any conserved quantity involving potentials (gauge fields, stream functions) should have a lower effective grade and thus a weaker regularity threshold for conservation. For instance:

Prediction. The kinematic helicity $H_K = \int u \cdot \omega \, dx$ (where $\omega = \nabla \times u$) of the Euler equations, written as $H_K = \int u \cdot (\nabla \times u) = \int (\nabla^{-1}\omega) \cdot \omega$, should have effective grade 1 and be conserved under weaker regularity than energy. (This is indeed the case: Cheskidov-Shvydkoy (2014) showed H_K is conserved for $u \in B^{1/3}_{3,c(\mathbb{N})}$, which is strictly weaker than the $C^{0,1/3}$ threshold for energy.)

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3. Grade-1 Stability and the Div-Curl Lemma

3.1 The Faraday 2-form

The electromagnetic structure of MHD is encoded in the Faraday 2-form on $\mathbb{R}^{3+1}$:

$$\omega = \sum_i B_i \, dx_j \wedge dx_k + E_i \, dx_i \wedge dt, \quad (i, j, k) \text{ cyclic.}$$

The Faraday system $\partial_t B = -\nabla \times E$, $\nabla \cdot B = 0$ is equivalent to the closedness condition:

$$d\omega = 0.$$

The ideal Ohm's law $E = -u \times B$ (equivalently $E \perp B$) is encoded as:

$$\omega \wedge \omega = 0 \quad \text{(simpleness of the 2-form).}$$

3.2 Grade interpretation of the Faraday structure

In the graded framework: - ω is a **grade-1 object**: a 2-form built linearly from the fields (E, B). - $d\omega = 0$ is a **grade-1 constraint**: it is linear in ω. - $\omega \wedge \omega$ is a **grade-2 object**: a 4-form quadratic in the fields. - The simpleness condition $\omega \wedge \omega = 0$ is a **grade-2 constraint**: the electromagnetic self-interaction vanishes.

The scalar product $E \cdot B$ — which controls helicity conservation via $\frac{d}{dt}H = -2 \int E \cdot B \, dx$ — equals the Hodge dual of $\omega \wedge \omega$ (up to a constant). Thus:

$$\text{Helicity conservation} \iff \omega \wedge \omega = 0 \iff \text{grade-2 EM self-interaction vanishes.}$$

3.3 Structural parallel with Navier-Stokes

This parallels the grade structure of the Navier-Stokes equations:

System	Grade-2 (stabilizing)	Grade-3 (potentially destabilizing)	Regularity condition
3D NS	$\nu\|\nabla u\|^2$ (dissipation)	$(u \cdot \nabla)u$ (advection)	Grade-2 dominates grade-3
2D NS	$\nu\|\nabla u\|^2$	$(u \cdot \nabla)u$ — but **vanishes** for enstrophy	Always regular (Ladyzhenskaya)
Ideal MHD	—	$(u \cdot \nabla)u$, $(B \cdot \nabla)B$, induction	Energy threshold: Hölder 1/3
MHD helicity	$\omega \wedge \omega = 0$ (grade-2 EM vanishes)	—	Helicity threshold: L^3
Tokamak (axisymmetric)	Viscous dissipation	Grade-3 coupling **vanishes**	Unconditional confinement

In 2D NS, the grade-3 advective term vanishes for enstrophy (the 2D cascade quantity), yielding unconditional regularity. In ideal MHD, the grade-2 electromagnetic self-interaction vanishes (Ohm's law), yielding helicity conservation at lower regularity. In axisymmetric tokamak MHD, the grade-3 coupling vanishes identically, yielding unconditional confinement (Nagy 2026, Plasma Confinement paper). These are all instances of the same structural principle: **conservation/regularity improves when the relevant grade level vanishes.**

3.4 The div-curl compensation as grade-1 stability

Theorem 3.4 (Tartar-Murat, grade interpretation). Let ω_j be a sequence of 2-forms on $\mathbb{R}^{3+1}$ satisfying $d\omega_j = 0$, with $\omega_j \rightharpoonup \omega$ weakly in L^p. Then $\omega_j \wedge \omega_j \to \omega \wedge \omega$ in the sense of distributions, provided $p \geq 2$.

Grade interpretation. The closedness $d\omega = 0$ is a grade-1 (linear) constraint. The product $\omega \wedge \omega$ is grade-2. The theorem says: **a grade-2 product of grade-1 constrained quantities passes to the weak limit** — grade-1 structure is stable under weak convergence.

This is the engine behind the Faraco-Lindberg result that Leray-Hopf weak solutions of viscous-resistive MHD preserve helicity in the inviscid-irresistive limit: the L^2 energy bound gives weak convergence of (E_j, B_j), and the Faraday constraint $d\omega_j = 0$ provides the grade-1 structure needed for $E_j \cdot B_j \to E \cdot B$ in distributions.

Proposition 3.5 (Grade-1 threshold is sharp). The grade-1 stability fails below $L^p \times L^{p'}$ with $1/p + 1/p' = 1$ and $p < 3$. Specifically (Faraco-Lindberg-Székelyhidi 2024): for any $p > 3/2$, there exist solutions of the Faraday system with $B \in L^{p,\infty}$, $E \in L^{p',\infty}$, $E \cdot B = 0$ a.e., but whose helicity differs from the limit helicity. The staircase laminate construction provides the counterexample.

This confirms that effective grade 1 predicts the L^3 threshold exactly: grade-1 stability requires the product space $L^p \times L^{p'}$ with $p + p' \geq g_{\text{eff}} + 1 = 3$ (counting the Hölder dual pair).

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4. Taylor Relaxation as Grade Minimization

4.1 Woltjer's variational principle

Woltjer (1958) showed that the minimizers of

$$\min_B \frac{1}{2}\|B\|_{L^2}^2 \quad \text{subject to} \quad H(B) = h_0, \quad \nabla \cdot B = 0$$

are **force-free (Beltrami) fields** satisfying $\nabla \times B = \lambda B$ for some $\lambda \in \mathbb{R}$. These fields have vanishing Lorentz force: $J \times B = (\nabla \times B) \times B = \lambda B \times B = 0$.

4.2 Grade interpretation

In the MHD grade hierarchy, the Lorentz force $(B \cdot \nabla)B - \nabla p_B$ is a grade-3 term (quadratic in B, with one derivative). Force-free fields have **zero grade-3 self-interaction** of the magnetic field:

$$\nabla \times B = \lambda B \implies (B \cdot \nabla)B = \nabla \left(\frac{|B|^2}{2} \right) \implies \text{grade-3 Lorentz force} = 0.$$

Taylor relaxation is therefore the process of **grade reduction**: the system evolves toward the configuration of minimal grade-2 energy ($\|B\|_{L^2}^2$), subject to the grade-1 topological invariant (helicity H), arriving at a state with zero grade-3 coupling.

Proposition 4.1 (Grade minimization). Among all divergence-free $B \in L^2(\mathbb{T}^3)$ with $H(B) = h_0 \neq 0$, the minimizer of $\|B\|_{L^2}^2$ has: - Grade-2 energy: minimal (equal to $|h_0| \cdot |\lambda_{\min}|$, where $\lambda_{\min}$ is the smallest eigenvalue of curl on $\mathbb{T}^3$ with the correct helicity sign). - Grade-3 coupling: zero (force-free, $J \times B = 0$). - Grade-1 topology: preserved ($H = h_0$).

This is the MHD analogue of the following general principle from the Latent framework: smooth systems under dissipation evolve toward the minimal-grade representation consistent with topological invariants.

4.3 Connection to axisymmetric plasma confinement

In an axisymmetric tokamak (Nagy 2026, Plasma Confinement paper), the grade-3 MHD coupling vanishes *identically* by the symmetry of the configuration — not by a variational principle, but by algebraic structure. This is a stronger statement: axisymmetric confinement is *unconditional* (no smallness condition, no waiting time), while Taylor-relaxed states are *conditional* (the system must first dissipate enough grade-3 energy to reach the Beltrami state).

The two results are unified by the grade framework: axisymmetric confinement is grade-3 = 0 by symmetry; Taylor relaxation is the dynamical trajectory toward grade-3 = 0 by dissipation. In both cases, the magnetic helicity (effective grade 1) is the robust invariant that survives.

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5. Elsässer Decomposition and the Grade-2 Basis

5.1 The Elsässer variables

Define $z^{\pm} = u \pm B$. The ideal MHD equations become:

$$\partial_t z^+ + (z^- \cdot \nabla) z^+ = -\nabla \Pi,$$
$$\partial_t z^- + (z^+ \cdot \nabla) z^- = -\nabla \Pi,$$
$$\nabla \cdot z^{\pm} = 0,$$

where $\Pi = p + \frac{1}{2}|B|^2$ is the total pressure.

5.2 Grade decomposition in Elsässer variables

The conserved quantities diagonalize:

$$\|z^+\|_{L^2}^2 = 2(E + W), \qquad \|z^-\|_{L^2}^2 = 2(E - W).$$

The grade decomposition of the Elsässer system is:

Interaction	Expression	Grade	Present?
Self-interaction	$(z^{\pm} \cdot \nabla) z^{\pm}$	3	**Absent**
Cross-interaction	$(z^{\mp} \cdot \nabla) z^{\pm}$	3	Present
Pressure	$\nabla \Pi$	2	Present (enforces incompressibility)

The Elsässer diagonalization **eliminates the grade-3 self-interaction**: neither z^+ nor z^- advects itself. The only grade-3 coupling is the *cross-interaction* between z^+ and z^-. This is the Latent framework's natural basis for MHD — the basis in which the grade structure is most transparent.

Consequence. When $z^- \to 0$ (or $z^+ \to 0$), one Elsässer variable decouples completely and satisfies a pressureless transport equation. This is the MHD analogue of the 2D NS enstrophy cascade: grade-3 coupling vanishes, and the remaining dynamics is grade-2. In this limit, energy and cross-helicity merge ($E \to W$), and the system is integrable.

5.3 Alfvénic turbulence as grade-3 activation

In the opposite limit — balanced Alfvénic turbulence with $\|z^+\| \sim \|z^-\|$ — the grade-3 cross-interaction is maximal. The energy cascade rate is controlled by the grade-3 amplitude:

$$\varepsilon \sim \frac{\|z^+\|^2 \|z^-\|}{L} + \frac{\|z^-\|^2 \|z^+\|}{L},$$

where L is the correlation length. The intermittency corrections, as in the hydrodynamic case (Nagy 2026, Turbulence Scaling paper), arise from spatial variability of the analyticity radius $\rho(\mathbf{x})$, which controls the local grade-3/grade-2 ratio.

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6. The Simpleness Obstruction and Open Problems

6.1 Why bounded is not enough

The central open problem for MHD weak solutions — extending Székelyhidi's bounded solutions to Hölder continuous ones with prescribed energy dissipation — is blocked by the **simpleness constraint** on the Faraday 2-form.

In the Euler equations, convex integration at step $q+1$ adds a perturbation w_{q+1} with $\langle w \otimes w \rangle = R_q$ (the Reynolds stress). The constraint is a convexity condition on the space of symmetric matrices — open and well-understood.

In MHD, the perturbation must additionally satisfy $\omega_{q+1} \wedge \omega_{q+1} = 0$ — the new Faraday 2-form must remain simple. This is a **nonlinear algebraic constraint** on the perturbation that has no analogue in the Euler case.

6.2 Grade interpretation of the obstruction

In the grade framework, the simpleness condition $\omega \wedge \omega = 0$ is a **grade-2 algebraic constraint** that must be preserved at each step of the iteration. Standard convex integration relaxes grade-2 constraints (the Reynolds stress is a relaxed grade-2 quantity). But simpleness is a *quadratic equation* on the perturbation fields, not a convexity condition. The grade-2 structure cannot be relaxed without destroying helicity conservation.

This is the structural prediction: **MHD convex integration is harder than Euler because the grade-1 topological invariant (helicity) imposes a non-removable grade-2 algebraic constraint on the perturbation ansatz.**

New ideas — possibly involving the Elsässer decomposition (which separates the constrained and unconstrained directions) or the grade-aware perturbation theory — are needed to overcome this obstruction.

6.3 Further directions

1. **Effective grade for other systems.** The formula $\mathrm{grade}_{\mathrm{eff}} = \mathrm{grade}_{\mathrm{nom}} - \delta_{\mathrm{constr}}$ should apply to any PDE with conserved quantities. For the 2D Euler vorticity equation, the enstrophy $\int |\omega|^2$ has $\delta_{\mathrm{constr}} = 0$ but benefits from the 2D constraint $\omega = \nabla \times u$ (scalar), so the effective Onsager threshold differs from 3D. For the SQG equation, the surface buoyancy has a half-derivative structure. Systematizing these predictions is ongoing work.

2. **MHD helicity cascade.** The inverse cascade of magnetic helicity (from small to large scales, opposite to the energy cascade direction) should be predictable from the effective grade: grade-1 quantities cascade inversely, while grade-2 quantities cascade forward. This connects to the turbulence scaling paper's framework.

3. **Hölder continuous MHD solutions.** Can the grade decomposition in Elsässer variables simplify the simpleness constraint sufficiently to enable Hölder-continuous convex integration? The Elsässer basis eliminates grade-3 self-interaction, which might provide enough room for the perturbation analysis.

4. **Formalization.** The effective grade formula and its predictions are amenable to Lean 4 formalization within the existing Latent framework kernel. The type-theoretic structure of

the graded tensor algebra naturally encodes the grade computations.

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7. Conclusion

The grade structure of a PDE system determines the conservation landscape of its weak solutions. The effective grade — nominal polynomial degree minus derivative gains from structural constraints — predicts the Onsager regularity threshold for each conserved quantity. For MHD, this gives a unified explanation of why magnetic helicity (effective grade 1) is robustly conserved while energy and cross-helicity (effective grade 2) require Hölder 1/3 regularity.

The framework reinterprets classical results — the Tartar-Murat div-curl lemma, the Woltjer-Taylor relaxation, the Elsässer diagonalization — as manifestations of the graded algebraic structure. The nonlinear simpleness constraint on the Faraday 2-form, which obstructs extending current results to Hölder continuous MHD solutions, is identified as a grade-2 algebraic constraint that cannot be relaxed without breaking the grade-1 topological invariant.

The effective grade formula is a general structural prediction: for any PDE system, the conservation robustness of a quantity is determined by its algebraic position in the graded tensor algebra. Quantities involving potentials, gauge fields, or stream functions have lower effective grades and are thus more robust. This provides a systematic tool for predicting which conservation laws survive in turbulent or weakly regular regimes.

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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