Pseudospectral Robustness of the Analyticity Parameter

Why Damping Rate Ratios Are More Stable Than Individual QNM Frequencies

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Scaffold

Abstract

Jaramillo et al. (2021) demonstrated that individual quasinormal mode frequencies of black holes are spectrally unstable — small perturbations to the effective potential can cause O(1) shifts in higher overtone frequencies. This raises fundamental concerns about the physical relevance of overtone measurements. We show that the damping rate ratio = / is pseudospectrally robust: under the same perturbations that destabilize individual _n, remains within [TBD]% of its unperturbed value. The mechanism is cancellation: systematic shifts in and are correlated (both respond similarly to potential modifications), so their ratio is protected. This identifies as the natural observable for black hole spectroscopy — more meaningful than individual frequencies, and directly measurable with current and future gravitational wave detectors.

1. Introduction

1.1 The QNM Instability Problem

- QNMs are eigenvalues of a non-self-adjoint operator
- Pseudospectrum of the Regge-Wheeler/Zerilli operator extends far from eigenvalues
- Higher overtones are exponentially unstable: _n ~ · exp(n)
- This threatens the physical significance of overtone extraction

1.2 The Hypothesis

- Paper I showed 3 is universal across Kerr spins, mass ratios, multipoles
- is a RATIO of damping rates from the SAME potential
- Perturbations that shift the potential affect both and
- Key conjecture: / _n/ _n

1.3 Significance

- If true: is the right quantity for spectroscopy (robust + universal)
- Resolves the tension between overtone instability and overtone extraction

2. Setup

2.1 The Regge-Wheeler Equation

- V(r*) = effective potential for gravitational perturbations
- QNM boundary conditions: outgoing at infinity, ingoing at horizon
- Standard eigenvalue problem for _n

2.2 Perturbation Model

- Perturbed potential: V_ (r*) = *V(r*) + · V(r*)
- Classes of perturbations:
  (a) Localized bumps: V = A · exp(-(r*-r)²/ ²)
  (b) Uniform shifts: V = const
  (c) Random smooth perturbations (draw from GP)
  (d) Astrophysically motivated: accretion disk, dark matter spike

2.3 Quantities Tracked

- Individual: | / |, | / |, | / |, | / |
- Ratio: | / | where = /
- Pseudospectral radius _ ( ) at each eigenvalue

3. Method

3.1 Spectral Computation

- Discretize Regge-Wheeler on Chebyshev grid (N = 200-400 points)
- Eigenvalue solver (SLEPc or direct QR for moderate N)
- Compute unperturbed spectrum + pseudospectrum

3.2 Perturbation Sweep

- For each perturbation class, sample 100 random realizations at each
- range: 10 to 10 ¹
- Track _n and for n = 0, 1, 2, 3

3.3 Statistical Analysis

- Compute: mean | / | vs mean | _n/ _n| at each
- Stability gain: G = | / | / | / |

4. Results

4.1 Individual Frequency Instability

[Reproduce Jaramillo et al.: _n grows exponentially with n] [Show pseudospectrum contours]

4.2 Robustness

[Key result: / is O( ) while / is O(· exp( ))] [Stability gain G 1 for higher overtones]

4.3 Physical Mechanism

- and respond to the SAME potential, with correlated shifts
- = / cancels the "common mode" of the perturbation
- Formal: expand _n = _n · Σ c_k(n) · ^k → ratio cancels leading terms

4.4 Implications for Spectroscopy

- Measuring instead of individual _n: robust to environmental effects
- = 3 test: even with O(1) shifts in individual frequencies, 3 persists
- This makes the Latent framework's predictions pseudospectrally protected

5. Discussion

5.1 Why Ratios Are Special

- General principle: ratios of eigenvalues of the same operator are more stable
- Analogy: the fine-structure constant in QED is more stable than individual energy levels
- = 3 as the "fine-structure constant of ringdown"

5.2 Beyond Schwarzschild

- Extend to Kerr: angular eigenvalue problem adds complexity
- But universality across spins (Paper I) suggests robustness persists

5.3 Experimental Proposal

- Measure from GW data directly (ratio of matched-filter SNRs for n=0 and n=1 templates)
- -based BH spectroscopy test: does = 3 hold event-by-event?

6. Conclusion

[is pseudospectrally robust. It is the natural observable for BH spectroscopy. Individual frequencies are unstable; their ratio is protected. The Latent framework identifies the right quantity.]

References

[Jaramillo et al. 2021 PRX, Nollert 1999, Berti et al. 2009, Paper I, Boyanov et al. 2024 pseudospectrum]