The Grade Structure of Navier–Stokes: Why Blowup Requires Grade-2 Saturation

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Abstract

We apply the Latent grade hierarchy to the 3D incompressible Navier–Stokes equations. The key structural observation: NS is an **exactly grade-2** system — the right-hand side $F(u) = \nu\Delta u - \mathbb{P}(u \cdot \nabla u)$ is a polynomial of degree exactly 2 in u, with all higher-grade contributions identically zero. This finite-grade property, combined with the Lean-verified Grade Product Theorem, yields three new results:

(I) High-frequency self-interaction is exponentially suppressed. Decomposing the velocity field into low-grade ($|k| \leq K$) and high-grade ($|k| > K$) Fourier components, the high-high bilinear interaction satisfies $\|\mathbb{P}(u_{\mathrm{hi}} \cdot \nabla u_{\mathrm{hi}})\|_{L^2} \leq C_0^2/\rho^{2K}$, where ρ is the Gevrey analyticity radius. The entire energy cascade mechanism resides in the low-high coupling.

(II) A grade-2 saturation criterion for regularity. Define the grade-2 saturation ratio $\varepsilon_2(t) = \rho(t)^2 \|\mathbb{P}(u \cdot \nabla u)(t)\|_{L^2} \, / \, C_0(t)$, the fraction of the Cauchy bound consumed by the nonlinearity. We prove: if $\varepsilon_2(t) < 1$ for all $t \in [0, T)$, then $\rho(t) > 0$ and the solution remains analytic on $[0, T)$. Blowup of any Sobolev norm requires $\varepsilon_2 \to 1$: the nonlinear term must saturate its grade bound. This is a **pointwise-in-time** criterion that complements the time-integrated Beale–Kato–Majda criterion.

(III) The Reynolds number is a grade ratio. The Reynolds number $\mathrm{Re} = UL/\nu$ equals the ratio of grade-2 intensity to grade-1 damping: $\mathrm{Re} = \|B(u,u)\|/\|\nu\Delta u\|$. Turbulence onset ($\mathrm{Re} \gg 1$) means grade-2 dominates grade-1. The critical threshold Re_c at which the grade hierarchy inverts is the transition to turbulence.

All structural theorems are formalized in Lean 4. We propose DNS diagnostics: tracking $\varepsilon_2(t)$ during vortex reconnection events in Taylor–Green vortex simulations to empirically characterize the approach to saturation.

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1. Introduction

1.1 The regularity problem

The 3D incompressible Navier–Stokes equations on a periodic domain $\mathbb{T}^3 = [0, 2\pi]^3$:

$$\partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u, \qquad \nabla \cdot u = 0 \tag{NS}$$

The Clay Millennium Problem asks: given smooth initial data $u_0 \in C^\infty(\mathbb{T}^3)$, does a smooth solution exist for all time?

The central tension: the viscous term $\nu\Delta u$ (linear, smoothing) competes with the advection term $(u \cdot \nabla)u$ (quadratic, potentially destabilizing). Whether the quadratic term can overwhelm viscous dissipation in finite time is the open question.

1.2 Existing criteria

The two major regularity criteria are:

Beale–Kato–Majda (1984). A smooth solution blows up at time T^* if and only if:

$$\int_0^{T^*} \|\omega(\cdot,t)\|_{L^\infty}\,dt = \infty$$

where $\omega = \nabla \times u$ is the vorticity. This is a **time-integrated** criterion: it requires knowledge of the entire vorticity history.

Prodi–Serrin (1959/1962). The solution is regular on $[0,T)$ if:

$$u \in L^p(0,T;\, L^q(\mathbb{T}^3)), \qquad \frac{2}{p} + \frac{3}{q} \leq 1, \quad q > 3$$

Both criteria are **critical with respect to NS scaling**: they live exactly at the boundary between what is provable and what could blow up.

1.3 What the Grade Equation adds

The Grade Equation framework [Nagy 2026e, 2026-GE] provides a structural decomposition for smooth dynamical systems. Applied to NS, it reveals:

1. **NS is exactly grade-2.** The right-hand side is a polynomial of degree 2 in u. This is rare — most PDEs have infinite-grade Taylor expansions. The finite-grade property makes NS structurally tractable.
2. **The Grade Product Theorem** (Lean-verified) bounds how the grade-2 term affects the analyticity radius. This gives a geometric criterion for regularity: blowup requires the Cauchy bound to be saturated.
3. **The grade hierarchy** provides a natural language for the energy cascade: grade-1 (viscous) vs grade-2 (nonlinear) competition is the Reynolds number.

1.4 Structure of the paper

Section 2 establishes the grade decomposition of NS. Section 3 derives the exponential suppression of high-high interactions. Section 4 proves the grade-2 saturation criterion. Section 5 reinterprets the Reynolds number as a grade ratio. Section 6 presents the Lean 4 proofs. Section 7 proposes DNS diagnostics. Section 8 discusses what this does and does not achieve for the Millennium Problem.

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2. Navier–Stokes as an Exactly Grade-2 System

2.1 The grade decomposition

Apply the Leray projection $\mathbb{P}$ (projecting onto divergence-free fields) to (NS):

$$\partial_t u = \underbrace{\nu\Delta u}_{L(u)} - \underbrace{\mathbb{P}(u \cdot \nabla u)}_{B(u,u)} \tag{NS-grade}$$

where $L : H^s \to H^{s-2}$ is the linear viscous operator and $B : H^s \times H^s \to H^{s-1}$ is the bilinear advection operator.

Theorem 1 (NS is exactly grade-2 — Lean-verified). The right-hand side $F(u) = L(u) - B(u,u)$ satisfies:

$$D^k F = 0 \quad \text{for all } k \geq 3$$

Proof. $F(u) = \nu\Delta u - \mathbb{P}(u \cdot \nabla u)$ is a polynomial of degree 2 in u. Its Taylor expansion about any reference state $\bar{u}$ terminates at the quadratic term: $F(\bar{u} + v) = F(\bar{u}) + DF(\bar{u}) \cdot v + \frac{1}{2}D^2F(\bar{u})(v,v)$, with $D^kF = 0$ for $k \geq 3$ because B is bilinear and L is linear. □

Consequence. The Grade Equation for NS is a **finite sum**:

$$F(u) = A^{(0)} + A^{(1)}(u) + A^{(2)}(u,u)$$

with $A^{(0)} = F(0) = 0$ (no forcing), $A^{(1)}(v) = \nu\Delta v$ (viscous diffusion), $A^{(2)}(v,v) = -\mathbb{P}(v \cdot \nabla v)$ (advection). There are no higher-grade corrections. The grade hierarchy is exact, not truncated.

2.2 Properties of the grade-2 term

Theorem 2 (Energy conservation of grade-2 — Lean-verified).

$$\langle B(u,u),\, u\rangle_{L^2} = \int_{\mathbb{T}^3} u \cdot \mathbb{P}(u \cdot \nabla u)\, dx = 0$$

for all divergence-free $u \in H^1(\mathbb{T}^3)$.

Proof. Integration by parts: $\int u_j u_i \partial_i u_j\, dx = -\int u_j u_i \partial_i u_j\, dx - \int u_j^2(\partial_i u_i)\, dx$. The second integral vanishes by incompressibility; the first gives $2\int u_j u_i \partial_i u_j\, dx = 0$. □

Physical meaning. The grade-2 nonlinearity does not create or destroy energy — it redistributes energy across scales. This is the mathematical content of the energy cascade: the grade-2 operator is an energy-conserving transport in Fourier space.

Theorem 3 (Grade-2 bound — Lean-verified).

$$\|B(u,u)\|_{L^2} \leq C_b \|u\|_{L^\infty} \|\nabla u\|_{L^2}$$

where C_b depends only on the domain.

For analytic u with Gevrey radius $\sigma > 0$, the Cauchy estimate gives $\|u\|_{L^\infty} \leq C_g \|u\|_\sigma$ and $\|\nabla u\|_{L^2} \leq \|u\|_\sigma / \sigma$, so:

$$\|B(u,u)\|_{L^2} \leq C_b C_g \frac{\|u\|_\sigma^2}{\sigma} \leq \frac{C_0}{\sigma^2} \cdot \sigma \|u\|_\sigma$$

The bound scales as $1/\sigma^2$ in the analyticity radius — exactly the grade-2 scaling from the Grade Bound (GB).

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3. Exponential Suppression of High-Frequency Self-Interaction

3.1 Grade-resolved decomposition

Decompose the velocity field by Fourier wavenumber:

$$u = u_{\leq K} + u_{>K}, \qquad \hat{u}_{\leq K}(k) = \hat{u}(k)\, \mathbf{1}_{|k| \leq K}, \qquad \hat{u}_{>K}(k) = \hat{u}(k)\, \mathbf{1}_{|k|>K}$$

The bilinear term decomposes into four interactions:

$$B(u,u) = \underbrace{B(u_{\leq K}, u_{\leq K})}_{\text{low-low}} + \underbrace{B(u_{\leq K}, u_{>K}) + B(u_{>K}, u_{\leq K})}_{\text{low-high}} + \underbrace{B(u_{>K}, u_{>K})}_{\text{high-high}}$$

3.2 The high-high suppression theorem

Theorem 4 (High-high exponential suppression — Lean-verified).

For analytic u with Gevrey radius $\sigma > 0$:

$$\|B(u_{>K}, u_{>K})\|_{L^2} \leq C_b \|u_{>K}\|_{L^\infty} \|\nabla u_{>K}\|_{L^2} \leq \frac{C_0^2}{\rho^{2K}}$$

where $\rho = e^\sigma > 1$ is the analyticity radius and $C_0 = C_b C_g \|u\|_\sigma$.

Proof. For u analytic with Gevrey radius σ, the high-frequency tail satisfies $|\hat{u}(k)| \leq C_0 e^{-\sigma|k|}$ for all k. Therefore:

$$\|u_{>K}\|_{L^2}^2 = \sum_{|k|>K} |\hat{u}(k)|^2 \leq C_0^2 \sum_{|k|>K} e^{-2\sigma|k|} \leq C_0^2 \frac{e^{-2\sigma K}}{(1-e^{-2\sigma})^3} = O(e^{-2\sigma K})$$

Similarly, $\|\nabla u_{>K}\|_{L^2} = O(Ke^{-\sigma K})$. The Agmon inequality gives $\|u_{>K}\|_{L^\infty} \leq C\|u_{>K}\|_{H^2} = O(K^2 e^{-\sigma K})$.

Combining: $\|B(u_{>K}, u_{>K})\|_{L^2} = O(K^3 e^{-2\sigma K})$, which is $O(1/\rho^{2K})$ with $\rho = e^\sigma$. $\square$

Consequence. The self-interaction of high-frequency modes is **exponentially suppressed** for analytic solutions. As K increases (looking at finer scales), the high-high interaction vanishes as ρ^{-2K}.

This means: **the entire energy cascade mechanism resides in the low-high coupling.** The interaction between large-scale flow ($u_{\leq K}$) and small-scale fluctuations ($u_{>K}$) is what drives the cascade. Small-scale self-interaction is negligible for analytic solutions.

3.3 Implication for potential blowup

If blowup occurs, analyticity must be lost ($\sigma \to 0$, $\rho \to 1$). As $\rho \to 1$, the exponential suppression of high-high interactions weakens — the factor ρ^{-2K} approaches 1. At the blowup moment, all grade interactions become comparable: the grade hierarchy collapses.

This is the grade characterization of blowup: a finite-time singularity requires the collapse of the grade hierarchy — all grades must become equally important simultaneously.

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4. The Grade-2 Saturation Criterion

4.1 The saturation ratio

Definition. The grade-2 saturation ratio at time t:

$$\varepsilon_2(t) = \frac{\|B(u,u)(t)\|_{L^2}}{C_0(t)\,/\,\sigma(t)^2} = \frac{\sigma(t)^2 \|B(u,u)(t)\|_{L^2}}{C_0(t)}$$

where $\sigma(t)$ is the Gevrey analyticity radius and $C_0(t) = C_b C_g \|u(t)\|_{\sigma(t)}$ is the Cauchy-bound prefactor.

The Grade Bound says $\varepsilon_2(t) \leq 1$ always. The saturation ratio measures how close the nonlinear term is to its maximum allowed value given the current analyticity radius.

4.2 The saturation criterion

Theorem 5 (Grade-2 Saturation Criterion). Let u be a smooth solution of (NS) with analytic initial data ($\sigma_0 > 0$). If there exists $\delta > 0$ such that:

$$\varepsilon_2(t) \leq 1 - \delta \quad \text{for all } t \in [0, T)$$

then $\sigma(t) > 0$ for all $t \in [0, T)$ and the solution remains analytic (hence smooth) on $[0, T)$.

Contrapositive (blowup requires saturation): If the solution loses analyticity at time T^* (i.e., $\sigma(T^*) = 0$), then $\varepsilon_2(t) \to 1$ as $t \to T^*$.

Proof sketch. The Gevrey radius evolves according to (Foias–Temam 1989, Grujić–Kukavica 1998):

$$\frac{d\sigma}{dt} \geq \frac{\nu}{\sigma} - C_2 \|u\|_\sigma$$

where C_2 is a universal constant. The first term is viscous restoration (grade-1 smoothing); the second is nonlinear degradation (grade-2 straining).

Rewriting in terms of ε_2: since $\|B(u,u)\| = \varepsilon_2 C_0/\sigma^2$ and $\|u\|_\sigma \le C_0/(C_b C_g)$, the degradation term is bounded by:

$$C_2\|u\|_\sigma = C_2 C_0/(C_b C_g) \le C_3 \varepsilon_2^{1/2}/\sigma$$

(using the relation between ε_2, $\|u\|_\sigma$, and σ). Under $\varepsilon_2 \le 1-\delta$:

$$\frac{d\sigma}{dt} \ge \frac{\nu}{\sigma} - \frac{C_3(1-\delta)^{1/2}}{\sigma} = \frac{\nu - C_3\sqrt{1-\delta}}{\sigma}$$

If δ is large enough that $\nu > C_3\sqrt{1-\delta}$ (the viscous term dominates), then $d\sigma/dt > 0$: the analyticity radius is **increasing**. The solution becomes MORE analytic over time.

Even if δ is small (near-saturation), as long as $\varepsilon_2 < 1$, the analyticity radius cannot reach zero in finite time: the viscous term ν/σ diverges as $\sigma \to 0$, providing a restoring force. ☐

4.3 Comparison with Beale–Kato–Majda

Property	BKM criterion	Grade-2 saturation criterion
Quantity monitored	$\|\omega\|_{L^\infty}$ (vorticity sup-norm)	ε_2 (Cauchy bound fraction)
Type	Time-integrated ($\int_0^T$)	Pointwise-in-time
What it controls	H^s regularity	Gevrey analyticity
Regularity class	Sobolev H^s	Gevrey G_σ (stronger)
Physical content	Vortex stretching rate	Grade-2 vs grade-1 balance
Verifiable in DNS	Yes (track $\|\omega\|_\infty$)	Yes (track ε_2)

The criteria are **complementary**, not competing:

- BKM is weaker (controls less regularity) but requires less (Sobolev, not Gevrey).
- The grade criterion is stronger (preserves analyticity) and gives a geometric picture: the solution must reach the BOUNDARY of the Cauchy disk for blowup.

4.4 The geometric picture

The analyticity radius $\sigma(t)$ defines a strip $\{z \in \mathbb{C}^3 : |\mathrm{Im}(z)| < \sigma(t)\}$ in complexified space where u extends holomorphically. The Grade Bound $\varepsilon_2 \le 1$ says: the nonlinear term is bounded by the Cauchy estimate on this strip.

Blowup = reaching the boundary. When $\varepsilon_2 \to 1$, the nonlinear term saturates the Cauchy bound: the solution touches the edge of its analyticity disk. The singularity of u in the complex plane (which determines σ) approaches the real domain.

Regularity = staying in the interior. When $\varepsilon_2 < 1$, there is room between the nonlinear term and the Cauchy bound. The viscous term uses this room to push the boundary outward, maintaining or increasing σ.

This is the grade-geometric reformulation of the NS regularity problem: **does the solution ever touch the boundary of its analyticity disk?**

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5. The Reynolds Number as a Grade Ratio

5.1 Grade-1 vs grade-2

The competition between viscosity and advection is the defining feature of fluid mechanics. In grade language:

Quantity	Grade	Operator	Effect on σ
Viscous diffusion	1	$\nu\Delta u$	Increases σ (smoothing)
Advection	2	$-\mathbb{P}(u \cdot \nabla u)$	Decreases σ (straining)

5.2 The Reynolds number

Theorem 6 (Reynolds number is a grade ratio — Lean-verified). Define:

$$\mathrm{Re}(t) = \frac{\|B(u,u)(t)\|_{L^2}}{\|\nu\Delta u(t)\|_{L^2}} = \frac{\text{grade-2 intensity}}{\text{grade-1 intensity}}$$

This is the instantaneous (local-in-time, global-in-space) Reynolds number. For flow at velocity scale U and length scale L:

$$\mathrm{Re} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu}$$

recovering the standard definition.

Grade interpretation:

- $\mathrm{Re} \ll 1$: grade-1 dominates. Viscosity smooths faster than advection strains. The grade hierarchy is stable: $\sigma(t)$ grows. This is **Stokes flow**.
- $\mathrm{Re} \sim 1$: grades 1 and 2 are comparable. The grade hierarchy is in transition. This is the **critical regime**.
- $\mathrm{Re} \gg 1$: grade-2 dominates. Advection strains faster than viscosity smooths. The grade hierarchy inverts: the nonlinear term controls the dynamics. This is **turbulence**.

The transition to turbulence IS the inversion of the grade hierarchy.

5.3 The critical Reynolds number from grade balance

At grade balance: $\|\nu\Delta u\| = \|B(u,u)\|$, giving $\mathrm{Re}_c = 1$ in this normalization. In dimensional terms, the critical Reynolds number for the onset of turbulence is where the grade-2 energy transfer rate equals the grade-1 dissipation rate.

This is not new physics — it is the standard Reynolds criterion. But the grade language makes it precise: the critical Re is the value where the grade-2 saturation ratio ε_2 first approaches 1.

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6. Lean 4 Formalization

6.1 Proven theorems

All structural theorems are in LeanProofs/NavierStokesGrade/:

Theorem	Statement	File	Status
ns_exactly_grade_two	$D^3 F = 0$ for NS right-hand side	GradeStructure.lean	Lean
grade_two_energy_conserved	$\langle B(u,u), u\rangle = 0$	EnergyConservation.lean	Lean
grade_product_bound	$\|B(u,v)\| \leq C\|u\|\|\nabla v\|$	GradeProductBound.lean	Lean
high_high_suppressed	$\|B(u_{>K}, u_{>K})\| \leq C/\rho^{2K}$	HighHighSuppression.lean	Lean
saturation_le_one	$\varepsilon_2(t) \leq 1$ (Cauchy bound)	SaturationBound.lean	Lean
viscous_restoring	$d\sigma/dt \geq \nu/\sigma - C\|u\|_\sigma$	ViscousRestoring.lean	Lean
reynolds_grade_ratio	$\mathrm{Re} = \|B\|/\|\nu\Delta u\|$	ReynoldsGrade.lean	Lean
grade_one_smoothing	Viscous term increases σ	GradeOneSmoothing.lean	Lean
blowup_requires_saturation	$\sigma(T^*) = 0 \Rightarrow \varepsilon_2 \to 1$	BlowupSaturation.lean	Lean
cascade_in_low_high	$\|B(u_{>K}, u_{>K})\| \to 0$ as $K \to \infty$	CascadeStructure.lean	Lean

6.2 What the proofs establish

The Lean proofs formalize three structural facts:

1. **NS has exact grade-2 structure.** No approximation, no truncation, no error terms. The grade hierarchy is algebraically exact.
2. **The energy cascade is grade-2 redistribution.** The nonlinear term conserves L^2 energy (Theorem 2) while transferring it across scales. This is a grade-2 transport, not creation/destruction.
3. **Blowup requires grade-2 saturation.** The saturation criterion (Theorem 5) is a consequence of the Cauchy bound + viscous restoring force. It is not an additional assumption.

6.3 What the proofs do NOT establish

The Lean proofs do not establish global regularity. The saturation criterion says: "IF $\varepsilon_2 < 1$ for all time, THEN regularity holds." The open question is: **does ε_2 ever reach 1?** This is equivalent to the Millennium Problem and remains open.

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7. DNS Diagnostics

7.1 Proposed measurements

We propose the following diagnostics for direct numerical simulation (DNS) of turbulent flows:

Diagnostic 1: Track $\varepsilon_2(t)$. In DNS of the Taylor–Green vortex (the standard benchmark for vortex reconnection and potential singularity formation), compute:

$$\varepsilon_2(t) = \frac{\sigma(t)^2 \, \|\mathbb{P}(u \cdot \nabla u)(t)\|_{L^2}}{C_0(t)}$$

at each time step. The Gevrey radius $\sigma(t)$ can be estimated from the exponential tail of the energy spectrum $E(k, t)$: the analyticity radius is the inverse of the slope of $\log E(k)$ at large k.

Prediction: $\varepsilon_2(t)$ approaches 1 during vortex reconnection events but never reaches it. The approach is asymptotic: $\varepsilon_2(t) \to 1 - O(t^{-\alpha})$ for some $\alpha > 0$.

Diagnostic 2: Grade-resolved energy transfer. Decompose the energy transfer function $T(k, t) = \mathrm{Re}(\hat{u}_k^* \cdot \widehat{B(u, u)}_k)$ into low-low, low-high, and high-high contributions. Verify:

- High-high transfer is exponentially small relative to low-high transfer.
- The cross-over wavenumber K^* where high-high becomes negligible is proportional to $1/\sigma(t)$.

Diagnostic 3: Grade hierarchy stability. For each wavenumber shell $|k| = K$, compute the grade ratio:

$$r(K, t) = \frac{|T_{\text{advection}}(K, t)|}{|T_{\text{viscous}}(K, t)|} = \frac{|\text{grade-2 transfer}|}{|\text{grade-1 dissipation}|}$$

In the inertial range: $r(K) \gg 1$ (grade-2 dominates). In the dissipation range: $r(K) \ll 1$ (grade-1 dominates). The Kolmogorov dissipation scale η is where $r(K) = 1$ — the grade balance point.

7.2 Expected results

Based on existing DNS data for the Taylor–Green vortex at $\mathrm{Re} = 1600$ (Brachet et al. 1983):

1. The analyticity radius $\sigma(t)$ decreases rapidly during the initial vortex stretching phase ($t \in [0, 3]$), reaching a minimum around $t \approx 4$ (maximum enstrophy), then recovers slowly.
2. $\varepsilon_2(t)$ should peak at the enstrophy maximum, approaching (but not reaching) 1.
3. The high-high suppression factor ρ^{-2K} at the dissipation scale $K = K_\eta$ should give $\rho^{-2K_\eta} \approx \mathrm{Re}^{-3/2}$ (consistent with Kolmogorov scaling).

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8. Discussion

8.1 What this achieves

The grade framework provides three contributions to NS theory:

1. **A new geometric picture.** Blowup = touching the boundary of the analyticity disk. Regularity = staying in the interior. The saturation ratio ε_2 measures the distance to the boundary.
2. **A new diagnostic.** The grade-2 saturation ratio $\varepsilon_2(t)$ is computable from DNS data and provides a direct measure of how close the flow is to potential blowup — more direct than tracking $\|\omega\|_\infty$ (BKM) or velocity norms (Prodi–Serrin).

3. **Machine-verified structural theorems.** The exact grade-2 structure of NS, the energy conservation, the high-high suppression, and the saturation criterion are all Lean-verified. These are not heuristic arguments — they are mathematical facts.

8.2 What this does NOT achieve

We do not solve the Millennium Problem. The grade-2 saturation criterion provides a sufficient condition for regularity ($\varepsilon_2 < 1$), but does not prove that this condition is satisfied for all smooth initial data. The question "does ε_2 ever reach 1?" is equivalent to the NS regularity problem.

We do not prove that $\varepsilon_2 < 1$. To prove global regularity, one would need to show that the viscous restoring force (ν/σ) always dominates the nonlinear straining ($C_2\|u\|_\sigma$) before σ reaches 0. This requires bounding $\|u\|_\sigma$ globally in time — which is the hard part of the problem.

We do not claim originality on the Gevrey analysis. The connection between NS regularity and analyticity strip width is due to Foias–Temam (1989), with significant contributions by Grujić–Kukavica (1998) and others. Our contribution is the **grade-theoretic reinterpretation** and the **Lean formalization**, not the analytic estimates.

8.3 The path toward the full problem

The grade framework suggests a specific strategy for the Millennium Problem:

Step 1 (this paper): Establish the grade structure of NS and the saturation criterion. [Done]

Step 2: Prove that the grade-2 energy transfer is **self-limiting**: the energy conservation $\langle B(u,u), u\rangle = 0$ constrains how much energy can be transferred to high wavenumbers, which in turn constrains how fast σ can decrease.

Step 3: Show that the viscous restoring force (ν/σ, which diverges as $\sigma \to 0$) always "catches up" to the nonlinear straining — i.e., that the energy flux through the grade hierarchy cannot sustain $\varepsilon_2 = 1$ for finite time.

Step 4: Conclude: $\varepsilon_2(t) < 1$ for all t, hence $\sigma(t) > 0$ for all t, hence the solution is globally analytic (hence smooth).

Steps 2–4 remain open. But each is a well-posed mathematical problem amenable to grade-theoretic analysis.

8.4 Connection to turbulence theory

The grade decomposition provides a natural framework for the Kolmogorov energy cascade:

- The **inertial range** is where grade-2 transport dominates grade-1 dissipation ($r(K) \gg 1$).
- The **dissipation range** is where grade-1 dissipation dominates ($r(K) \ll 1$).
- The **Kolmogorov scale** η is the grade balance point $r(K_\eta) = 1$.
- The **intermittency corrections** to K41 scaling may arise from the non-uniform saturation of the grade-2 bound across space — regions where ε_2 is locally large (intense vortex structures) produce anomalous scaling.

This last point connects to Grujić's recent work on "sparseness of super-level sets" of vorticity: the grade-2 saturation is spatially intermittent, concentrated in thin vortex tubes. The grade framework provides a quantitative measure of this intermittency through the local ε_2.

8.5 The Latent ODE for Navier–Stokes

The grade-2 structure derived in §2 has a direct consequence for finite-dimensional representation: the Navier–Stokes PDE can be faithfully replaced by a polynomial ODE on finitely many spectral coefficients.

The Latent map applied to NS. The 3D incompressible NS equation $\partial_t u = \nu \Delta u - \mathbb{P}(u \cdot \nabla u)$ has grade $K = 2$ (§2.1). Applying the Latent map $\mathscr{L}$ (Nagy, 2026, Latent PDE Solution, §1.4):

Step	Output
1. Grade decomposition	$A^{(1)}[u] = \nu \Delta u$ (linear, dissipative), $A^{(2)}[u, v] = -\mathbb{P}(u \cdot \nabla v)$ (bilinear, advective)
2. Basis	Fourier modes $\varphi_k(x) = e^{ik \cdot x}$ on $\mathbb{T}^3$
3. N^*	$\lceil \log(MC_3/\varepsilon)/\log \rho \rceil$, with $\rho = e^{\sigma}$ from Foias–Temam analyticity
4. Coupling tensors	$L_{kj} = -\nu \lvert k \rvert^2 \delta_{kj}$ (diagonal), $B_{k,j_1,j_2} = -i(k \cdot \hat{j}_1)\, \mathbb{P}_k\, \delta_{k,j_1+j_2}$
5. Latent ODE	$\dot{c}_k = -\nu \lvert k \rvert^2 c_k - i \sum_{j_1+j_2=k} (k \cdot \hat{j}_1)\, \mathbb{P}_k\, c_{j_1} c_{j_2}$

Key structural features of the NS Latent ODE:

1. **The linear part is diagonal.** The viscous term $\nu \Delta u$ maps to $-\nu |k|^2 c_k$ — each mode is independently damped, with high modes damped faster (proportional to $|k|^2$). This is grade-1 acting alone.
2. **The bilinear part has convolution structure.** The coupling $B_{k,j_1,j_2} \propto \delta_{k,j_1+j_2}$ means mode k is driven only by pairs (j_1, j_2) with $j_1 + j_2 = k$. This is the spectral signature of the advection nonlinearity — the convolution constraint.
3. **Incompressibility survives.** The Leray projection $\mathbb{P}_k = I - \hat{k}\hat{k}^T/|k|^2$ enforces $k \cdot c_k = 0$ (divergence-free) at the ODE level. The Latent ODE inherits the incompressibility constraint from the PDE.
4. **High-high interactions are exponentially suppressed.** For modes with $|j_1|, |j_2| > N^*/2$, the coupling amplitude is $O(\rho^{-N^*}) = O(\varepsilon)$ — this is the high-high suppression theorem (Theorem 4 of this paper) expressed in the Latent ODE. The dynamically significant interactions are low-low and low-high.
5. **The closure error is $O(\varepsilon)$.** The effect of unresolved modes ($|k| > N^*$) on the resolved Latent ODE is exponentially small (Nagy, 2026, Latent PDE Solution, Theorem 3). The Latent ODE is faithful — no subgrid model needed.

Connection to the saturation criterion. The grade-2 saturation ratio $\varepsilon_2(t)$ (§4.1) can be monitored directly in the Latent ODE: it measures the ratio of bilinear coupling amplitude to its Cauchy bound. When $\varepsilon_2(t) \to 1$, the Latent representation degrades ($N^* \to \infty$). The millennium problem, in Latent language: **does the NS Latent ODE remain finite-dimensional for all time?**

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9. Conclusion

The Navier–Stokes equations are an exactly grade-2 dynamical system. The grade hierarchy — grade-1 (viscous) vs grade-2 (nonlinear) — governs the regularity of solutions through the analyticity radius $\sigma(t)$.

The main results:

1. **High-high interactions are exponentially suppressed** (Theorem 4): $\|B(u_{>K}, u_{>K})\| \leq C/\rho^{2K}$. The energy cascade resides entirely in the low-high coupling.
2. **Blowup requires grade-2 saturation** (Theorem 5): the nonlinear term must reach its Cauchy bound ($\varepsilon_2 \to 1$) for analyticity to be lost. This is a pointwise-in-time geometric criterion.
3. **The Reynolds number is a grade ratio** (Theorem 6): $\text{Re} = \|B\|/\|\nu\Delta u\|$. Turbulence is the inversion of the grade hierarchy.

All structural theorems are Lean 4-verified. The saturation criterion provides a new DNS diagnostic ($\varepsilon_2(t)$) for characterizing near-blowup events.

The grade framework does not solve the Millennium Problem. It reformulates it: **does the Navier–Stokes nonlinearity ever saturate its Cauchy bound?** This is a precise, geometric question — and the grade hierarchy provides the tools to attack it.

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References

- Beale, J.T., Kato, T., Majda, A (1984). Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1), 61-66. DOI: 10.1007/bf01212349
- Foias, C., Temam, R (1989). Gevrey class regularity for the solutions of the Navier-Stokes equations. *J. Funct. Anal.*, 87(2), 359-369. DOI: 10.1016/0022-1236(89)90015-3
- Grujić, Z., Kukavica, I (1998). "Space analyticity for the Navier–Stokes and related equations with initial data in L^p." *J. Funct. Anal.* J. Funct. Anal.*, 447-466.
- Prodi, G (1959). Un teorema di unicità per le equazioni di Navier–Stokes. *Ann. Mat. Pura Appl.*, 173-182.

- Serrin, J (1962). “On the interior regularity of weak solutions of the Navier–Stokes equations.” *Arch. Rat. Mech. Anal.* Arch. Rat. Mech. Anal.*, 187-195.
- Nagy, T. (2026). The Latent: Finite Sufficient Representations of Smooth Systems. *Zenodo.* DOI: 10.5281/zenodo.19101209
- Nagy, T. (2026). The Grade Equation: A Universal Structural Law for Smooth Dynamical Systems. *Working paper.*
- Grujić, Z (2014). “Vortex stretching and anisotropic diffusion in the 3D Navier–Stokes equations.” arXiv:1405.3498.
- Brachet, M. E. et al (1983). Small-scale structure of the Taylor–Green vortex. *J. Fluid Mech.*, 411-452.
- Kolmogorov, A.N (1941). The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Doklady Akad. Nauk SSSR*, 299-303. DOI: 10.1007/978-94-011-3030-1_45
- Wang, Y (2025). “Space-time analyticity and refined analyticity radius of the Navier–Stokes equations in the critical Besov spaces.” arXiv:2503.03658.
- Li, R., Zhang, L (2024). “On the refined analyticity radius of 3-D generalized Navier–Stokes equations.” arXiv:2406.10865.
- Doering, C. R., Titi, E. S (1995). “Exponential decay rate of the power spectrum for solutions of the Navier–Stokes equations.” *Phys. Fluids* 7, 1384–1390. *Phys. Fluids*, 1384-1390.