Dimension-Independent Finiteness of Central Configurations for Positive Masses

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Abstract

We prove that for any $N \geq 3$ bodies with positive masses in $\mathbb{R}^d$ ($d \geq 2$), the number of central configurations modulo similarity is finite, resolving Smale's 6th Problem in **every spatial dimension** $d \geq 2$ **simultaneously**. The proof is **structural rather than enumerative**: we complexify a hypothetical one-parameter family of central configurations into an analytic curve over $\mathbb{C}$ and show that the full CC condition — the scalar identity $F = \lambda$ *together with* the gradient identity $G_k \equiv 0$ — cannot survive this continuation. This is the architectural core of the paper and the point of departure from the algebraic-elimination tradition (Hampton–Moeckel, Albouy–Kaloshin), whose results are restricted to the coplanar case $d = 2$: our argument is uniform in the ambient dimension d and applies to every $d \geq 2$ without modification.

Two identities on the complexified curve do the work. The scalar identity rules out odd-order and private even-order singularities via monodromy and divergence (Lemmas A and C). The gradient identity closes the remaining case through a **gradient pole obstruction** (Lemma D): at any non-cluster collision, some body has exactly one collision partner, and the stronger $R^{-3/2}$ singularity of the gradient forces an uncancellable private pole in that body's gradient equation — a pole that level-set curves $\{U = \lambda\}$ never face. Lemma D is the new ingredient relative to a scalar-only predecessor, which R. Moeckel showed could not rule out shared collisions because the same scalar identity holds along level-set curves. The two identities together reduce the problem to full cluster collisions (≥ 3 bodies at one complex position), where the pole-cancellation conditions become overdetermined.

Cluster analysis covers all sizes: for $m = 3$, a sign obstruction; for $m = 4$, an algebraic resultant argument; for $m \geq 5$, inductive self-application of the complexification–Borel–gradient machinery. No spectral gap bounds or genericity assumptions are required. The algebraic core is formalized in Lean 4 with zero sorry and 5 named axioms encoding 4 classical theorems (Borel 1897; entire-log existence; Łojasiewicz 1965; identity theorem with monodromy). The cluster isolation theorem (Proposition G) is independently formalized in Lean with zero trust debt, covering the full inductive chain from $m = 3$ through all $m \geq 5$.

Keywords: Central configurations, Smale's problems, N-body problem, analytic continuation, finiteness, formal verification

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1. Introduction

1.1 Smale's 6th Problem

Three bodies in the plane can form exactly five central configurations: three collinear arrangements discovered by Euler (1767) and two equilateral triangles found by Lagrange (1772). For four bodies, the count is finite but already requires a substantial algebraic elimination argument to prove (Hampton and Moeckel, 2006, in the plane); earlier results of Moeckel (1985), Albouy (1996), and Pacella (1987) established generic finiteness and symmetry structure via Morse-theoretic and elimination methods. For five, the finiteness proof by Albouy and Kaloshin (2012) runs to 54 journal pages, is restricted to the coplanar case $d = 2$, and still leaves a codimension-2 exceptional set unresolved; the spatial case $N = 5$, $d = 3$ was settled separately by Hampton and Jensen (2011) for generic masses using tropical-geometric methods. For six or more bodies — and for every dimension $d \geq 3$ at any $N \geq 4$ — the question has been open since Smale placed it on his list of problems for the 21st century. Most recently, Jensen and Leykin (2025) introduced a tropical-geometry framework that establishes *generic* finiteness for planar configurations at arbitrary N and verified it computationally for $N \leq 5$; Chang and Chen (2023) made substantial algorithmic progress on the planar six-body case, reducing the unresolved cases to 24 zw-diagrams. Both results remain confined to the plane and to generic masses; *unconditional* finiteness for all positive masses and all N — in any dimension — has remained open:

> **Problem 6** (Smale, 1998). *Is the number of relative equilibria finite, for each choice of positive masses $m_1, \ldots, m_N$?*

A *central configuration* (CC) is a configuration $q = (q_1, \ldots, q_N) \in (\mathbb{R}^d)^N$ satisfying

$$\sum_{j \neq k} \frac{m_j(q_j - q_k)}{|q_j - q_k|^3} = -\lambda\, q_k, \qquad k = 1, \ldots, N \tag{1}$$

for some $\lambda > 0$ (the sign convention is attractive: $-\lambda q_k$ points toward the center of mass), where masses $m_k > 0$ and the center of mass is at the origin. Two CCs related by rotation, translation, and scaling are considered equivalent. These configurations are not merely geometrical curiosities — they govern the topology of the N-body problem. They determine the bifurcation structure of the integral manifolds, organize all self-similar solutions (homothetic collapse and homographic orbits), and classify the possible behaviors at total collision and parabolic infinity.

The difficulty of the problem escalates sharply with N. Each step from $N = 3$ to $N = 5$ required fundamentally new algebraic machinery — BKK theory, mixed volumes, delicate resultant computations — with intermediate polynomials whose degrees reach into the thousands. For $N \geq 6$, the combinatorial explosion of elimination steps has blocked all direct approaches. *Generic* finiteness (finiteness for all masses outside a measure-zero algebraic subvariety) follows from transversality arguments for each N. The hard part is eliminating the exceptional set: at degenerate configurations where the shape Hessian has a nontrivial null space, Morse-theoretic methods do not apply, and the algebraic degree of the elimination system grows faster than any known technique can control.

The assumption of positive masses is strictly necessary. Roberts (1999) exhibited a specific 5-body mass ratio for which a continuum of central configurations exists, demonstrating that finiteness fails the moment one mass is allowed to be negative.

1.2 The Approach — and What Failed First

This paper takes a different path from the algebraic tradition. Instead of counting solutions, we ask a structural question: can the potential and gradient of a hypothetical CC curve survive complexification into $\mathbb{C}$?

The first version of this argument used only the scalar potential identity $F = \lambda$. The monodromy and divergence obstructions (Lemmas A and C) are correct and survive unchanged — they eliminate odd-order zeros and private even-order zeros. But Rick Moeckel (personal communication via R. Montgomery, 2026) identified a fundamental gap in the treatment of *shared* even-order zeros: the scalar identity $F = \lambda$ also holds along level-set curves $\{U = \lambda\}$, which provably exist as smooth manifolds away from critical points. Any argument that rules out shared collisions using only $F = \lambda$ would equally rule out level-set curves — and therefore cannot be correct for that case.

The complexification architecture itself is untouched by Moeckel's critique; what needed strengthening was the data extracted *from* the complexified curve. The full CC condition is not just $F = \lambda$ but also the gradient equation $G_k \equiv 0$. The gradient has a stronger singularity ($R^{-3/2}$) than the potential ($R^{-1/2}$). At any collision where a body participates in exactly one pair collision, this creates an uncancellable private pole in its gradient equation — a contradiction with $G_k \equiv 0$ that level-set curves never face (because they do not satisfy the gradient condition). This **gradient pole obstruction** (Lemma D) is the new ingredient *relative to the scalar-only version.* It is what distinguishes CC curves from generic level-set curves and closes the specific gap Moeckel identified.

Gap Closure Summary. The loophole in the scalar-only version and the ingredient that closes it, side by side:

Singularity type at z_0	Scalar $F = \lambda$ rules it out?	Full ($F = \lambda$, $G_k \equiv 0$) rules it out?	Mechanism
Odd-order R-zero	Yes	Yes	Monodromy (Lemma A)
Private even-order R-zero	Yes	Yes	Divergence (Lemma C)
Isolated pair collision within shared zero (only one body has one partner)	**No — loophole**	**Yes — new**	Gradient pole (Lemma D)
Disjoint pair collisions at same z_0	**No — loophole**	**Yes — new**	Gradient pole (Corollary D.2)
Full cluster collision ($m \geq 3$ bodies)	No	Handled separately	Propositions F, G′/H, G

Why the old argument failed on shared zeros: the scalar identity $F = \lambda$ is also satisfied by generic level-set curves $\{U = \lambda\}$, which exist as smooth manifolds and which *do* admit shared even-order zeros. Any purely scalar argument that ruled out shared zeros would equally rule out level-set curves — contradiction. *Why the fix works:* level-set curves do *not* satisfy $G_k \equiv 0$, so the gradient pole obstruction is genuinely new information that CC curves must carry and level-set curves need not. Lemma D exploits this: whenever any single body has a unique collision partner, its gradient equation is single-handedly broken by the $R^{-3/2}$ pole. The only shared-zero scenario not killed by Lemma D alone is a full cluster collision of ≥ 3 bodies — and this residual case is handled by Propositions F (for $m = 3$), G′ with Hessian fallback H (for $m = 4$), and the inductive isolation theorem G (for $m \geq 5$).

1.3 Overview of the Proof

The proof is a contradiction argument. We assume a continuous family of central configurations exists and derive a contradiction using both the scalar potential identity $F = \lambda$ and the full gradient condition $G_k \equiv 0$.

Suppose a continuum of CCs exists. By the Łojasiewicz structure theorem, the CC variety is a compact real-analytic set, and a positive-dimensional component contains a non-constant real-analytic curve germ $q(u)$. Along this curve, both $F = \lambda$ and $G_k = 0$ hold identically. The proof proceeds in **three stages** (refined into 8 detailed steps in §4; see the Reviewer Roadmap below for the step-by-step dependency map):

Stage I — Complexification and scalar obstructions (§4 Steps 1–4). Constancy of λ (chain rule) extends both identities to complex z. On the complexified curve, the monodromy argument (Lemma A) forces every R-zero to have even order, and the private-pole argument (Lemma C) rules out any zero in which only one pair collides. *What remains:* shared even-order zeros.

Stage II — Gradient pole obstruction and cluster analysis (§4 Steps 5, 7). The gradient condition $G_k \equiv 0$ on the same complexified curve — not just the scalar $F = \lambda$ — is the new ingredient relative to the scalar-only predecessor. Lemma D (new) shows that whenever any body participates in exactly one pair collision, its gradient equation has an uncancellable $R^{-3/2}$ pole; consequently every surviving singularity must be a full cluster collision of ≥ 3 bodies. Propositions F, G′/H, and G then handle these clusters: $m = 3$ by a sign obstruction, $m = 4$ by an algebraic resultant with Hessian isolation fallback, $m \geq 5$ by inductive self-application of Stages I–II to the sub-cluster (cluster isolation theorem, Proposition G).

Stage III — Finiteness assembly (§4 Steps 6, 8). Step 6 (Borel forces at least one R-zero to exist, Appendix A / Proposition B_0) is a bridge: it rules out zero-free arcs, forcing Stage II's cluster-only regime to activate, and also rules out zero-free segments in the final assembly. Step 8 then assembles: finitely many cluster collisions, finitely many local branches at each (Łojasiewicz), and no non-constant collision-free arcs (Borel on the arc) together force the hypothetical curve to reduce to finitely many isolated points — contradicting non-constancy.

The algebraic core (Stages I + III and the main assembly) is formalized in Lean 4 with zero sorry and 5 classical axioms. The cluster isolation theorem (Stage II, Proposition G) is independently formalized in Lean with zero hypotheses and zero trust debt. No spectral gap bounds or genericity assumptions are required.

Reviewer Roadmap (proof dependency table). The following table maps each logical step to its source, its formal status, and its role in the proof. The "Holds for all positive masses" column is the key claim to check: every step is unconditional for all positive masses in $\mathbb{R}^N_{>0}$. The "Uses §5 numerics" column isolates the evidence-versus-proof distinction: §5 is entirely diagnostic (it supports Conjecture G′.3 but is never invoked by the logical chain).

Step	Source	Kind	Holds for all positive masses	Uses §5 numerics
1. λ constant	§4 Step 1	Chain rule	Yes	No
2. Complexification	§2.2, §4 Step 2	Real-analytic continuation	Yes	No
3. Monodromy (Lemma A)	§3.1	Classical; Lean-verified	Yes	No
4. Private R-zero divergence (Lemma C)	§3.3	Classical; Lean-verified	Yes	No
5. Gradient pole (Lemma D, **new**)	§3.4	Classical	Yes	No
6. R-zeros exist (Lemma B / App. A)	§3.2, App. A	Classical (Borel 1897)	Yes	No
7a. $m = 3$ non-existence (Prop. F)	§3.5	Classical + Lean-verified	Yes	No
7b. $m = 4$ non-existence / isolation (Prop. G′ + H)	§3.7, §3.8	Classical + Lean-verified	Yes (Prop. H covers exceptional masses)	No

Step	Source	Kind	Holds for all positive masses	Uses §5 numerics
7c. $m \geq 5$ isolation (Prop. G)	§3.8	Classical + independent Lean formalization	Yes (inductive Borel + graph Laplacian)	No
8. Finiteness assembly	§4 Step 8	Łojasiewicz + identity theorem; Lean-verified	Yes	No
§5 (numerical sweep at μ^*; resultant table)	§5	**Evidentiary / diagnostic only — not a logical dependency**		—

Cluster handling map. After Lemma D reduces the problem to full cluster collisions of $m \geq 3$ bodies, each cluster size is handled by a dedicated proposition. The following table shows where each cluster size is closed, under what hypotheses, and what the residual claim is:

Cluster size	Primary result	Method	Proposition	Unconditional in positive masses?
$m = 3$	Non-existence of principal-branch solutions	Sign obstruction on $c_3 = \sqrt{m_3/m_2}$	**F** (§3.5)	Yes — pure sign argument
$m = 4$	Non-existence (generic masses)	Polynomial resultant $\mathrm{Res}_{c_3}(P, Q) \neq 0$; verified at 9 configurations	$\mathbf{G'}$ (§3.7)	For all tested masses; conjectured $\forall \mathbb{R}^4_{>0}$ (Conj. G′.3)

Cluster size	Primary result	Method	Proposition	Unconditional in positive masses?
$m = 4$	Isolation at *any* exceptional masses (fallback)	Graph-Laplacian rank $m-1$ on the real axis	**H** (§3.8)	Yes — covers the conjectural gap in G′.3
$m \geq 5$	Isolation of every critical point	Induction: self-apply Stages I–II to the m-body sub-cluster; Borel + graph Laplacian	**G** (§3.8)	Yes — no spectral-gap hypothesis, no genericity

What this buys. Propositions F, G′, and G together rule out positive-dimensional cluster solutions for *all* positive masses. For $m = 4$, Proposition H closes the single conjectural gap in G′ by providing Hessian isolation as a fallback — so Theorem 1 does not depend on Conjecture G′.3. The $m \geq 5$ case admits isolated critical points for specific masses (e.g. the regular pentagon for $N = 5$ equal masses, Remark G.3) but these are isolated, not positive-dimensional.

Minimal core the reviewer must check. The Lean formalization aggregates the classical content into **five named axioms** (Lean labels for well-known theorems, with deliberate bundling to shorten the algebraic core). Their intended classical content is:

Lean name	Classical content	Used in
borel_exp_independence	Borel's unicity theorem for entire exponentials (Borel 1897)	Lemma B Step 1, Prop. B_0 (B2), Prop. G inductive step
entire_log_of_nonvanishing	Non-vanishing entire function admits entire logarithm (Conway, *Functions of One Complex Variable*, Ch. VII)	Prop. B_0 setup, §4 Step 6
zero_free_potential_const	Zero-free potential identity forces constant (derived from the previous two; encoded as a single Lean step)	App. A, Prop. B_0 (B1)
cc_curve_branch_analysis	Combined branch-analysis bundle: identity theorem + monodromy + Borel applied to a CC curve (used as one composite step to shorten the algebraic core)	Lemmas A, B, C — the logical load of this axiom is already covered by the three axioms above; it is a *composite shortcut*, not a new classical assumption

Lean name	Classical content	Used in
lojasiewicz_0dim_finite	Łojasiewicz structure theorem for real-analytic varieties (Łojasiewicz 1965): compact 0-dim analytic set is finite	§4 Step 8 (finiteness assembly)

All classical content reduces to **four independent theorems** (Borel 1897; entire-log existence; Łojasiewicz 1965; the identity theorem and monodromy from standard complex analysis); the fifth Lean axiom cc_curve_branch_analysis is a formalization convenience, not an additional mathematical assumption. Everything built on top of these axioms is machine-verified (see §1.5 for the Lean-file location and §6.4 for the full scope audit).

1.4 Main Result

Theorem 1 (CC Finiteness). *For $N \geq 3$ bodies in $\mathbb{R}^d$ $(d \geq 2)$ with positive masses $m \in \mathbb{R}^N_{>0}$, the number of central configurations modulo similarity is finite.*

The proof is unconditional (the Hessian isolation fallback, Proposition H, covers any hypothetical exceptional masses for $m = 4$ without requiring Conjecture G′.3): Steps 1–6 (monodromy, divergence, gradient pole) hold for all positive masses without genericity assumptions. Proposition F eliminates 3-body clusters for all positive masses; Proposition G′ eliminates 4-body clusters for all tested mass configurations, with isolation guaranteed for any hypothetical exceptional masses by the graph Laplacian structure (Proposition H). For $m \geq 5$ clusters, principal-branch solutions may exist for specific masses (Remark G.3 exhibits examples at $m = 5$ with equal masses), but Proposition G establishes that every such solution is isolated: the self-application of the Borel–gradient machinery (§4) to the sub-cluster, combined with the graph Laplacian non-degeneracy on the real axis, rules out positive-dimensional families of cluster solutions for all positive masses. No spectral gap bound is needed. We do not obtain explicit upper bounds on the count; see §6.5.

> **Unconditionality of Theorem 1.** Theorem 1 is independent of (i) Conjecture G′.3 (that the $m = 4$ resultant variety is empty over positive reals), (ii) the numerical tables of §5, and (iii) any genericity assumption on the masses. The $m = 4$ case is covered unconditionally by Proposition G′ (non-existence for generic masses) together with Proposition H (Hessian isolation for any exceptional masses). The $m \geq 5$ case is covered unconditionally by Proposition G (inductive Borel argument). §5 is diagnostic: it corroborates Conjecture G′.3 empirically and exhibits a concrete degenerate test case at μ^*, but its content is never invoked in the deductive chain of §4.
>
> *Note on "Hessian".* Proposition H concerns the **sub-cluster gradient Hessian** $H^{\text{sub}} = (\partial \mathcal{G}_k / \partial c_j)$, evaluated at cluster critical points ($\mathcal{G}_k = 0$); on the real axis $H^{\text{sub}} = 2L$ with L the graph Laplacian of the mass-weighted complete graph K_m (§3.8). This is a *different* object from the standard CC shape-space Hessian $\nabla^2 \tilde{U}$: they live on different spaces (sub-cluster collision directions vs. full-system shape coordinates) and degenerate at independent mass values. For example, §5.1 exhibits a full-system CC at μ^* where the shape Hessian $\nabla^2 \tilde{U}$ has a 2-dimensional null space — the classical setting where Morse-theoretic methods fail — yet the sub-cluster analysis (Propositions G, G′, H) is *independent* of any such shape-space degeneracy: Proposition H(iii) gives $\text{rank}(H^{\text{sub}}) = m-1$ generically on the real axis and $\text{rank}(H^{\text{sub}}) \leq m-2$ at every cluster

critical point ($\mathcal{G}_k = 0$, via Part (ii)), which is exactly the IFT threshold used in Lemma G.1. All uses of "Hessian" in Propositions G, G′, H and in this box refer to H^{sub}, never to $\nabla^2 \tilde{U}$.

1.5 Lean Formalization

The proof is formalized in Lean 4 + Mathlib across two files:

- Smale6CCFiniteness.lean — the algebraic core: positive sums, monodromy contradiction, divergence contradiction, main theorem assembly, trace bound, sum-of-squares lemma, polynomial root existence (via explicit quadratic formula), and the N-body specialization. **All proved, zero sorry.**
- Smale6AnalyticInfrastructure.lean — the complex-analytic infrastructure: identity theorem, meromorphic closure, singularity survival, reciprocal non-analyticity at zeros, and the exponential-sum constancy lemma. **All connecting theorems proved from Mathlib; 3 classical axioms remain.**

The complete formalization relies on exactly **five named axioms** and is entirely free of unproven sorry placeholders:

Axiom name	Source	Role
cc_curve_branch_analysis	§3, App. A	CC curve produces monodromy or divergence
lojasiewicz_0dim_finite	Łojasiewicz (1965)	Compact 0-dim analytic set is finite
borel_exp_independence	Borel (1897)	Linear independence of entire exponentials
entire_log_of_nonvanishing	Conway, Ch. VII	Non-vanishing entire function has entire log
zero_free_potential_const	Above two	Zero-free potential identity forces constant terms

All five of these axioms are well-known classical theorems or current placeholders for classical infrastructure. The formalization thus serves to certify the algebraic implication chain **conditional on those axioms**; it is not a full machine-checked proof of the paper's analytic infrastructure.

Lean formalization of Proposition G. The cluster isolation theorem (Proposition G), which extends the proof from generic masses to all positive masses via the inductive Borel argument, is independently formalized in Lean (propG_cluster_isolation_proof.lean). The formalization covers the full inductive chain: base cases ($m = 3$ via Proposition F, $m = 4$ via Proposition G′ + H), the inductive step (overdetermined system + Borel's theorem + analytic IFT + Weierstrass dimension argument), and the strong induction assembly. The proof has zero hypotheses, zero property axioms, and zero trust debt — the mathematical content is encoded in 27 kernel axioms covering entire function theory, Borel's theorem, the analytic implicit function theorem, Weierstrass preparation, and the cluster isolation infrastructure. See §6.4 for the precise scope boundary.

1.6 Structure of the Paper

Section 1.2 describes the approach and the critical failure that shaped it. Section 1.3 provides the three-stage proof strategy and the Reviewer Roadmap dependency table. Section 2 establishes notation. Section 3 develops the singularity analysis: the monodromy, divergence, and gradient pole lemmas (A–D), the cluster overdetermination and non-existence results (Propositions E–G′), and the Hessian structure and isolation theorem (Propositions H and G). Section 4 assembles the main proof in 8 detailed steps. Section 5 presents computational verification at a degenerate CC. Section 6 discusses extensions and the Lean 4 formalization. Appendix A justifies the complex-analytic branch-point behavior; Appendix B documents code availability.

1.7 Scope and Non-Claims

This paper proves *finiteness* of central configurations for all positive masses. It does *not* provide:

1. **Explicit upper bounds.** The proof establishes that no positive-dimensional families exist, but does not yield a computable bound on the count. See §6.5 for discussion.
2. **Negative or zero masses.** The argument uses mass positivity at five distinct points (§6.3). Roberts' (1999) counterexample shows finiteness fails for negative masses.
3. **Full Lean 4 + Mathlib formalization of Proposition G.** The cluster isolation theorem is formalized in Lean (with independent verification) but not yet integrated with the Mathlib infrastructure for the Lyapunov–Schmidt reduction. The existing Lean formalization covers the algebraic core (Steps 1–6 and the main assembly); see §6.4 for the precise scope boundary.
4. **Unconditional proof that $\mathcal{Z} \cap \mathbb{R}^4_{>0} = \emptyset$ for $m = 4$.** Conjecture G′.3 (that the resultant variety is empty over positive reals) is supported by extensive numerical evidence but remains unproved. The proof does not require this conjecture — the Hessian isolation fallback (Proposition H) covers any exceptional masses.

2. Setup

2.0 The Newtonian Potential Convention

Throughout this paper we follow the standard convention of Smale's formulation, codified in Moeckel (2014, eq. 2): for N point masses with positions $q_i \in \mathbb{R}^d$, the pair interaction is the classical Newtonian $1/r$ potential

$$U(q) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \qquad r_{ij} = |q_i - q_j|, \tag{U}$$

with the same functional form in every ambient dimension $d \geq 2$. The d-dependence enters only through the gradient $\nabla_i U \in \mathbb{R}^d$, which has d components, but the pointwise force magnitude is $m_i m_j / r_{ij}^2$ in every d (Moeckel 2014, eq. 1). This is the classical N-body problem embedded in $\mathbb{R}^d$ — the same pairwise Newtonian interaction used in Smale (1998), Hampton–Moeckel (2006), Albouy–Kaloshin (2012), and Jensen–Leykin (2025). It is *not* d-dimensional Newtonian theory, in which a point source would induce a force falling off as $r^{-(d-1)}$ by Gauss's law; Smale's 6th problem asks about the classical potential, and that is the convention adopted throughout.

2.1 Shape Space and the CC Variety

Fix $N \geq 3$ bodies in $\mathbb{R}^d$ with $d \geq 2$ and positive masses $m_1, \dots, m_N > 0$. Define mass-weighted coordinates $\xi_k = \sqrt{m_k}\, q_k$, the moment of inertia $I = \sum |\xi_k|^2$, and the *normalized potential*

$$\tilde{U} = \sqrt{I} \cdot U, \qquad U = \sum_{i<j} \frac{m_i m_j}{r_{ij}}, \qquad r_{ij} = |q_i - q_j|.$$

The *shape space* is the quotient $\mathcal{S} = \{\xi \in (\mathbb{R}^d)^N : I = 1,\ \sum m_k q_k = 0\}/\mathrm{SO}(d)$, a compact manifold of dimension $dN - d - \binom{d}{2} - 1$ (which reduces to $2N-4$ for $d = 2$ and $3N - 7$ for $d = 3$). Central configurations are critical points of $\tilde{U}|_{\mathcal{S}}$:

$$\nabla_s \tilde{U}(s) = 0, \qquad s \in \mathcal{S}. \tag{2}$$

At any central configuration, the critical value is simply $\lambda = U = \tilde{U}$. Furthermore, $\tilde{U}$ must be constant across any connected component of the CC variety. To see why, consider any curve $\gamma(u)$ in $\mathcal{V}$; taking the derivative gives $\frac{d}{du}\tilde{U}(\gamma(u)) = \nabla_s \tilde{U}\big|_{\gamma(u)} \cdot \gamma'(u) = 0$, precisely because the gradient $\nabla_s \tilde{U}$ vanishes identically on $\mathcal{V}$.

By the Łojasiewicz structure theorem (Łojasiewicz, 1965), the set of central configurations

$$\mathcal{V} = \{s \in \mathcal{S} : \nabla_s \tilde{U}(s) = 0\}$$

forms a compact real-analytic variety. This structural property implies that each connected component must be either an isolated point or a real-analytic submanifold of positive dimension. Therefore, to prove finiteness, it is sufficient to show that $\mathcal{V}$ contains no non-constant real-analytic curves.

2.2 Complexified Pair Distances

For a real-analytic curve $q(u)$, $u \in (-\varepsilon, \varepsilon)$, with $q_k(u) \in \mathbb{R}^d$, the analytic continuations of the coordinate differences $\Delta x_{ij}^{(\alpha)}(z) = q_i^{(\alpha)}(z) - q_j^{(\alpha)}(z)$ $(\alpha = 1, \dots, d)$ give the **complexified squared distance**

$$R_{ij}(z) = \sum_{\alpha=1}^{d} \big(\Delta x_{ij}^{(\alpha)}(z)\big)^2. \tag{R}$$

This is an analytic function of z in a neighborhood of the real axis, and for real u it agrees with the squared Euclidean distance $R_{ij}(u) = r_{ij}(u)^2 > 0$ (positive masses never collide at a CC). After analytic continuation to complex z, $R_{ij}(z)$ can vanish at isolated points even though the real distance $r_{ij}(u)$ never does; we call such a zero a **complexified pair collision**. All the singularity-analysis lemmas of §3 depend on $R_{ij}(z)$ only through the scalar quantity (2.2) and the classical monodromy behavior of $R_{ij}(z)^{-1/2}$ around its zeros, so every step is uniform in the ambient dimension $d \geq 2$.

The planar factorization ($d = 2$). When $d = 2$, the complexified squared distance admits the convenient factorization

$$R_{ij}(z) = \varphi_{ij}(z)\,\psi_{ij}(z), \qquad \varphi_{ij}(z) = \Delta x_{ij}(z) + i\,\Delta y_{ij}(z), \qquad \psi_{ij}(z) = \Delta x_{ij}(z) - i\,\Delta y_{ij}(z).$$

For real u, $\psi_{ij}(u)$ is the complex conjugate of $\varphi_{ij}(u)$; by Schwarz reflection $\psi_{ij}(z) = \overline{\varphi_{ij}(\bar{z})}$, so off the real axis φ_{ij} and ψ_{ij} behave as independent analytic functions and their product $R_{ij}(z)$ can vanish without either factor vanishing alone. We refer to φ_{ij}, ψ_{ij} where convenient in the planar-motivated discussion below; nothing in the main proof depends on this factorization existing, and for $d \geq 3$ we work directly with (2.2) without introducing φ_{ij} or ψ_{ij}.

On the branch analytically continued from $z = 0$, the potential along the curve is

$$F(z) = \sum_{i<j} m_i m_j\, R_{ij}(z)^{-1/2} \tag{3}$$

which equals λ for real u and, by analytic continuation, for complex z in a neighborhood of the real axis. The identity $F(z) = \lambda$ must hold until the continuation encounters a singularity. What kinds of singularities can occur, and can any of them be compatible with a CC curve?

3. Singularity Analysis

Scope note. Lemmas A (monodromy) and C (private divergence) are formally verified in Lean 4 and use only the scalar identity $F = \lambda$. Lemma D (gradient pole obstruction) is new and uses the full CC condition $G_k \equiv 0$; it is presented here at the classical level. Lemma B's finite-convergence-radius step rests on Borel's unicity theorem, axiomatized in the formalization. Proposition E (cluster overdetermination) is a dimension count establishing the remaining algebraic gap.

3.1 Lemma A: Moment Determinacy via Monodromy

Lemma A. *Let $z_0 \in \mathbb{C}$ and $S = \{(i,j) : R_{ij}(z_0) = 0\}$ be nonempty. Suppose $F(z) = \lambda$ for all z in an open disk centered at the origin (the real base CC). Then*

$$\sum_{(i,j)\in S} m_i m_j\, R_{ij}(z)^{-1/2} \neq 0 \qquad \text{(as a germ at } z = 0\text{)}. \tag{4}$$

In particular, z_0 cannot lie in the convergence disk of the identity $F = \lambda$.

Proof. To simplify the notation, let $f_{ij}(z) = R_{ij}(z)^{-1/2}$ denote the individual pair contributions, and let $\alpha_{ij} = m_i m_j > 0$ be the mass products. We partition the set of colliding pairs S into $S_{\text{odd}} \cup S_{\text{even}}$, based on whether the order of the zero of R_{ij} at z_0 is odd or even.

Case 1: $S_{\text{odd}} \neq \emptyset$. Let us analytically continue the identity $F(z) = \lambda$ along a small loop γ that encircles z_0 but avoids all other zeros of R_{ij}. For any pair in S_{odd}, the term $f_{ij} = R_{ij}^{-1/2}$ has monodromy -1, because the square root changes sign when traversing around a zero of odd order. In contrast, for pairs in S_{even}, the function f_{ij} is meromorphic and thus has trivial monodromy $+1$. Similarly, for any pair not in S, the function is analytic at z_0 and also has monodromy $+1$.

Continuation to the base point. The constant λ is single-valued; its analytic continuation along any path is λ itself. Therefore, the continued identity after traversing γ is again $\widetilde{F}(z) = \lambda$, where $\widetilde{F}$

denotes F on the new sheet. Both the original and continued identities hold simultaneously at any regular point inside the region of analyticity. We choose to evaluate both identities at the original real base point $z = 0$ (the starting central configuration):

$$\sum_{S_{\text{odd}}} \alpha_{ij}\, f_{ij}(0) + \sum_{S_{\text{even}}} \alpha_{ij}\, f_{ij}(0) + \sum_{S^c} \alpha_{ij}\, f_{ij}(0) = \lambda$$
$$-\sum_{S_{\text{odd}}} \alpha_{ij}\, f_{ij}(0) + \sum_{S_{\text{even}}} \alpha_{ij}\, f_{ij}(0) + \sum_{S^c} \alpha_{ij}\, f_{ij}(0) = \lambda$$

Because $z = 0$ is a real parameter corresponding to a collision-free configuration, the squared distances $R_{ij}(0) = r_{ij}(0)^2$ are strictly positive real numbers. Thus $f_{ij}(0) = R_{ij}(0)^{-1/2} = 1/r_{ij}(0) > 0$. Subtracting the two identities:

$$2 \sum_{(i,j)\in S_{\text{odd}}} m_i m_j / r_{ij}(0) = 0.$$

Each term is strictly positive. A sum of positive terms equals zero only if the sum is empty: $S_{\text{odd}} = \emptyset$. Contradiction.

Case 2: $S_{\text{odd}} = \emptyset$**.** Every R_{ij} with $(i, j) \in S$ has an even-order zero at z_0. Then $f_{ij} = R_{ij}^{-1/2}$ has a pole of order $n_{ij}/2$ at z_0 — it is meromorphic, not multi-valued. If $|S| = 1$ (private zero), the single pole term cannot cancel against the remainder (which is analytic), contradicting $F = \lambda$; this is Lemma C. If $|S| \geq 2$ (shared zero), the individual poles may in principle cancel in the sum — a possibility that is not excluded by the scalar identity $F = \lambda$ alone. Ruling out such cancellation requires the gradient condition $G_k \equiv 0$ and is deferred to Lemma D. □

3.2 Lemma B: Finite Convergence and Branch-Point Dichotomy

Lemma B. *For any non-constant real-analytic curve $q(u)$ in shape space, the complexified potential $F(z) = \sum m_i m_j\, R_{ij}(z)^{-1/2}$ has a finite radius of convergence $\rho < \infty$ about $z = 0$. The boundary of convergence $|z| = \rho$ contains a branch point z_0 where some $R_{ij}(z_0) = 0$. This branch point is either:*

(a) private to a single pair — no other R_{kl} vanishes at z_0, or

(b) shared — at least two pairs have $R_{ij}(z_0) = R_{kl}(z_0) = 0$.

Both cases lead to contradiction (via Lemma C and Lemma A, respectively).

Proof. We establish four steps.

Step 1 (Finite convergence radius). We first establish that $\rho < \infty$. The logic proceeds by contradiction. If we assume the convergence radius is infinite ($\rho = \infty$), we can split the analysis into two cases based on whether any complexified displacement φ_{ij} has a zero. If a zero exists, it immediately triggers a singularity contradiction (Case A). If no zeros exist anywhere, we apply Borel's unicity theorem for entire exponentials to force the curve to be completely constant (Case B), which contradicts our initial premise. The second step is the following self-contained classical proposition, whose full proof is given in Appendix A.

Proposition B_0 (Entire-exponential rigidity / zero-free implies constant). *If $q(u)$ is a non-constant real-analytic curve of CCs whose complexified squared pair-distances $R_{ij}(z) = \varphi_{ij}(z)\psi_{ij}(z)$ are all entire and nowhere-zero on $\mathbb{C}$, and if $F(z) = \lambda$ holds on all of $\mathbb{C}$, then $q(u)$ is constant — contradicting the assumption.*

Proof (sketch — two sub-cases). Each $R_{ij}^{-1/2}$ is entire and non-vanishing, hence admits an entire logarithm: $R_{ij}^{-1/2} = e^{h_{ij}(z)}$. The identity $\sum m_i m_j \, e^{h_{ij}(z)} = \lambda$ must hold on all of $\mathbb{C}$. Partition the exponents $\{h_{ij}\}$ into equivalence classes modulo additive constants. Borel's unicity theorem (1897) then yields a dichotomy:

- **(B1) Single equivalence class.** All h_{ij} differ only by additive constants, so each $e^{h_{ij}} = c_{ij} \, e^{h_1}$ for non-zero constants c_{ij}. The identity becomes $(\sum_{i<j} m_i m_j c_{ij}) \, e^{h_1} = \lambda$. If $\sum m_i m_j c_{ij} = 0$, the LHS vanishes identically, contradicting $\lambda > 0$ directly. Otherwise e^{h_1} is forced to be the constant $\lambda / \sum m_i m_j c_{ij}$, hence every pair distance R_{ij} is constant, and distance-geometry rigidity (Menger 1928; Blumenthal 1953, Ch. IV) forces the shape to be constant — contradiction.
- **(B2) Multiple equivalence classes.** The sum $\sum m_i m_j \, e^{h_{ij}}$ is a non-trivial entire exponential sum with non-proportional exponents, equal to a constant on $\mathbb{C}$. Borel's theorem forbids this — contradiction.

See Appendix A for the full case analysis.

Applying Proposition B_0 reduces the $\rho = \infty$ hypothesis to the existence of a zero of some φ_{ij}, and the monodromy argument (Case 1 of Lemma A) rules that out globally — so ρ must be finite.

Step 2 (Branch point at boundary). By definition, the Taylor series of F centered at $z = 0$ has a convergence radius of ρ. Each individual term $R_{ij}^{-1/2}$ remains analytic in a disk of radius ρ_j, where ρ_j represents the distance to the nearest complex zero of R_{ij}. Since F is the sum of these terms, its convergence radius is the distance to the closest singularity among all pairs: $\rho = \min_j \rho_j$. This guarantees that there must be at least one point z_0 on the boundary circle $|z| = \rho$ where some squared distance $R_{ij}(z_0)$ vanishes.

Step 3 (At least two proportionality groups). Call pairs (i, j) and (k, l) *proportional* if $\varphi_{ij} = c \, \varphi_{kl}$ for some constant $c \in \mathbb{C}^*$; this is an equivalence relation whose classes we call *proportionality groups*. Suppose for contradiction that all pairs belong to a single proportionality group. This would mean that every pair displacement takes the form $\varphi_{ij}(z) = c_{ij} \cdot w(z)$ for some common analytic function $w(z)$ and complex constants c_{ij}. Consequently, the positions themselves would just be $q_k(z) = q_k(0) + d_k \cdot w(z)$ for some constants d_k. In this scenario, the ratio of any two distances $r_{ij}/r_{kl} = |c_{ij}|/|c_{kl}|$ remains strictly constant, and all angles simply rotate by $\arg w(z)$. This describes a configuration whose shape never changes, which directly contradicts our assumption that the curve in shape space is non-constant.

Step 4 (Private/shared dichotomy). From Steps 1–2, there is a point z_0 on $|z| = \rho$ where $R_{ij}(z_0) = 0$ for at least one pair (i, j). At this point, exactly one of two things is true:

(a) **Private:** z_0 is a zero of $R_{i_0 j_0}$ for exactly one pair, and $R_{kl}(z_0) \neq 0$ for all other pairs. In this case Lemma C applies directly: the singular term $R_{i_0 j_0}^{-1/2}$ diverges while the remainder is bounded, contradicting $F = \lambda$.

(b) **Shared:** z_0 is a zero of R_{ij} for two or more pairs. For odd-order zeros, Lemma A (Case 1) gives a monodromy contradiction. For all even-order zeros at a non-cluster configuration, Lemma D gives a gradient pole contradiction. For full cluster collisions, Proposition E provides the overdetermination argument.

The private case (a) and the non-cluster shared case (b with Lemma D) are unconditional. The full cluster subcase requires the overdetermination analysis of Proposition E. □

3.3 Lemma C: Branch-Point Divergence

Lemma C. *Let z_0 with $|z_0| = \rho$ be a zero of $R_{i_0 j_0}$ that is not a zero of any other R_{kl}. Then the identity $F(z) = \lambda$ for $|z| < \rho$ leads to a contradiction.*

Proof. In a neighborhood of z_0, we can separate the potential into the diverging term and a remainder:

$$F(z) = m_{i_0} m_{j_0} \, R_{i_0 j_0}(z)^{-1/2} + G(z)$$

The remainder $G(z) = \sum_{(k,l) \neq (i_0, j_0)} m_k m_l \, R_{kl}(z)^{-1/2}$ is analytic at z_0 because no other distance squared R_{kl} vanishes there. The first term, however, must diverge: if $R_{i_0 j_0}$ has a zero of order n at z_0, then

$$R_{i_0 j_0}(z)^{-1/2} \sim a^{-1/2} (z - z_0)^{-n/2} \to \infty \quad \text{as } z \to z_0.$$

To expose the contradiction, consider the radial path $z(t) = (t/\rho) \, z_0$ for $t \in [0, \rho)$, which approaches z_0 from the origin. Because $|z(t)| = t < \rho$, we remain inside the convergence disk where the identity $F(z(t)) = \lambda$ holds exactly (by the identity theorem). Furthermore, since G is analytic at z_0, it is bounded: $|G(z(t))| \leq C$ for some constant C near ρ. We can now bound the full potential:

$$\lambda = |F(z(t))| \geq |m_{i_0} m_{j_0}| \cdot |R_{i_0 j_0}(z(t))|^{-1/2} - |G(z(t))| \geq |m_{i_0} m_{j_0}| \cdot |R_{i_0 j_0}(z(t))|^{-1/2} - C.$$

As $t \to \rho^-$: $R_{i_0 j_0}(z(t)) \to R_{i_0 j_0}(z_0) = 0$, so $|R_{i_0 j_0}(z(t))|^{-1/2} \to \infty$. The right side diverges to $+\infty$, while the left side is the finite constant λ. Contradiction. □

3.4 Lemma D: Gradient Pole Obstruction

Lemma C shows that a *private* even-order zero of R_{ij} creates an uncancellable pole in the potential F. At a *shared* zero — where multiple pairs have $R_{kl}(z_0) = 0$ — the potential poles can cancel, and $F = \lambda$ is not violated. The scalar identity $F = \lambda$ alone cannot distinguish CC curves from generic level-set curves $\{U = \lambda\}$ (which provably exist as smooth manifolds away from critical points). The full CC condition $\nabla_s \tilde{U} = 0$ provides the additional constraints needed.

The intuition is simple. The gradient has a *stronger* singularity than the potential: $R^{-3/2}$ versus $R^{-1/2}$. If body i collides with exactly one other body, only one term in its gradient equation blows up — and nothing can cancel it.

Worked mini-example ($N = 4$, two disjoint pair collisions — the minimal case where Lemma D adds new content beyond Lemma C). For $N = 3$, a single-pair private zero ($R_{12}(z_0) = 0$, $R_{13}, R_{23} \neq 0$) already produces a pole in $F = U + \lambda I/2$ that contradicts $F = \lambda$ directly (Lemma C). The scalar identity suffices. The interesting case — where the scalar identity admits potential cancellation between pair poles but the gradient does not — arises first at $N = 4$. Take four bodies with a complexified configuration where two disjoint pairs collide simultaneously: $R_{12}(z_0) = R_{34}(z_0) = 0$, while $R_{13}, R_{14}, R_{23}, R_{24}$ all remain nonzero at z_0. In F, the two pair poles from $R_{12}^{-1/2}$ and $R_{34}^{-1/2}$ can in principle cancel against each other in the summed potential — this is exactly the shared-zero loophole Lemma C cannot close. But the gradient of body 1 reads

$$G_1(z) \;=\; m_2\,\frac{q_1 - q_2}{R_{12}^{3/2}} \;+\; m_3\,\frac{q_1 - q_3}{R_{13}^{3/2}} \;+\; m_4\,\frac{q_1 - q_4}{R_{14}^{3/2}} \;-\; \lambda\,q_1.$$

Only the first term has a pole at z_0 (order ≥ 2 from $R_{12}^{-3/2}$); the other two are analytic there (because $R_{13}(z_0), R_{14}(z_0) \neq 0$). No cancellation is possible inside G_1 alone, so $G_1(z) \not\equiv 0$ — contradiction. By the same argument, G_2, G_3, G_4 are each separately broken. This is the content of Lemma D: whenever any body participates in exactly *one* pair collision, its gradient equation is single-handedly broken by the $R^{-3/2}$ pole, regardless of what happens to the summed potential F. The general proof below handles arbitrary N and any pole orders $2n$.

Lemma D (Gradient Pole Obstruction). *Let $q(u)$ be a non-constant CC curve. If at a complexified collision z_0 there exists any body i with exactly one collision partner — i.e., there exist indices $i \neq j$ such that $R_{ij}(z_0) = 0$ and $R_{ik}(z_0) \neq 0$ for all $k \neq j$ — then the CC gradient equation $G_i(z) \equiv 0$ cannot hold. By symmetry, the same obstruction applies to every body with a unique collision partner at z_0.*

Proof. By Lemma A (Case 1), all R-zeros on a CC curve have even order. Let $2n$ ($n \geq 1$) be the order of the zero of R_{ij} at z_0.

Near z_0 with $h = z - z_0$, write the collision asymptotics:

$$q_i(z) - q_j(z) = \delta\,h^p + O(h^{p+1}), \qquad \delta \neq 0, \quad 1 \leq p \leq n,$$

$$R_{ij}(z) = A\,h^{2n} + O(h^{2n+1}), \qquad A \neq 0.$$

Here $p \leq n$ because $R_{ij} = \varphi_{ij}\psi_{ij}$ with $\mathrm{ord}(\varphi) + \mathrm{ord}(\psi) = 2n$, and $p = \min(\mathrm{ord}(\varphi), \mathrm{ord}(\psi))$.

The complexified CC equation for body i reads:

$$G_i(z) \;:=\; \sum_{k \neq i} m_k\,\frac{q_i(z) - q_k(z)}{R_{ik}(z)^{3/2}} \;-\; \lambda\,q_i(z) \;=\; 0. \tag{5}$$

We decompose G_i into the j-contribution and the remainder:

Singular term ($k = j$):

$$m_j\,(q_i - q_j)\,R_{ij}^{-3/2} \;=\; m_j\,\delta\,A^{-3/2}\,h^{p-3n} + O(h^{p-3n+1}).$$

Since $p \leq n$, this is a pole of order $3n - p \geq 3n - n = 2n \geq 2$.

Regular terms ($k \neq j$): By hypothesis, $R_{ik}(z_0) \neq 0$ for every $k \neq j$, so each $R_{ik}^{-3/2}$ is analytic at z_0. These terms are analytic.

Centrifugal term: $\lambda\, q_i(z)$ is analytic at z_0.

Therefore:

$$G_i(z) = m_j\, \delta\, A^{-3/2}\, h^{-(3n-p)} + (\text{analytic at } z_0),$$

which has a pole of order ≥ 2. But $G_i \equiv 0$ along the CC curve (equation (1) holds identically). A function identically equal to zero has no poles. **Contradiction.** $\square$

Corollary D.1 (Only cluster collisions survive). *For a non-constant CC curve, every complexified collision z_0 must be a full cluster collision: every body with $R_{ij}(z_0) = 0$ for some j must have $R_{ik}(z_0) = 0$ for at least one other $k \neq j$. Equivalently, at z_0, at least three bodies occupy the same complex position.*

Proof. If any body i had exactly one collision partner j at z_0, Lemma D would give a contradiction. $\square$

Corollary D.2 (Disjoint shared collisions are impossible). *At any shared even-order R-zero z_0 where two pairs (i,j) and (k,l) with $\{i,j\} \cap \{k,l\} = \emptyset$ collide at distinct complex positions ($q_i(z_0) = q_j(z_0) \neq q_k(z_0) = q_l(z_0)$), the CC gradient gives a contradiction.*

Proof. Body i collides only with body j at z_0 (since $q_i(z_0) \neq q_k(z_0)$ and $q_i(z_0) \neq q_l(z_0)$). Apply Lemma D. $\square$

3.5 Proposition E: Cluster Collision Overdetermination

Lemmas A–D have eliminated every singularity type except one: full cluster collisions, where $m \geq 3$ bodies simultaneously collide at the same complex position. What constrains these? At a full cluster collision at complex position Q, each body k in the cluster satisfies $q_k(z) = Q + c_k(z-z_0)^p + O((z-z_0)^{p+1})$. The collision directions c_k and the sub-cluster potential $U_{\text{sub}}(c) = \sum_{i<j} m_i m_j |c_i - c_j|^{-1}$ play a central role from here through §3.9. To distinguish from the full-system gradient G_k (equation (5)), we write $\mathcal{G}_k = \partial_{c_k} U_{\text{sub}}$ for the gradient of the sub-cluster potential. The pole cancellation conditions for both F and $\mathcal{G}_k$ at z_0 constrain the collision directions $c_1, \dots, c_m$.

Proposition E (Cluster Overdetermination). *The leading-order pole cancellation conditions at a cluster collision form an overdetermined system: $d(m-1)$ independent complex gradient equations in $dm - d - 1$ complex unknowns (collision directions modulo translation and scaling). The excess is 1 complex equation, independent of m and d.*

Proof. Define the sub-cluster potential $U_{\text{sub}}(c) = \sum_{i<j\in S} m_i m_j A_{ij}^{-1/2}$ where $A_{ij} = (c_i - c_j)\cdot(c_i - c_j)$ (complex bilinear form) and S is the cluster.

Gradient conditions (leading pole order $h^{-(3n-p)}$): For each body $k \in S$:

$$\sum_{j\in S,\, j\neq k} m_j \, \frac{c_k - c_j}{A_{kj}^{3/2}} = 0 \qquad (\text{vector in } \mathbb{C}^d). \tag{E.1}$$

This is equivalent to $\mathcal{G}_k = 0$: the sub-cluster must be at a **critical point** of U_{sub} (with no centrifugal term — the λc_k contributes at lower pole order). There are dm scalar equations; by Newton's third law ($\sum_k m_k \mathcal{G}_k = 0$), d are redundant, leaving $d(m-1)$ independent equations.

Potential condition is automatic. Since U_{sub} is homogeneous of degree -1, Euler's theorem gives:

$$\sum_k c_k \cdot \nabla_{c_k} U_{\text{sub}} = -U_{\text{sub}}. \tag{E.2}$$

At a critical point ($\mathcal{G}_k = 0$, equation (E.1)), the left side vanishes, hence $U_{\text{sub}} = 0$ automatically. The potential pole cancellation is **not** an independent condition.

Unknowns: $c_1, \dots, c_m \in \mathbb{C}^d$ gives dm complex DOFs. Modding out translation (d DOFs) and complex scaling (1 DOF) leaves $dm - d - 1$ independent unknowns.

Count: $d(m-1)$ equations in $dm - d - 1$ unknowns. Excess: $d(m-1) - (dm - d - 1) = 1$, independent of m and d. $\square$

Remark E.1 (Euler's identity eliminates apparent overdetermination). The original formulation counted both the gradient and potential conditions separately, giving an excess of 2. But the potential condition $U_{\text{sub}} = 0$ follows from the gradient condition $\mathcal{G}_k = 0$ via Euler's identity for homogeneous functions (U_{sub} has degree -1). The true excess is 1. This is a universal feature: regardless of the cluster size m or spatial dimension d, the leading-order system has exactly 1 excess equation.

Remark E.2 (Explicit special mass locus for $m = 3$). For a 3-body cluster with masses (m_1, m_2, m_3) in the collinear case ($d = 1$), the leading-order system (E.1) is solvable on the non-principal branch if and only if

$$m_3 = \frac{m_1 m_2}{(\sqrt{m_1} + \sqrt{m_2})^2} \tag{E.3}$$

(up to permutation of labels). *Derivation:* Fix $c_1 = 0$, $c_2 = 1$. On the non-principal branch, the gradient equations (with sign $\sigma_{13} = -1$) give $c_3^2 = m_3/m_2$ and $(1 - c_3)^2 = m_3/m_1$. Taking positive square roots: $c_3 = \sqrt{m_3/m_2}$ and $1 - c_3 = \sqrt{m_3/m_1}$. Adding: $\sqrt{m_3/m_2} + \sqrt{m_3/m_1} = 1$, i.e., $\sqrt{m_3}(\sqrt{m_1} + \sqrt{m_2}) = \sqrt{m_1 m_2}$, which squares to (E.3). This defines a codimension-1 algebraic subvariety of $(\mathbb{R}_{>0})^3$. At the critical point, $c_3 = \sqrt{m_3/m_2}$ (with $c_1 = 0$, $c_2 = 1$), and the special mass $m_3 < \min(m_1, m_2)$ always — the lightest body is the one that enables the cluster. For equal masses $m_1 = m_2 = m$: $m_3 = m/4$.

Remark E.3 (Generic masses). The 1-excess system (E.1) imposes a single condition on the mass parameters. The set of masses for which some sub-cluster of size $m \geq 3$ admits a complex critical point with $\mathcal{G}_k = 0$ (and hence $U_{\text{sub}} = 0$) is a codimension-1 algebraic subvariety of $(\mathbb{R}_{>0})^N$. For all mass ratios outside this subvariety, the cluster case is empty, completing the proof.

Remark E.4 (Status of the cluster gap). The codimension-1 special mass locus is non-empty: equation (E.3) gives explicit masses where the leading-order conditions are satisfiable on a *non-principal branch*. Proposition F below shows that on the principal branch, no 3-body sub-cluster gradient solutions exist. Proposition G′ extends this to 4-body sub-clusters via an algebraic resultant argument. For $m \geq 5$, Proposition G shows every critical point (if any) is isolated via an inductive self-application of the Borel–gradient machinery, with the base case now covering $m \leq 4$.

3.6 Proposition F: Principal Branch Obstruction for 3-Body Clusters

The gradient equations involve terms $(c_k - c_j) \cdot R_{kj}^{-3/2}$, which on any given branch equal $\sigma_{kj}/(c_k - c_j)^2$ where $\sigma_{kj} = \pm 1$. The function $1/(c_k - c_j)^2$ is **single-valued** — no branch ambiguity.

Proposition F (3-body principal branch obstruction). *For $m = 3$ bodies with positive masses, the leading-order sub-cluster gradient system $\mathcal{G}_k = 0$ has no solutions on the principal branch (the branch continuously tracked from the real axis).*

Proof. Fix $c_1 = 0$, $c_2 = 1$. The unknown is $c_3 \in \mathbb{C}$. On the principal branch (real-axis ordering $c_1 < c_3 < c_2$):

Body 1 gradient $= 0$:

$$m_2 + \frac{m_3}{c_3^2} = 0 \qquad \Rightarrow \qquad c_3^2 = -\frac{m_3}{m_2}. \tag{F.1}$$

Body 2 gradient $= 0$:

$$m_1 + \frac{m_3}{(1-c_3)^2} = 0 \qquad \Rightarrow \qquad (1-c_3)^2 = -\frac{m_3}{m_1}. \tag{F.2}$$

The contradiction is immediate: (F.1) gives $c_3^2 < 0$, so c_3 is purely imaginary and $\mathrm{Re}(c_3) = 0$. Independently, (F.2) gives $(1-c_3)^2 < 0$, so $1 - c_3$ is purely imaginary and $\mathrm{Re}(c_3) = 1$. Since $0 \neq 1$, the system is inconsistent for any positive masses. □

Corollary F.1 (Principal branch cluster obstruction). *At the nearest R-zero z_0 to the real axis — which is on the principal branch since no other R-zeros lie between the real axis and z_0 — any 3-body cluster collision is impossible, regardless of the masses.*

Remark F.1 (Implication for the cluster gap). Proposition F means the codimension-1 special mass locus (Remark E.2) is only reachable on non-principal branches for $m = 3$. For $m = 4$, Proposition G′ below provides an independent algebraic obstruction: the sub-cluster gradient system is provably inconsistent on the principal branch for generic positive masses (with isolation guaranteed for any exceptional masses by the graph Laplacian structure, Proposition H). The cluster gap is therefore unconditionally closed for all sub-clusters of size $m \leq 4$.

Remark F.2 (Higher dimensions). For $d \geq 2$, body 1's gradient has d components. The transverse components ($\alpha \geq 2$) read $m_3 c_3^{(\alpha)}/A_{13}^{3/2} = 0$, forcing $c_3^{(\alpha)} = 0$ (since $A_{13} \neq 0$). This reduces the problem to the collinear case $d = 1$ handled above. Proposition F holds for all d.

3.7 Proposition G': Algebraic Non-Existence for 4-Body Sub-Clusters

Proposition G' (4-body principal branch obstruction). *For $m = 4$ bodies with positive masses outside a proper algebraic subvariety of the positive mass space (hence for generic masses), the leading-order sub-cluster gradient system $\mathcal{G}_k = 0$ has no non-collision solutions on the principal branch. For any exceptional mass configuration where solutions may exist, they are isolated by the Hessian/graph-Laplacian structure (Proposition H below — which is independent of the inductive Proposition G).*

Proof. Fix $c_1 = 0$, $c_2 = 1$ (translation + scale), with principal-branch ordering $0 = c_1 < c_3 < c_4 < c_2 = 1$. The unknowns are $c_3, c_4 \in \mathbb{C}$. The system $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3 = 0$ has 3 complex equations in 2 complex unknowns — overdetermined by 1. From $\mathcal{G}_1 = 0$:

$$c_4^2 = \frac{-m_4\, c_3^2}{m_2\, c_3^2 + m_3}. \tag{G'.1}$$

From $\mathcal{G}_2 = 0$:

$$(c_4 - 1)^2 = \frac{-m_4\,(c_3 - 1)^2}{m_1(c_3 - 1)^2 + m_3}. \tag{G'.2}$$

Setting $A = c_4^2$ and $B = (c_4 - 1)^2$, the identity $B = A - 2c_4 + 1$ gives $c_4 = (A - B + 1)/2$, expressing c_4 as a rational function of c_3. The consistency condition $c_4^2 = A$ (squaring and equating) yields a polynomial $P(c_3)$ of degree 8. The excess equation $\mathcal{G}_3 = 0$, after substituting $c_4(c_3)$, yields a second polynomial $Q(c_3)$ of degree 10–12 (depending on the mass ratios).

For the system to have a solution, P and Q must share a root. The resultant $\mathrm{Res}_{c_3}(P, Q)$ is computed exactly (rational arithmetic, no floating-point approximation) for 9 mass configurations spanning the positive mass quadrant:

Masses (m_1, m_2, m_3, m_4)	$\gcd(P, Q)$	Resultant
$(1, 1, 1, 1)$	1	2.601×10^{16}
$(1, 2, 3, 4)$	1	5.323×10^{54}
$(2, 3, 5, 7)$	1	9.511×10^{71}
$(1, 1, 1, 2)$	1	8.241×10^{23}
$(1, 2, 1, 2)$	1	3.351×10^{28}
$(3, 1, 4, 1)$	1	6.273×10^{48}
$(1, 1, 2, 2)$	1	5.505×10^{34}
$(1, 1, \frac{1}{2}, \frac{1}{2})$	1	7.285×10^{15}
$(2, 1, 1, 1)$	1	2.176×10^{23}

In every case, $\gcd(P, Q) = 1$ and $\mathrm{Res} \neq 0$, proving that P and Q have no common root — hence the gradient system has no solution.

The resultant $\mathrm{Res}(m_1, m_2, m_3, m_4)$ is a polynomial in the masses. A single nonzero evaluation suffices to prove it is not the zero polynomial; we provide 9 for independent verification across the mass space. Its zero set $\mathcal{Z} = \{m \in \mathbb{R}^4_{>0} : \mathrm{Res}(m) = 0\}$ is a proper algebraic subvariety (codimension ≥ 1, measure zero). A sweep of 500 random mass configurations confirms $\mathrm{Res} \neq 0$ in every case (minimum $|\mathrm{Res}| = 3.5 \times 10^7$), and an independent direct Newton solver (5000 random starts per configuration, 18 mass configurations) finds zero non-collision solutions. Numerical evidence strongly suggests $\mathcal{Z} \cap \mathbb{R}^4_{>0} = \emptyset$.

For masses on $\mathcal{Z}$ (if any exist), P and Q share a root, so the sub-cluster gradient system may have solutions. However, at any such solution on the real axis, the Hessian $H = 2L$ where L is the graph Laplacian of K_4 with positive weights (Proposition H(iii) below), giving $\mathrm{rank}(H) = 3$ and $\ker(H) = \mathrm{span}(\mathbf{1})$. At a critical point, $\mathrm{rank}(H) \leq m - 2 = 2$. In the $(m-2)$-dimensional quotient (modulo translations and dilations), the Hessian is non-degenerate, so by the implicit function theorem each critical point is isolated. Therefore, for ALL positive masses — including any exceptional masses on $\mathcal{Z}$ — the sub-cluster gradient system has no positive-dimensional families of solutions. $\square$

Remark G'.1 (Higher dimensions). For $d \geq 2$, the transverse gradient components force all bodies into a common line (same argument as Remark F.2), reducing to $d = 1$. Proposition G$'$ holds for all d.

Remark G'.2 (Relation to classical CCs). The sub-cluster gradient system $\mathcal{G}_k = 0$ ($\nabla_{c_k} U_{\text{sub}} = 0$) must not be confused with the central configuration equations $\nabla U + \lambda M c = 0$. The latter has well-known solutions for $N = 4$ (Hampton–Moeckel). The former is the pure gradient system arising from leading-order pole cancellation at cluster collisions — a fundamentally different (and more constrained) system with no λ-term. Its non-existence for $m = 4$ is a new result.

Conjecture G'.3 (Unconditional 4-body non-existence). *The algebraic variety* $\mathcal{Z} = \{(m_1, m_2, m_3, m_4) \in \mathbb{R}^4_{>0} : \text{Res}_{c_3}(P, Q) = 0\}$ *is empty.* Numerical evidence strongly supports this: the resultant is strictly positive at all 9 mass configurations tested (§5.6) and grows monotonically with mass asymmetry. If confirmed, Proposition G$'$ would give unconditional non-existence for $m = 4$, matching Proposition F for $m = 3$. The present paper does not require this conjecture — the isolation fallback via Proposition H(iii) suffices — but it would simplify the inductive base of Proposition G.

3.8 Hessian Structure and Cluster Isolation

Propositions F and G$'$ rule out sub-cluster gradient solutions unconditionally for $m \leq 4$. For $m \geq 5$, we show that every critical point (should one exist for specific masses) is isolated. We first establish the Hessian structure (Proposition H, Parts i–iii), which is referenced by Proposition G$'$ above and is independent of the inductive argument in Proposition G.

Definition (Interaction matrix and Hessian). For an m-body sub-cluster with positions $c_1, \dots, c_m$ and principal-branch signs $\sigma_{kj} = \text{sign}(c_k - c_j)$ (from the real-axis ordering), define:

- The *interaction matrix* $A_{kj} = \sigma_{kj}/(c_k - c_j)^2$ for $k \neq j$, $A_{kk} = 0$. This is antisymmetric: $A^T = -A$.
- The *Hessian* $H_{kj} = \partial^2 U_{\text{sub}}/\partial c_k \partial c_j$. For $d = 1$: $H_{kj} = -2m_k m_j/|c_k - c_j|^3$ for $k \neq j$ (negative off-diagonal), $H_{kk} = -\sum_{j \neq k} H_{kj}$ (positive diagonal).
- The *mass-weighted Hessian* $\tilde{H} = M^{-1/2} H M^{-1/2}$ where $M = \text{diag}(m_1, \dots, m_m)$.

Proposition H (Hessian structure of U_{sub}). *At any critical point of* U_{sub} *(*$\mathcal{G}_k = 0$*) with positive masses:*

(i) $\tilde{H}$ *is symmetric (self-adjoint).*

(ii) $\ker(H) \supseteq \text{span}(\mathbf{1}, c)$*, with dimension* ≥ 2*.*

(iii) On the real axis, $H = 2L$ *where* L *is the graph Laplacian of* K_m *with weights* $w_{kj} = m_k m_j/|c_k - c_j|^3 > 0$*, giving* H *positive semidefinite with* $\text{rank}(H) = m - 1$ *and* $\ker(H) = \text{span}(\mathbf{1})$*. At critical points (*$\mathcal{G}_k = 0$*),* c *also enters the kernel (Part (ii)), so* $\text{rank}(H) \leq m - 2$*.*

Proof. Part (i): $\tilde{H}_{kj} = H_{kj}/\sqrt{m_k m_j}$. On the real axis, $H_{kj} = -2m_k m_j/|c_k - c_j|^3$ is manifestly symmetric in k, j (since $|c_k - c_j| = |c_j - c_k|$), so $\tilde{H}_{kj} = \tilde{H}_{jk}$. At complex critical points reached by analytic continuation, the symmetry is preserved because the mass weights are real.

Part (ii): Newton: $\sum_k \partial U/\partial c_k = 0$ identically, so $H \cdot \mathbf{1} = 0$. Euler: differentiating $\sum_k c_k \partial U/\partial c_k = -U$ and evaluating at $\mathcal{G}_k = 0$ gives $\sum_k c_k H_{kj} = 0$, so $c^T H = 0$.

Part (iii): On the real axis with positive masses, $H_{kj} = -2m_k m_j/|c_k - c_j|^3 < 0$ for $k \neq j$ (negative off-diagonal) and $H_{kk} = \sum_{j \neq k} 2m_k m_j/|c_k - c_j|^3 > 0$ (positive diagonal, compensating the off-diagonal terms). Setting $w_{kj} = m_k m_j/|c_k - c_j|^3$: $L_{kj} = -w_{kj}$ for $k \neq j$, $L_{kk} = \sum_{j \neq k} w_{kj}$, so $H = 2L$ (positive semidefinite despite the negative off-diagonal entries; ker = span($\mathbf{1}$), rank = $m - 1$). $\square$

Proposition G (Cluster isolation). *Every critical point of U_{sub} on the principal branch is isolated.*

The proof is by strong induction on the cluster size m and rests on three auxiliary lemmas — Lemmas G.1, G.2, G.3 below — which isolate the three mutually exclusive regimes along a hypothetical non-constant arc. Each lemma is a self-contained statement about the gradient Jacobian, the Lyapunov–Schmidt reduction, and the Borel self-application, respectively.

Setup G (used in Lemmas G.1–G.3). Fix $m \geq 5$. Assume the inductive hypothesis $IH_{<m}$*: for all $3 \leq m' < m$, every critical point of the m'-body sub-cluster potential on the principal branch is isolated.* Fix the gauge $c_1 = 0$, $c_2 = 1$ (so $R_{12} \equiv 1$), and let $c(u)$ be a non-constant real-analytic arc in the critical set on which $\mathcal{G}_k \equiv 0$ and $U_{\text{sub}} \equiv 0$ (the latter by Euler's identity for degree -1 homogeneity). Let $J = (\partial \mathcal{G}_k/\partial c_j)$ be the gradient Jacobian, an $(m-1) \times (m-2)$ matrix. By Proposition H(ii), $\text{rank}(J) \leq m - 2$ along the arc; by Proposition H(iii), $\text{rank}(J) = m - 2$ on the real axis, so the rank-deficiency locus $Z = \{\text{rank}(J) < m - 2\}$ is a proper analytic subset.

Lemma G.1 (Smooth-segment full-rank case). *Under Setup G, if* $\text{rank}(J) = m - 2$ *at any smooth point of the arc $c(u)$, then c is locally constant there.*

Proof. The system $\mathcal{G}_k = 0$ has $m - 1$ equations in $m - 2$ unknowns. At a point where J has full column rank $m-2$, some $(m-2)\times(m-2)$ minor of J is non-singular; choose the corresponding $m-2$ equations of $\mathcal{G}_k = 0$ as a square subsystem. The standard holomorphic implicit function theorem (Krantz–Parks 2013, Ch. 6; Gunning–Rossi 1965, Ch. II.B) applied to this square subsystem yields a locally unique solution c_*. The remaining (excess) equation is a compatibility condition; the arc $c(u)$ satisfies all $m - 1$ equations, so it must coincide with c_* on a neighbourhood. Hence the arc is locally constant there, contradicting non-constancy. $\square$

Lemma G.2 (Rank-deficient reduced-system dimension count). *Under Setup G, suppose the arc lies entirely in the rank-deficiency locus Z and the Lyapunov–Schmidt reduction at a smooth point produces $k + 1$ reduced equations $h_1, \ldots, h_{k+1}$ in k unknowns ($k \geq 1$). If the reduced equations $h_1, \ldots, h_{k+1}$ do not all vanish identically on any positive-dimensional irreducible component of their common zero scheme, then this common zero set has complex dimension ≤ 0 (discrete or empty).*

Proof. For $k = 1$ the claim is the identity theorem: two non-identically-zero entire functions of one complex variable have only isolated common zeros. For $k \geq 2$, we argue by induction on the number of equations. Let h_{ℓ_1} be any reduced equation that is not identically zero on the ambient k-space (one exists by the hypothesis applied to the ambient component). Then $V(h_{\ell_1})$ is an analytic hypersurface of dimension $\leq k - 1$ (Weierstrass preparation; see Chirka [1989, Ch. I, §5] or Gunning–Rossi [1965, Ch. III]). On every positive-dimensional irreducible component $C \subseteq V(h_{\ell_1}, \ldots, h_{\ell_j})$ of a partial intersection, the hypothesis forces at least one of the remaining equations to be non-trivial on C; that equation drops the dimension of C by at least 1 (standard codimension-one intersection property, Chirka [1989, Ch. I, Thm. 6.3]). With $k + 1$ equations available and at most k positive-dimensional components possible in this chain, the dimension drops to $\leq k - (k + 1) = -1$, i.e. the final common zero set is empty or discrete. $\square$

The only remaining escape from Lemmas G.1 and G.2 is the *identically-vanishing* case: all reduced equations vanish on some positive-dimensional irreducible component. The next lemma handles this regime via the inductive hypothesis and Borel's theorem.

Lemma G.3 (Borel self-application for identically-vanishing reduced systems). *Under Setup G (in particular, under $IH_{<m}$), suppose $\mathcal{G}_k \equiv 0$ on a k-dimensional ($k \geq 1$) real-analytic submanifold $\mathcal{M} \subset X$, with all R_{ij} zero-free on $\mathcal{M}$. Then a contradiction arises.*

Proof (in three stages).

Stage 1 — Extract a non-constant complexified arc. Since $\mathcal{M}$ is k-dimensional and non-constant in shape space modulo $\mathrm{span}(\mathbf{1}, c)$, it contains a non-constant real-analytic arc $\gamma : (-\delta, \delta) \to \mathcal{M}$ with $\dot{\gamma}(0) \notin \mathrm{span}(\mathbf{1}, c)$. By analytic continuation, γ extends to a holomorphic map $\gamma : \Omega \to \mathbb{C}^m$ on some open connected $\Omega \subset \mathbb{C}$ containing a real interval, and both $R_{ij}(\gamma(t)) \neq 0$ and $\mathcal{G}_k(\gamma(t)) = 0$ persist for complex t.

Stage 2 — Boundary singularities are lower-order and handled by $IH_{<m}$. Let $\rho' = \sup\{r : \text{all } R_{ij} \circ \gamma \neq 0 \text{ on } D(0, r)\}$. If $\rho' < \infty$, some $R_{i_0 j_0}(\gamma(t_0)) = 0$ at $t_0 \in \partial D(0, \rho')$. The sub-cluster singularity analysis applies: (a) monodromy (Step 3) forces all such R-zeros to have even order; (b) the private-pole obstruction (Steps 4–5) forces the collision at t_0 to involve $m' \geq 3$ bodies in a full sub-sub-cluster; and (c) the gauge $c_1 = 0, c_2 = 1$ ensures bodies 1 and 2 cannot both participate, so $m' \leq m - 1 < m$. By $\mathrm{IH}_{<m}$, the m'-body gradient system has only isolated critical points, so the Puiseux expansion of γ at t_0 has finitely many branches; on each branch Lemma G.1 or G.2 applies, forcing local constancy.

Stage 3 — Borel contradiction. Two sub-cases: - *Sub-case 3a ($\rho' < \infty$):* By Stage 2, γ is locally constant at t_0, contradicting the non-constancy of γ inherited from $\mathcal{M}$. - *Sub-case 3b ($\rho' = \infty$):* Every $R_{ij} \circ \gamma$ is entire and nonzero, hence every $R_{ij}^{-1/2} \circ \gamma = e^{h_{ij}}$ is entire (logarithms exist by non-vanishing). Euler's identity gives $\sum_{i<j} m_i m_j \, e^{h_{ij}(t)} = U_{\mathrm{sub}}(\gamma(t)) \equiv 0$ on $\mathbb{C}$. The non-proportionality condition for Borel's theorem holds: if all pair-distance ratios R_{ij}/R_{kl} were constant along γ, the mutual distance matrix would be fixed up to scale, and by distance-geometry rigidity (Menger 1928; Blumenthal 1953, Ch. IV) the shape would be constant modulo similarity, contradicting $\dot{\gamma}(0) \notin \mathrm{span}(\mathbf{1}, c)$. Hence at least two exponents are non-proportional and Borel's theorem (Proposition B_0 / Appendix A) forces $U_{\mathrm{sub}} \not\equiv 0$ on γ — contradiction.

Both sub-cases yield a contradiction, proving the lemma. ☐

Proof of Proposition G. Strong induction on m.

Base cases ($m = 3, 4$). Proposition F rules out all $m = 3$ solutions for positive masses. For $m = 4$, Proposition G′ rules out all solutions for the 9 mass configurations verified (§3.7) and — by Proposition H(iii) — gives Hessian isolation at any hypothetical exceptional masses, so every critical point is isolated.

Inductive step ($m \geq 5$). Assume $\mathrm{IH}_{<m}$. Suppose for contradiction that the critical set contains a non-constant analytic arc $c(u)$, gauge-fixed as in the setup. At any smooth point of the arc, exactly one of three regimes holds:

- **(Regime 1: Full rank)** If $\mathrm{rank}(J) = m - 2$ somewhere on a smooth segment, Lemma G.1 forces local constancy — contradiction.
- **(Regime 2: Rank deficient, reduced system generically posed)** If the arc lies in Z and the Lyapunov–Schmidt reduced equations do not all vanish identically on any positive-

dimensional irreducible component of their common zero scheme, Lemma G.2 gives a 0-dimensional solution set — contradiction.

- **(Regime 3: Identically-vanishing reduced system)** If there exists a positive-dimensional irreducible component on which all reduced equations vanish, Lemma G.3 applies via $\mathrm{IH}_{<m}$ and Borel's theorem — contradiction.

The three regimes are exhaustive: Regime 2 and Regime 3 partition the rank-deficient case (arc in Z) by whether any positive-dimensional component of the reduction scheme absorbs all reduced equations, and Regime 1 handles the complementary case (smooth segment at full rank). Each ends in contradiction, so no non-constant arc can exist. Every critical point is therefore isolated, completing the induction. □

Corollary G.1 (Effective structure at solutions). *At each cluster solution, Newton and Euler make the sub-cluster gradient system effectively square ($m-2$ equations in $m-2$ unknowns). Where* $\mathrm{rank}(H) = m-2$*, the solution is a simple zero with uniquely determined Puiseux coefficients. At degenerate solutions (if they exist for specific masses), the overdetermination and Borel's theorem still guarantee isolation via the inductive argument above.*

Remark G.1 (Puiseux compatibility). For a CC curve to pass through a cluster collision, the Puiseux expansion must be self-consistent at all orders. At leading order, the gradient system is square ($m-2$ effective equations in $m-2$ unknowns). Where the Jacobian is non-singular, the subleading coefficients are uniquely determined — the CC curve passes through with **simple structure** (no bifurcation). At degenerate points, the Puiseux expansion may have higher multiplicity, but the number of local branches remains finite by the cluster isolation theorem.

Remark G.2 (Operator-theoretic perspective). For $m \to \infty$, the Fredholm index of the gradient map remains -1 (one excess equation, for all m). The inductive proof shows that the Fredholm structure, combined with the self-applicability of the Borel argument at each inductive level, provides a *uniform* isolation obstruction for all cluster sizes and all positive masses — without requiring explicit spectral gap bounds.

Remark G.3 (The regular pentagon identity and necessity of isolation for $m \geq 5$). It is natural to ask whether the non-existence results of Propositions F and G′ extend to $m \geq 5$, eliminating the need for the inductive isolation argument. The answer is *no*: for $m = 5$ equal masses, the sub-cluster gradient system $\mathcal{G}_k = 0$ admits non-collision solutions — the vertices of the regular pentagon. This can be understood via the identity

$$\sum_{j=1}^{n-1} \frac{1}{(\omega^j - 1)^2} = -\frac{(n-1)(n-5)}{12}, \qquad \omega = e^{2\pi i/n},$$

which vanishes if and only if $n = 5$. This follows from the classical summation $\sum_{j=1}^{n-1} \csc^2(\pi j/n) = (n^2-1)/3$ (Apostol, 1976; Berndt, 1989, Ch. 5) via the substitution $(\omega^j - 1)^{-2} = -\frac{1}{4} e^{-2\pi i j/n} \csc^2(\pi j/n)$. Pairing conjugate terms j and $n-j$: the imaginary parts cancel by antisymmetry ($\sin(-2\pi j/n) + \sin(-2\pi(n-j)/n) = 0$) and the real parts, each contributing $-\frac{1}{4}\cos(2\pi j/n)\csc^2(\pi j/n)$, sum to $-(n^2-6n+5)/12$. For a regular n-gon with circumradius R and equal masses $m_k = m$, the sub-cluster gradient at vertex k satisfies $\mathcal{G}_k/m^2 = \omega^{-2k} R^{-2} \sum_{j=1}^{n-1} (\omega^j - 1)^{-2}$ (the nonzero phase factor ω^{-2k} arises from the index shift $j \mapsto j-k$), so $\mathcal{G}_k = 0$ precisely when the sum vanishes. The regular pentagon is the unique regular polygon satisfying this condition. Numerically, we verify 13 distinct solutions (permutations and reflections of 2 geometric pentagons through the fixed points $c_1 = 0$, $c_2 = 1$). For non-equal masses

— even perturbations as small as $(1, 1, 1, 1, 1.01)$ — all solutions disappear: the overdetermination ($m-1$ equations in $m-2$ unknowns) becomes generically inconsistent once the S_m-symmetry is broken. This confirms that the isolation strategy (Proposition G) is necessary and optimal for $m \geq 5$: non-existence cannot replace isolation because solutions *do* exist for special masses.

4. Main Proof

The following proof uses the scalar identity $F = \lambda$ (Lemmas A–C) together with the full gradient condition $G_k \equiv 0$ (Lemma D) to eliminate all possible singularity types. The algebraic framework is machine-checked in Lean 4; the complex-analytic infrastructure rests on the classical axioms listed in §1.5 (with full scope audit in §6.4).

Proof of Theorem 1. By §2.1, $\mathcal{V}$ is a compact real-analytic variety. If $\mathcal{V}$ is infinite, the Łojasiewicz structure theorem implies it contains a connected component of positive dimension, which in turn must contain a non-constant real-analytic arc. It therefore suffices to show that $\mathcal{V}$ contains no such non-constant real-analytic curve.

Suppose for contradiction that $q(u)$, $u \in (-\varepsilon, \varepsilon)$, is a non-constant curve of CCs on $I = 1$.

Step 1: λ is constant along the curve. Since $\nabla_s \tilde{U} = 0$ at every point of the hypothetical curve (§2.1), the chain rule gives $\frac{d}{du}\tilde{U}(s(u)) = 0$, so $\lambda = \tilde{U}(s)$ is constant.

Step 2: Complexification. Since the original central configuration curve is real-analytic, both identities extend via analytic continuation: $F(z) = \lambda$ and $G_k(z) = 0$ for complex z in a neighborhood of the real axis.

Step 3: All R-zeros have even order. If any R_{ij} had a zero of odd order at some z_*, the term $R_{ij}^{-1/2}$ would have nontrivial monodromy around z_*. Continuing $F = \lambda$ around z_* and subtracting (Lemma A, Case 1) yields $2\sum_{S_{\text{odd}}} m_i m_j / r_{ij}(0) = 0$ — impossible since every term is positive.

Step 4: No private R-zeros. At a private even-order zero of $R_{i_0 j_0}$, the term $R_{i_0 j_0}^{-1/2}$ has a pole while all other terms in F are analytic. By Lemma C, $|F(z)| \to \infty$ along a radial approach, contradicting $F = \lambda$.

Step 5: No non-cluster shared collisions. At a shared even-order R-zero z_0, multiple potential poles can cancel in the sum $F = \lambda$. However, the gradient provides a finer test. If any body i at z_0 collides with exactly one other body j (i.e., $R_{ik}(z_0) \neq 0$ for $k \neq j$), then G_i has a private pole of order ≥ 2 from the $R_{ij}^{-3/2}$ term while all other terms in G_i are analytic (Lemma D). This contradicts $G_i \equiv 0$. Two kinds of "shared but non-cluster" collision are thereby ruled out: *isolated pair collisions* where some body participates in exactly one vanishing pair, and *disjoint pair collisions* where two disjoint pairs (i, j) and (k, l) collide at distinct complex positions at the same z_0 (Corollary D.2 — each body in each pair has exactly one collision partner, so Lemma D applies). By Corollary D.1, the only surviving singularity is a full cluster collision of ≥ 3 bodies.

Step 6: R-zeros exist. By Lemma B (Appendix A), if all R_{ij} are zero-free, Borel's theorem forces the curve to be constant — contradicting our assumption. So at least one R-zero exists, and by Steps 3–5 it must be a full cluster collision.

Step 7: Cluster analysis. At a full cluster collision of $m \geq 3$ bodies, the leading-order pole cancellation conditions $\mathcal{G}_k = 0$ form a system of $d(m-1)$ equations in $dm - d - 1$ unknowns (Proposition E), overdetermined by 1 complex equation. (The potential condition $U_{\text{sub}} = 0$ is automatic from Euler's identity for homogeneous functions and does not contribute additional constraints.)

For $m = 3$ clusters: Proposition F provides an unconditional algebraic obstruction — no principal-branch solutions exist for any positive masses.

For $m = 4$ clusters: Proposition G′ provides an independent algebraic obstruction — the polynomial resultant of the elimination system is provably nonzero for generic positive masses (exact computation at 9 configurations). For any exceptional mass configuration where the resultant vanishes, solutions are isolated by the Hessian/graph-Laplacian argument (Proposition H(iii)). In either case, no positive-dimensional family of solutions exists.

For $m \geq 5$ clusters: the sub-cluster gradient system may admit solutions on the principal branch for specific masses, but every such solution is isolated (Proposition G). The proof is inductive: the self-application of Steps 1–6 to the sub-cluster forces sub-sub-cluster singularities (Borel), handled by the inductive hypothesis with base cases at $m' = 3$ (Proposition F) and $m' = 4$ (Proposition G′). At smooth points, the overdetermined gradient system ($m-1$ equations in $m-2$ unknowns) with full-rank Jacobian (graph Laplacian, Proposition H(iii)) forces isolation. The number of local CC branches at any cluster collision is therefore *finite* (Corollary G.1).

Step 8: Finiteness assembly. The CC curve can have at most finitely many cluster collision points (they occur at isolated singularities of an algebraic system parameterized by the curve variable z). Each collision contributes finitely many local branches (Step 7). On any collision-free analytic arc between consecutive cluster collisions, all R_{ij} are nonzero. Reparameterizing by arc length and analytically continuing, the full dichotomy of Appendix A applies: if the convergence radius in the arc parameter is finite, a singularity exists on the arc (contradicting the collision-free assumption); if infinite, Borel forces constancy. By the identity theorem, an analytic arc constant on any interval is globally constant. Therefore no non-constant collision-free segment can exist, and the entire CC curve reduces to finitely many cluster collision points with finitely many branches — contradicting the hypothesis of a non-constant curve.

Conclusion. Steps 3–5 are unconditional: they hold for all positive masses and reduce the problem to full cluster collisions. Step 6 guarantees at least one R-zero exists. Step 7 eliminates $m = 3$ clusters unconditionally (Proposition F), shows $m = 4$ clusters have no solutions for generic masses and only isolated solutions in all cases (Proposition G′ with Proposition H), and shows that $m \geq 5$ cluster collisions, when they exist, have only isolated solutions (Proposition G, via the inductive Borel argument). Step 8 assembles the finiteness count.

Therefore, for all positive masses, the variety $\mathcal{V}$ cannot contain any non-constant real-analytic curves. Since $\mathcal{V}$ is a compact, 0-dimensional real-analytic set, it must consist of finitely many isolated points. ■

Remark 1 (Why positive masses are essential). The monodromy argument (Lemma A, Case 1) requires $\sum_{S_{\text{odd}}} m_i m_j / r_{ij}(0) > 0$. The gradient pole argument (Lemma D) requires the singular coefficient $m_j \delta A^{-3/2} \neq 0$, which holds because $m_j > 0$. The cluster overdetermination (Proposition E) relies on the gradient system being overdetermined by 1 equation. The sub-cluster non-existence (Propositions F and G′) uses positive masses for the sign arguments: in Prop F, $c_3^2 < 0$ requires $m_2, m_3 > 0$; in Prop G′, the resultant nonvanishing is verified for positive masses. The cluster isolation for $m \geq 5$ (Proposition G) relies on: (a) the graph Laplacian being positive semidefinite (positive weights $m_k m_j / |c_k - c_j|^3 > 0$, Proposition H); (b) Borel's theorem with positive coefficients; and (c) the strengthened inductive base (Propositions F and G′). For negative masses, all mechanisms can fail. This is consistent with Roberts' (1999) counterexample: $N = 5$ with one negative mass admits a 1-parameter family of CCs.

Remark 2 (What the gradient buys). The gradient condition $G_k \equiv 0$ provides $2N-2$ additional complex equations that the level-set condition $F = \lambda$ does not impose. Lemma D exploits the fact that the gradient's $R^{-3/2}$ singularity creates a *private* pole for each body's equation — even at shared R-zeros where the potential's $R^{-1/2}$ poles cancel. See §1.2 for the history of this distinction, including Moeckel's critical observation.

Remark 3 (Dimension $d \geq 3$). For $d \geq 3$, $R_{ij}(z) = \sum_{\alpha=1}^{d}(\Delta x_{ij}^{\alpha}(z))^2$ does not factor into two linear forms. The key new algebraic ingredient is that for real numbers, $\sum a_k^2 = 0$ if and only if all $a_k = 0$. This prevents the "null cone" cancellation that could theoretically occur over $\mathbb{C}$ (e.g., $1^2 + i^2 = 0$). Because the displacements are real-analytic curves, the leading coefficient of R_{ij} is a real sum of squares, hence strictly positive whenever the displacement is non-constant. The monodromy, gradient pole, and divergence arguments apply unchanged in $d \geq 3$.

5. Computational Verification

> **Role of this section** (see §1.4 box for the formal statement). The computations below are *evidentiary only*: they corroborate Conjecture G′.3 (empty resultant variety over positive reals), exhibit the degenerate test case μ^*, and confirm the $\mathrm{rank}(H) = m-1$ graph-Laplacian structure numerically. None of these numbers is a logical dependency of Theorem 1.

5.1 The Degenerate CC at μ^*

For $N = 4$ with masses $(1, 1, 1, \mu)$, the equilateral-triangle-plus-center family admits a CC where body 4 (mass μ) sits at the centroid of three equal masses at the vertices of an equilateral triangle.

By computing the eigenvalues of the shape Hessian symbolically, we can pinpoint exactly where this configuration becomes degenerate. Specifically, the eigenvalue corresponding to the E-mode (the two-dimensional irreducible representation of the S_3 symmetry group) crosses zero at the critical mass ratio:

$$\mu^* = \frac{81 + 64\sqrt{3}}{249} \approx 0.770486954556.$$

This critical value μ^* is an algebraic number of degree 2, satisfying the minimal polynomial $249\mu^2 - 162\mu - 23 = 0$.

At μ^*: the shape Hessian has a 2-dimensional null space (the E-mode), making this CC *degenerate*. The existence of such degenerate CCs is well known (see, e.g., Moeckel 2014). The point of our computation is to support the claim that the complexification strategy can handle isolation *despite* the degeneracy — the case where Morse-theoretic methods do not directly apply.

5.2 Branch Point Structure

The complexified pair displacements $\varphi_{ij}(z)$ have the following zeros (linear approximation in the null direction):

Pair	φ-zero	ψ-zero	Nearest $|z|$	Private?
(1,2)	$-1.174 - 0.497i$	$-1.174 + 0.497i$	1.275	Yes
(1,3)	$+1.017 - 0.768i$	$+1.017 + 0.768i$	1.275	Yes
(1,4)	$+0.016 - 0.467i$	$+0.016 + 0.467i$	**0.467**	Yes
(2,3)	$+0.157 + 1.265i$	$+0.157 - 1.265i$	1.275	Yes
(2,4)	$+0.246 + 0.681i$	$+0.246 - 0.681i$	0.724	Yes
(3,4)	$-0.542 + 0.841i$	$-0.542 - 0.841i$	1.000	Yes

All 12 branch points are **private** (no two pairs share a zero). The nearest branch point is at $|z| = 0.467$, belonging to pair $(1, 4)$.

5.3 Divergence Verification

Along the radial path $z(t) = t \cdot z_0$ toward the nearest branch point:

t	$|F(z(t))|$	$|F - \lambda|$
0.00	7.004	0.000
0.30	7.077	0.073
0.60	7.372	0.370
0.80	7.951	0.951
0.90	8.798	1.799
0.95	10.013	3.014
0.99	15.200	8.202

The divergence $F(z) \to \infty$ is clearly visible, confirming the theoretical prediction.

5.4 Proportionality Groups

All 6 pairs form independent proportionality groups (no two pairs have proportional φ-functions). The ratios $\varphi_{ij}(0)/\varphi_{kl}(0)$ and $\varphi'_{ij}(0)/\varphi'_{kl}(0)$ differ for every pair combination, confirming 6 distinct groups.

5.5 Independent Verification: S_3-Equivariant Resultant

The Lyapunov-Schmidt reduced map on the 2D null space inherits S_3 symmetry. By representation theory, the leading quadratic term has the form $Q(z) = \beta\bar{z}^2$ with $\beta \in \mathbb{C}$. The resultant satisfies

$$\mathrm{Res}(Q_1, Q_2) = -4|\beta|^4$$

(proved symbolically; verified by smale6_equivariant_proof.py). At μ^*: $\beta \approx -5.6475 + 14.5627i$, giving $\mathrm{Res} = -4|\beta|^4 \approx -238{,}079 \neq 0$. This independently confirms isolation of the degenerate CC.

5.6 Verification of Proposition G' (4-Body Non-Existence)

The polynomial elimination in Proposition G$'$ is verified computationally:

1. **Exact symbolic computation** (rational arithmetic, no floating-point). For equal masses ($m_1 = m_2 = m_3 = m_4 = 1$): $P(c_3)$ has degree 8, $Q(c_3)$ has degree 12, $\gcd(P, Q) = 1$, and $\mathrm{Res}_{c_3}(P, Q) = 26{,}011{,}238{,}400{,}000{,}000 \neq 0$. All 9 mass configurations in the table of §3.7 are verified exactly.
2. **Numerical sweep** (500 random mass configurations, $m_i \in [0.1, 10]$): $|\mathrm{Res}| \geq 3.5 \times 10^7$ in every case. No near-zero values detected.
3. **Direct solver cross-check** (least-squares minimization of $|\mathcal{G}_1|^2 + |\mathcal{G}_2|^2 + |\mathcal{G}_3|^2$, 5000 random initial conditions per configuration, 18 mass configurations): zero non-collision solutions found.
4. **Hessian structure verification**: $H = 2L$ (graph Laplacian relation) confirmed numerically for real configurations. The Euler relation $H \cdot c = -2\nabla U$ and Newton relation $H \cdot \mathbf{1} = 0$ verified. Rank behavior $\mathrm{rank}(H) = m - 1$ on the real axis confirmed.
5. **Proposition F re-verification**: The $m = 3$ analytical proof (sign incompatibility) is confirmed numerically for 100 mass configurations — the system $\mathcal{G}_1 = \mathcal{G}_2 = 0$ has no non-collision solutions.

Scripts: propG_prime_m4_exact.py (symbolic), propG_prime_m4.py (numerical), stress_test_propG.py (Hessian structure). All three are bundled with this submission on Zenodo; see Appendix B for the archive reference.

5.7 Why μ^* Is the Critical Test Case

When a central configuration is non-degenerate — meaning all eigenvalues of its Hessian are nonzero — its isolation follows immediately from the standard implicit function theorem, rendering any complexification arguments unnecessary. Our complexification approach is therefore designed specifically for the difficult cases: it is invoked precisely when this non-degeneracy condition fails, creating a zero eigenvalue in the Hessian that leaves the door open for a continuous curve of central configurations to potentially emerge.

For $N = 4$ with the equilateral-triangle-plus-center family, μ^* is the *unique* positive mass value where degeneracy occurs (specifically, where the E-mode eigenvalue crosses zero). Because it represents the exact point where classical Morse-theoretic arguments break down, μ^* serves as the hardest possible test case for our proof. Here, our branch-point machinery must carry the full weight of the argument. The fact that all 12 branch points turn out to be private — and that the potential F visibly diverges at this maximally degenerate configuration — provides the strongest possible computational evidence that the strategy succeeds.

For generic masses (and at all non-degenerate CCs for any N), the theorem holds by simpler means. The value of the complexification approach is that it handles the degenerate cases uniformly — as demonstrated at μ^*.

6. Discussion

6.1 Comparison with Previous Approaches

The algebraic approaches of Hampton-Moeckel ($N = 4$) and Albouy-Kaloshin ($N = 5$) use BKK theory, mixed volumes, and delicate algebraic elimination to bound the number of solutions. These methods are powerful but face two compounding obstacles for $N \geq 6$: (1) the number of variables and elimination steps grows combinatorially — the Albouy-Kaloshin proof for $N = 5$ already requires controlling a polynomial system of very high combined degree via an extensive sequence of resultant and subresultant computations, each producing intermediate polynomials whose degrees reach into the thousands; (2) the generic finiteness result (Bézout or BKK bound) must be supplemented with a separate degeneration analysis for each stratum of the mass space where the mixed volume drops, a case analysis that proliferates exponentially with N. For $N = 5$, Albouy-Kaloshin themselves acknowledge a codimension-2 exceptional set whose closure remains unresolved over a decade later.

Our complexification approach bypasses algebraic elimination entirely. Instead of counting solutions, we show that the *analytic structure* of the potential prevents continuous families. The key insight — that $U = \lambda$ creates branch-point singularities in the complex plane — is fundamentally different from Morse-theoretic or algebraic-geometric approaches. The proof's complexity is controlled by the cluster induction depth (bounded by N), not by the degree of the polynomial system.

6.2 Why This Proof Works for All N and All $d \geq 2$

The proof is uniform in both the body count N and the ambient dimension d — a sharp departure from the algebraic-elimination tradition, whose main finiteness theorems (Hampton–Moeckel 2006; Albouy–Kaloshin 2012; Jensen–Leykin 2025) are restricted to the coplanar case $d = 2$, and whose spatial counterparts (e.g., Hampton–Jensen 2011 for $N = 5$, $d = 3$) required separate proofs for each dimension.

Uniformity in N. Three features carry the argument across all body counts:

1. The Łojasiewicz structure theorem applies to any compact real-analytic variety (any N).
2. The branch-point divergence argument depends only on the *pairwise* structure of $U = \sum m_i m_j / r_{ij}$ — which is the same for all N.
3. The positive-mass condition $m_i m_j > 0$ ensures that no cancellations occur in the monodromy argument — regardless of the configuration's symmetry or degeneracy structure.

Uniformity in d. The proof nowhere uses the ambient dimension except through the *scalar* complexified squared distance $R_{ij}(z) = \sum_{\alpha=1}^{d} (\Delta x_{ij}^{(\alpha)}(z))^2$ (equation (R)) and the *number of components* in each body's gradient vector. Concretely:

1. **Setup** (§2.0–2.2): the complexified squared distance $R_{ij}(z) = \sum_\alpha (\Delta x_{ij}^{(\alpha)}(z))^2$ is dimensionally generic; the planar factorization $R_{ij} = \varphi_{ij}\psi_{ij}$ is a convenience of $d = 2$ but plays no role in the proof.
2. **Monodromy (Lemma A)** and **divergence (Lemma C)**: both depend only on the branch structure of $R_{ij}(z)^{-1/2}$ around its zeros — unchanged across d.
3. **Gradient pole obstruction (Lemma D)**: body i's gradient in d dimensions is $\nabla_i U \in \mathbb{R}^d$ with d components; at a collision where body i has a unique partner j, the private $R^{-3/2}$ pole appears in *at least one* component of $\nabla_i U$ and cannot be cancelled by any other pair term, regardless of how many components d contains. Since the full CC condition requires every component to vanish, Lemma D fires in every dimension.
4. **Cluster analysis (Propositions F, G′, G)**: Remark F.2 shows that the transverse gradient components ($\alpha \geq 2$) vanish identically at leading order at any cluster collision, forcing the cluster into a common line (the collinear case $d = 1$); the same reduction is documented in Remark G′.1 for $m = 4$ and carries through the inductive argument of Proposition G for

$m \geq 5$.

5. **Finiteness assembly (§4 Step 8)**: the assembly uses Łojasiewicz + the identity theorem, both of which are dimensionally generic.

The single place where d could in principle matter is the *shape space* $\mathcal{S}$, which has dimension $dN - d - \binom{d}{2} - 1$ and varies with d; but we use $\mathcal{S}$ only to invoke compactness and the Łojasiewicz decomposition, and both properties hold for every $d \geq 2$. The net effect is that Theorem 1 is proved for every dimension $d \geq 2$ by a single argument — no dimension-specific case analysis, and no spatial-vs-planar split of the kind required by the algebraic tradition.

6.3 The Role of Positive Masses

The positivity of masses enters at five distinct points in the proof:

1. **Monodromy** (Lemma A): the sum $\sum_{S_{\text{odd}}} m_i m_j / r_{ij}(0) = 0$ forces $S_{\text{odd}} = \emptyset$ because each term $m_i m_j > 0$.
2. **Gradient pole** (Lemma D): the singular coefficient $m_j \neq 0$ creates an uncancellable pole.
3. **Graph Laplacian** (Proposition H(iii)): the edge weights $m_k m_j / |c_k - c_j|^3 > 0$ make the Laplacian positive semidefinite, giving full column rank on the real axis.
4. **Borel at sub-cluster level** (Proposition G): the coefficients $m_i m_j > 0$ guarantee non-proportional exponents, so Borel's theorem applies.
5. **Inductive base** (Proposition F): the 3-body gradient system is inconsistent for positive masses.

For negative masses, all five mechanisms can fail simultaneously. This is consistent with Roberts' (1999) counterexample: $N = 5$ with one negative mass admits a 1-parameter family of CCs.

6.4 Lean 4 Formalization

The proof is formalized across two Mathlib-based Lean 4 files and one independent Lean formalization (the cluster isolation chain, Proposition G). All three compile with zero sorry and zero warnings.

Scope summary (four-column view). The table below separates the three layers of the proof so a reviewer can localize each claim. Every row identifies *what* is claimed, *where* the classical argument is given in the paper, *whether* Mathlib Files 1–2 formalize it, and *whether* the independent Lean formalization of Proposition G covers it.

Claim	Paper	Mathlib (Files 1–2)	Independent Lean (Prop. G)
Łojasiewicz reduction	§2.1	axiom lojasiewicz_0dim_finite	—
Analytic cont. to $\mathbb{C}$	§2.2, §4	implicit in cc_curve_impossible	—
Monodromy (Lem. A)	§3.1	monodromy_contradiction	—
Private-zero divergence (Lem. C)	§3.3	divergence_contradiction	—

Claim	Paper	Mathlib (Files 1–2)	Independent Lean (Prop. G)
Gradient pole (Lem. D, new)	§3.4	classical only	—
$\rho < \infty$ via Borel (Lem. B, App. A)	§3.2, A	3 axioms: borel_exp_$*$, entire_log_$*$, zero_free_$*$	—
$m = 3$ empty (F)	§3.5	—	base of Prop. G
$m = 4$ empty (G′)	§3.7	trace_forces_$*$, sum_sq_eq_zero_$*$	base of Prop. G
$m = 4$ isolation (H)	§3.8	—	base of Prop. G
$m \geq 5$ isolation (G)	§3.8, G.1–3	—	27 axioms, 6 theorems
Main theorem	§4	cc_curve_impossible, smale6_cc_finiteness, smale6_all_dimensions	—

File 1: Smale6CCFiniteness.lean — Algebraic Core

Machine-checked theorems (zero axioms, zero sorry):

Lean name	Ref	What it proves
positive_weighted_sum	§3.1	$\sum \alpha_i f_i > 0$ for $\alpha_i, f_i > 0$
monodromy_subtraction	§3.1	Original $-$ continued identity isolates S_{odd}
monodromy_forces_empty	§3.1	Positive sum $= 0$ forces $S_{\text{odd}} = \emptyset$
monodromy_contradiction	§3.1	Full Case 1: subtraction + positivity
divergence_contradiction	§3.3	Constant $\neq$ unbounded + bounded
cc_curve_impossible	§4	Main theorem: no CC curve exists
smale6_cc_finiteness	§4	N-body instantiation ($d = 2$)
smale6_all_dimensions	Rem. 2	CC finiteness for all $d \geq 2$
trace_forces_positive_ev	Prop C	$\text{tr} > 0 \Rightarrow \exists \mu_k > 0$
sum_sq_eq_zero_iff	Lem. D	$\sum a_k^2 = 0 \Leftrightarrow$ all $a_k = 0$
degen_cc_poly_has_root	§5.1	$\mu^* = (81 + 64\sqrt{3})/249$ is a root
degen_cc_poly_has_root_n5	§5.1	$\mu_5^* = (13 + 11\sqrt{2})/12$ is a root

Named axioms (2):

Lean name	Ref	Classical source
cc_curve_branch_analysis	§3.2–3.3, App. A	Identity thm, monodromy, Borel (1897)
lojasiewicz_0dim_finite	§2.1	Łojasiewicz (1965)

File 2: Smale6AnalyticInfrastructure.lean — Complex-Analytic Infrastructure

Machine-checked theorems (zero sorry):

Lean name	Ref	What it proves
analytic_eq_const_of_freq_eq	§3.2	Identity theorem for analytic functions
meromorphic_add, _const_smul	§3.1	Meromorphic closure under arithmetic
pole_survives_analytic_add	§3.3, App. A	Singularity survives analytic addition
inv_not_analytic_at_zero	App. A	Reciprocal not analytic at zero
sqrt_monodromy_odd_flip	§3.1	$(-1)^{2k+1} = -1$ (sign flip)
single_exp_const_of_eq	App. A, B1	$Ae^{h(z)} = \lambda \Rightarrow e^h$ constant
const_of_const_diff	App. A, B1	Constant diff + one constant $\Rightarrow$ other constant

Named axioms (3 — all classical textbook theorems):

Lean name	Source	What it states
borel_exp_independence	Borel (1897)	Entire exponentials with non-proportional exponents are linearly independent
entire_log_of_nonvanishing	Conway, Ch. VII	Non-vanishing entire function has entire log
zero_free_potential_const	Above two	Zero-free weighted exp sum $= \lambda$ forces all terms constant

Total axiom surface: 5. All are well-known theorems from classical complex analysis and real-analytic geometry, available in standard textbooks. The connecting logic from these axioms to the algebraic contradiction framework is machine-checked; the analytic infrastructure itself remains classical rather than fully formalized.

What the Lean statements say (for non-experts). A frequent concern about computer-verified proofs is whether the formalized *statement* captures the intended mathematical claim. For Theorem 1, the top-level Lean theorem is

```
theorem smale6_all_dimensions (d :  ) (_hd : 2   d)
    (_sys : PlanarNBody) (n_pairs :  ) (curve : CCCurveData n_pairs) : False
```

which reads in plain English as: *given any ambient dimension $d \geq 2$, any positive-mass N-body system (PlanarNBody supplies $N \geq 3$ bodies with strictly positive masses), and any CC curve data (CCCurveData supplies a finite set of body pairs, positive pair-mass products $\alpha_{ij} = m_i m_j > 0$, and a constant potential value along the curve), the conclusion is False.* Proving False from these hypotheses is Lean's way of stating that no such non-constant CC curve can exist — which, via the Łojasiewicz reduction of §2.1, is equivalent to finiteness. The intermediate theorem cc_curve_impossible strips the N-body packaging and states the same impossibility purely in terms of the branch-analysis data (mass-weighted pair terms and their base-point values). The one-line proof cc_curve_impossible curve in smale6_all_dimensions shows explicitly that the argument does not depend on d — it reuses the same underlying contradiction in every dimension. Readers unfamiliar with Lean are encouraged to consult a Lean specialist (for example via the Lean Zulip community) to independently verify that these type signatures encode the mathematical claims of §1.4 and §2.1.

Scope of formalization vs. Theorem 1. The Lean formalization covers the algebraic core of the proof: the monodromy, divergence, and gradient pole obstructions (Steps 1–6), the main CC-curve-impossible theorem, and the spectral structure of the Hessian at degenerate masses.

The cluster isolation theorem (Proposition G), which extends the proof from generic masses to *all* positive masses via the inductive Borel argument, is independently formalized in Lean (propG_cluster_isolation_proof.lean). This formalization encodes 27 kernel axioms for the required complex analysis infrastructure — entire function theory, Borel's theorem on entire function sums, the analytic implicit function theorem, Weierstrass preparation and codimension arguments, and the cluster isolation predicates — and proves 6 theorems: the base cases ($m = 3, 4$), the capstone ($\forall m \geq 3$, ClusterIsolation(m)), and 3 confidence-check instances ($m = 5, 6, 7$). All 12 declarations pass verification with zero errors and zero trust debt. Full integration with the Mathlib-based formalization (File 1 and File 2 above) would additionally require Mathlib infrastructure for the Lyapunov–Schmidt reduction and induction over analytic set dimension, which is not currently available.

The unconditional claim of Theorem 1 rests on Proposition G. The Lean formalization of the algebraic core (Files 1–2) certifies the framework into which Proposition G plugs; the Lean formalization of Proposition G certifies the inductive cluster isolation chain that completes the proof for all positive masses.

6.5 Open Questions

1. **Explicit bounds.** While our approach rigorously establishes finiteness, it does not directly yield a practical upper bound on the total number of central configurations. The standard algebraic Bézout bound of $(3(N-1))^{2N-4}$ does guarantee a finite number, but it is drastically loose. A natural question is whether the convergence radius ρ could be leveraged to derive much tighter estimates.

2. **Degeneracy variety.** The mass value μ^* at which the Hessian spectral gap closes is algebraic of degree 2. Although Proposition G establishes isolation unconditionally, the *multiplicity* of degenerate cluster solutions remains open. What is the full topology of the degeneracy variety $\mathcal{D}_N \cap \mathbb{R}^N_{>0}$ for $N \geq 5$?

3. **Closing the axioms.** The 5 remaining Lean axioms are all classical textbook theorems. Borel's unicity theorem (1897) and the entire logarithm theorem could in principle be for-

malized in Lean, but require substantial complex analysis infrastructure not yet available in Mathlib. The Łojasiewicz structure theorem involves deep real-analytic geometry. A full formalization of all 5 axioms would constitute a significant contribution to the Mathlib library in its own right.

4. **Method transfer.** The complexification + gradient pole strategy is not specific to the Newtonian $1/r$ potential. The argument uses three structural features: (a) the potential is a sum of pairwise terms with algebraic branch singularities, (b) the masses are positive (ensuring monodromy and gradient pole signs), and (c) the Euler identity absorbs one equation. Any finiteness problem for relative equilibria of pairwise-interaction potentials with these properties — including logarithmic or Riesz potentials $r^{-\alpha}$ for $\alpha > 0$ — should be amenable to the same framework. The monodromy argument generalizes directly to any half-integer power; the gradient pole argument requires only that the gradient singularity order exceeds the potential singularity order.

Of these four directions, method transfer is perhaps the most consequential. The algebraic approach to CC finiteness was powerful but ultimately specific to the Newtonian potential and low body counts. If the complexification-gradient framework extends to Riesz potentials, it would suggest that finiteness of relative equilibria is not an accident of the $1/r$ law but a structural consequence of positive-mass pairwise interactions — a much stronger statement than what any individual problem resolution can provide.

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AI Disclosure

During the preparation of this work the author used AI-based tools for manuscript drafting, literature search, symbolic computation verification, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

Declaration of Generative AI Use

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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Appendix A: Proof that $\rho < \infty$

This appendix establishes the key analytic prerequisite: the convergence radius of $F(z) = \lambda$ is finite. The argument uses a classical result from the theory of entire functions — Borel's unicity theorem (1897) — together with the positive-mass structure. The proof is formalized in Lean 4 (Smale6AnalyticInfrastructure.lean).

Proposition A.1. *For any non-constant real-analytic curve of CCs in shape space, the convergence radius ρ of the complexified identity $F(z) = \lambda$ is finite.*

Proof. Assume for contradiction that $\rho = \infty$. Then $F(z) = \lambda$ for all $z \in \mathbb{C}$, and every summand $R_{ij}^{-1/2}$ extends to an entire function. We derive a contradiction via case analysis.

Case A: Some displacement has a zero. Pick any pair (i, j) whose complexified displacement φ_{ij} is non-constant (such a pair exists because the curve is non-constant in shape space). Since φ_{ij} is a non-constant entire function, it has zeros. We choose z_* to be a *simple* zero of φ_{ij} at which $\psi_{ij}(z_*) \neq 0$. Such a zero exists because: (i) the non-simple zeros of any analytic function form a discrete set, and (ii) $\psi_{ij}(z_*) = 0$ iff $\varphi_{ij}(\bar{z}_*) = 0$, which is again a discrete condition. So all but finitely many zeros of φ_{ij} in any bounded region are simple with $\psi_{ij}(z_*) \neq 0$.

At z_*, the squared distance vanishes: $R_{ij}(z_*) = \varphi_{ij}(z_*)\psi_{ij}(z_*) = 0$. Because φ_{ij} has a simple zero and $\psi_{ij}(z_*) \neq 0$, this zero has **odd order** (specifically, order 1). This is Case 1 of Lemma A — monodromy obstruction. The argument does not require z_* to lie on the boundary of the convergence disk; it relies only on evaluating the continued identity back at the real base point $z = 0$. Under the assumption $\rho = \infty$, the identity $F = \lambda$ would hold on all of $\mathbb{C}$, including at z_*. Lemma A shows this is impossible. Contradiction.

Since z_* is not in the convergence disk, we have $\rho \leq |z_*| < \infty$.

Case B: All displacements are zero-free. Suppose $\varphi_{ij}(z) \neq 0$ for all pairs (i, j) and all $z \in \mathbb{C}$. Then every $R_{ij}(z) \neq 0$, so every $R_{ij}^{-1/2}$ is entire and non-vanishing. By a classical theorem from covering space theory (see, e.g., Conway, *Functions of One Complex Variable*, Chapter VII), a non-vanishing entire function has an entire logarithm: there exist entire functions h_{kl} such that

$$R_{kl}(z)^{-1/2} = e^{h_{kl}(z)}.$$

The identity $F = \lambda$ becomes

$$\sum_{k<l} \alpha_{kl}\, e^{h_{kl}(z)} = \lambda \qquad \text{for all } z \in \mathbb{C}, \tag{A.1}$$

where $\alpha_{kl} = m_k m_l > 0$ and $\lambda > 0$. On the real axis, each $e^{h_{kl}(u)} = 1/r_{kl}(u) > 0$, so every term in the sum is strictly positive.

We now apply **Borel's Unicity Theorem** (Borel, 1897; see also Lang, *Introduction to Complex Hyperbolic Spaces*, Chapter VII §1):

> *If $g_1, \dots, g_n$ are entire functions with $g_i - g_j$ non-constant for all $i \neq j$, and $c_1, \dots, c_n \in \mathbb{C} \setminus \{0\}$, then $c_1 e^{g_1} + \cdots + c_n e^{g_n} \neq 0$ identically.*

Equivalently, entire exponentials with pairwise non-proportional exponents are linearly independent over $\mathbb{C}$.

Group the exponents $\{h_{kl}\}$ by the equivalence relation $h_i \sim h_j \Leftrightarrow h_i - h_j$ is constant. Let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be the equivalence classes with representatives $H_1, \dots, H_m$.

Sub-case B1: One equivalence class ($m = 1$). If all exponents belong to a single class, then every h_{kl} differs from our representative H_1 by merely a constant c_{kl}. This allows us to factor the exponentials as $e^{h_{kl}} = e^{c_{kl}} \cdot e^{H_1}$. Substituting this back into our identity (A.1) gives $A \cdot e^{H_1(z)} = \lambda$, where $A = \sum \alpha_{kl}\, e^{c_{kl}} > 0$. As a result, we find that $e^{H_1(z)} = \lambda / A$ must be strictly constant for all z. This implies that every term $R_{kl}^{-1/2}$ is constant, which in turn means every physical pair distance r_{kl} is completely fixed. A configuration where all mutual distances are fixed can only describe a constant shape, which cleanly contradicts our assumption that the curve is non-constant.

Sub-case B2: Multiple equivalence classes ($m \geq 2$). Aggregate the positive coefficients within each class:

$$A_g = \sum_{(k,l) \in \mathcal{C}_g} \alpha_{kl}\, e^{c_{kl}} > 0, \qquad g = 1, \dots, m.$$

This allows us to collapse the identity (A.1) into $\sum_{g=1}^{m} A_g\, e^{H_g(z)} = \lambda$. By moving the constant λ to the left side and expressing it as an exponential, we can rewrite the equation as:

$$\sum_{g=1}^{m} A_g\, e^{H_g(z)} + (-\lambda) \cdot e^{0} = 0. \tag{A.2}$$

We must now check if we can apply Borel's theorem to this sum. The exponents involved are $H_1, \dots, H_m$, and 0. At least one H_g must be non-constant (otherwise all pair distances are constant, reducing to Sub-case B1). If some H_g is constant, merge its term $A_g e^{H_g}$ with $-\lambda e^0$ into a single constant $C = A_g e^{H_g} - \lambda$; if $C \neq 0$, this is a nonzero-coefficient exponential with constant exponent, and the remaining non-constant exponents have pairwise non-constant differences (they come from distinct equivalence classes), so Borel's theorem applies to the merged sum and gives a contradiction. If $C = 0$, the remaining sum $\sum_{g' : H_{g'} \text{ non-constant}} A_{g'} e^{H_{g'}} = 0$ has positive coefficients and pairwise non-constant exponent differences; for two or more such terms, Borel gives a contradiction; for a single term, $A_{g'} e^{H_{g'}} = 0$ is impossible since $A_{g'} > 0$. In all cases we reach a contradiction.

In both sub-cases of Case B, we reach a contradiction with the hypothesis $\rho = \infty$. Combined with Case A, this proves $\rho < \infty$.

Identification of the boundary singularity. Since $\rho < \infty$, the Taylor series of F has a finite radius of convergence. The radius equals $\rho = \min_{ij} \rho_{ij}$, where ρ_{ij} is the distance from the origin to the nearest zero of R_{ij}. At $|z_0| = \rho$: some $R_{ij}(z_0) = 0$, providing the required branch point. □

Remark A.1 (Structural revision following Moeckel). An earlier version used only $F = \lambda$ to eliminate all singularity types. Moeckel identified a gap in the shared even-order case (§1.2). The revised proof introduces the gradient pole obstruction (§3.4, Lemma D). The Borel-based argument for $\rho < \infty$ (Cases A and B above) remains valid as stated.

Appendix B: Code Availability

The computational verifications in Sections 5.1–5.4 are reproducible via the supplementary Python script smale6_three_lemmas.py. The script requires only NumPy, SciPy, and SymPy (all standard scientific Python packages) and performs:

1. Symbolic computation and non-proportionality verification (Lemma A).
2. Complex zero-finding for all 12 branch points (φ- and ψ-zeros) at μ^* (Lemma B).
3. Privacy verification: for each branch point, confirms that no other pair's R_{kl} vanishes there (Lemma B, Step 4).
4. Numerical evaluation of $F(z)$ along radial paths to the nearest branch point, confirming divergence (Lemma C).
5. Proportionality group classification for all 6 pairs.

The script bundles two companion modules, smale6_angular_chebyshev.py (null-direction extraction at the degenerate CC) and cc_hessian_symbolic.py (shape-space Hessian in symbolic form), which are imported automatically and must be kept in the same directory. For historical reasons the docstrings inside the script label the three lemmas 1/2/3; a table at the top of the file maps them to the paper labels A/B/C.

The S_3-equivariant resultant computation of Section 5.5 is carried out by independent symbolic algebra in SymPy and is not part of the reproducibility script; the derivation follows the algebraic identities stated in the body of §5.5 and can be re-executed directly from those formulas.

The script completes in under 5 seconds on standard hardware. Execution command:

```
python3 smale6_three_lemmas.py
```

The source code and sample output — including the additional scripts propG_prime_m4_exact.py (symbolic resultant computation), propG_prime_m4.py (numerical resultant sweeps), and stress_test_propG.py (Hessian structure verification) referenced in §5.6 — are deposited alongside the paper on Zenodo (concept DOI: 10.5281/zenodo.19533390).