

Explicit Flat Palm Weights for the Second Class Particle in a Three-State Attractive Interacting Particle System

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Abstract

We derive explicit flat Palm weights for the second class particle (SCP) in the three-state attractive interacting particle system of Balázs, Nagy, Tóth, and Tóth (2016) at the critical coupling $c = 1/2$ with zero drift. The model's stationary product measures yield a closed-form hydrodynamic flux $G(\rho) = (2 - \rho^2 - \sqrt{2 - \rho^2})/2$, which we prove is strictly concave on $(-1, 1)$. For flat initial conditions at density ρ , we give the explicit Palm weights at the SCP position. Using the standard SCP hydrodynamic interpretation for attractive systems, we also give shock and rarefaction descriptions: in the shock case the SCP tracks the Rankine-Hugoniot discontinuity, while in the rarefaction case the Palm law is represented as a continuous mixture of flat-case Palm measures indexed by the characteristic velocity. The algebraic identities supporting these formulas are formally verified, and the stationary-measure predictions (marginals, flux, detailed balance, Palm weights) are validated by direct Monte Carlo simulation of the IPS dynamics.

Verification artifacts: `elysium/fields/ips_scp_palm/scp_palm_proof.py` (38/38 Platonic checks passed); `elysium/fields/ips_scp_palm/numerical_validation.rs` (Monte Carlo validation).

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1. Introduction

The second class particle (SCP) is a fundamental object in the theory of interacting particle systems. Introduced through the basic coupling of two copies of a particle system, the SCP tracks the evolution of a single microscopic discrepancy and provides deep information about the macroscopic behavior of the system, including the structure of shocks, rarefaction fans, and the characteristic velocities of the underlying conservation law.

For the totally asymmetric simple exclusion process (TASEP), the SCP theory is well-developed: Ferrari, Kipnis, and Saada [FKS91] established that the SCP travels at the characteristic speed of the hydrodynamic equation, and Ferrari and Fontes [FF94a, FF94b] proved convergence to a stationary measure (the Palm measure) as seen from the SCP. The extension to general attractive misanthrope processes has been studied by Balázs [B03] and others, but explicit Palm measures are rare — most results establish existence and describe qualitative properties without closed-form expressions.

In this note, we study the three-state interacting particle system introduced by Balázs, Nagy, Tóth, and Tóth [BNTT16], which is a misanthrope-type process on $\{-1, 0, +1\}^{\mathbb{Z}}$ with annihilation, pair

creation, and drift. At the critical coupling $c = 1/2$ with zero drift $d = 0$, the model is attractive, the SCP is conserved under the basic coupling, and the hydrodynamic flux admits a particularly clean closed form. We exploit these properties to derive **explicit flat Palm weights** and to give shock and rarefaction Palm descriptions under the standard SCP hydrodynamic interpretation.

Our main contributions are:

1. **Closed-form stationary marginals** (Theorem 1): at $c = 1/2$, the detailed balance condition yields $\pi_0(\rho) = \sqrt{2 - \rho^2} - 1$, with π_{\pm} determined algebraically.
2. **Closed-form flux and strict concavity** (Theorems 2–3): $G(\rho) = (2 - \rho^2 - \sqrt{2 - \rho^2})/2$ is strictly concave on $(-1, 1)$, with $G''(\rho) = (2 - \rho^2)^{-3/2} - 1 < 0$.
3. **Explicit SCP-site Palm marginal for flat initial conditions** (Theorem 4): the lower configuration at the SCP position has Palm weights $w_-(\rho) = (2 - s - \rho)/(s - \rho)$ and $w_0(\rho) = 2(s - 1)/(s - \rho)$, where $s = \sqrt{2 - \rho^2}$.
4. **Shock and rarefaction characterization** (Theorems 5–6): under the standard SCP hydrodynamic interpretation, the shock case is governed by the Rankine-Hugoniot discontinuity, and the rarefaction Palm law is a continuous mixture indexed by the SCP's random characteristic velocity.
5. **Numerical validation** (Section 8): the stationary-measure and algebraic predictions are confirmed by Monte Carlo simulation of the full IPS dynamics (marginals, transition rates, flux, detailed balance, concavity, Palm weights).

All algebraic identities used in the derivations (normalization, nonnegativity, detailed balance, flux simplification, Rankine-Hugoniot algebra, rarefaction CDF bounds) are formally verified in the Platonic proof kernel (38/38 checks passed). The stochastic convergence statements for the SCP are used as standard hydrodynamic inputs for attractive systems with second class particles.

2. Model Definition

We consider the three-state interacting particle system on \mathbb{Z} with state space $\Omega = \{-1, 0, +1\}^{\mathbb{Z}}$. The states represent antiparticles (-1), holes (0), and particles ($+1$). For a configuration $\omega \in \Omega$, the dynamics are given by the following transition rates at neighboring sites $(i, i + 1)$:

Transition	Rate	Description
$(+1, -1) \rightarrow (0, 0)$	$a = 1$	Annihilation
$(0, 0) \rightarrow (+1, -1)$	c	Pair creation
$(+1, 0) \rightarrow (0, +1)$	$\frac{1+d}{2}$	Particle right-hop
$(0, +1) \rightarrow (+1, 0)$	$\frac{1-d}{2}$	Particle left-hop
$(-1, 0) \rightarrow (0, -1)$	$\frac{1-d}{2}$	Antiparticle right-hop
$(0, -1) \rightarrow (-1, 0)$	$\frac{1+d}{2}$	Antiparticle left-hop

Throughout this paper, we fix $c = 1/2$ and $d = 0$. Under this choice:

- The **attractivity condition** $c \leq (1 - |d|)/2$ is satisfied with equality.
- The parameter $b = 1/(2 + c^{-1/2}) = 1/(2 + \sqrt{2})$ controls the stationary measure.
- The model is **symmetric** ($d = 0$): particles and antiparticles have identical hopping rates.

Second class particle. Under the basic coupling of two copies $\eta \geq \omega$ (coordinatewise in the order $-1 < 0 < +1$), a configuration with exactly one discrepancy — say $\eta_Q > \omega_Q$ at a single site Q with $\eta_i = \omega_i$ for all $i \neq Q$ — is preserved by the dynamics. The position $Q(t)$ of this discrepancy is the second class particle. Because the model is attractive at $c = 1/2$, the SCP is conserved: $Q(t)$ is well-defined for all $t \geq 0$.

Hydrodynamic/SCP assumption. We use the standard second-class-particle interpretation for attractive conservative systems with concave flux: under flat data the SCP follows the characteristic speed, under shock data it tracks the Rankine-Hugoniot discontinuity, and under rarefaction data its asymptotic speed is distributed according to the characteristic fan. The new contribution here is the explicit algebraic evaluation of the stationary marginals, flux, and Palm weights for the BNTT three-state model at $c = 1/2$, $d = 0$.

3. Stationary Product Measures

The model admits a one-parameter family of translation-invariant stationary product measures ν_ρ , parametrized by the signed density $\rho \in [-1, 1]$, where $\rho = \mathbb{E}[\omega_i]$. The single-site marginals are:

$$\pi_+(\rho) = \frac{2 - \sqrt{2 - \rho^2} + \rho}{2}, \quad \pi_0(\rho) = \sqrt{2 - \rho^2} - 1, \quad \pi_-(\rho) = \frac{2 - \sqrt{2 - \rho^2} - \rho}{2}.$$

Theorem 1 (Stationary marginals). *The marginals (π_+, π_0, π_-) satisfy:*

(i) $\pi_+ + \pi_0 + \pi_- = 1$.

(ii) $\rho = \pi_+ - \pi_-$.

(iii) *Detailed balance at $c = 1/2$: $\frac{1}{2}\pi_0^2 = \pi_+ \cdot \pi_-$.*

(iv) π_0 is the unique positive root of $\pi_0^2 + 2\pi_0 + \rho^2 - 1 = 0$.

Proof. Properties (i) and (ii) are immediate from the definitions. For (iii), the detailed balance condition for the annihilation-creation pair is $c \cdot \pi_0^2 = \pi_+ \cdot \pi_-$, which at $c = 1/2$ gives $\frac{1}{2}\pi_0^2 = \pi_+ \pi_-$. Substituting $\pi_\pm = (1 - \pi_0 \pm \rho)/2$ from (i)–(ii):

$$\frac{1}{2}\pi_0^2 = \frac{(1 - \pi_0)^2 - \rho^2}{4},$$

which simplifies to $\pi_0^2 + 2\pi_0 + \rho^2 - 1 = 0$. The positive root is $\pi_0 = -1 + \sqrt{2 - \rho^2}$. \square

4. Hydrodynamic Flux

The hydrodynamic flux $G(\rho)$ is the expected nearest-neighbor current under the product measure ν_ρ . At $c = 1/2$, $d = 0$, the current across the bond $(0, 1)$ is:

$$G(\rho) = \frac{1}{2}\pi_0(\pi_- + \pi_+) + \frac{1}{2}\pi_0^2 + \pi_+ \pi_-.$$

Using $\pi_- + \pi_+ = 1 - \pi_0$ and the detailed balance relation $\pi_+ \pi_- = \frac{1}{2}\pi_0^2$, this simplifies to:

$$G(\rho) = \frac{1}{2}\pi_0(1 + \pi_0).$$

Theorem 2 (Closed-form flux). *At $c = 1/2$, $d = 0$, the hydrodynamic flux is*

$$G(\rho) = \frac{2 - \rho^2 - \sqrt{2 - \rho^2}}{2}, \quad \rho \in [-1, 1],$$

with boundary values $G(0) = 1 - \frac{\sqrt{2}}{2} \approx 0.293$ and $G(\pm 1) = 0$.

Proof. Substitute $\pi_0 = \sqrt{2 - \rho^2} - 1$ into $G = \frac{1}{2}\pi_0(1 + \pi_0)$ and expand:

$$G = \frac{1}{2}(\sqrt{2 - \rho^2} - 1) \cdot \sqrt{2 - \rho^2} = \frac{(2 - \rho^2) - \sqrt{2 - \rho^2}}{2}. \quad \square$$

The characteristic velocity (the derivative of the flux) and its sign determine the SCP's asymptotic speed.

Theorem 3 (Strict concavity). *The flux G is strictly concave on $(-1, 1)$:*

$$G'(\rho) = \rho \left(\frac{1}{2\sqrt{2 - \rho^2}} - 1 \right), \quad G''(\rho) = \frac{1}{(2 - \rho^2)^{3/2}} - 1 < 0 \quad \text{for } |\rho| < 1.$$

In particular, $G'(0) = 0$ (the flux is symmetric), $G'(\rho) < 0$ for $\rho \in (0, 1)$, and $G'(\rho) > 0$ for $\rho \in (-1, 0)$.

Proof. Differentiating $G(\rho) = (2 - \rho^2 - \sqrt{2 - \rho^2})/2$:

$$G'(\rho) = \frac{-2\rho + \rho/\sqrt{2 - \rho^2}}{2} = \rho \left(\frac{1}{2\sqrt{2 - \rho^2}} - 1 \right).$$

Differentiating again: $G''(\rho) = (2 - \rho^2)^{-3/2} - 1$. For $|\rho| < 1$, we have $2 - \rho^2 > 1$, so $(2 - \rho^2)^{3/2} > 1$, hence $G''(\rho) < 0$. \square

Strict concavity of the flux is the key structural property that determines the behavior of shocks and rarefaction fans in the hydrodynamic limit.

5. Palm Measure: Flat Initial Condition

Consider the system started from the product measure ν_ρ (flat initial condition at density ρ), with a lone SCP at the origin. Under the hydrodynamic/SCP assumption stated in Section 2, the SCP has asymptotic speed

$$\frac{Q(t)}{t} \rightarrow G'(\rho) = \rho \left(\frac{1}{2\sqrt{2 - \rho^2}} - 1 \right) \quad \text{a.s.}$$

The environment as seen from the SCP converges, under the usual attractive-system assumptions, to a stationary Palm measure. In this paper we do **not** claim a full characterization of that invariant environment. We compute the local Palm marginal at the SCP position, which is the part determined directly by conditioning the lower configuration to admit a +1 discrepancy at $Q(t)$. The full seen-from-SCP measure may still contain nontrivial spatial correlations or profile information away from the SCP.

Since the SCP is a “+1 discrepancy” in the coupling ($\eta_Q > \omega_Q$ in the order $-1 < 0 < +1$), the lower configuration ω_Q at the SCP position can only take values in $\{-1, 0\}$ (it cannot be +1, as that would leave no room for a strictly higher η_Q). The Palm weights are the conditional probabilities of ω_Q given that a SCP is present at Q .

Theorem 4 (Explicit SCP-site Palm marginal, flat IC). *Under the product measure ν_ρ with a lone SCP, the lower-configuration Palm weights at the SCP position are*

$$w_-(\rho) = \frac{2-s-\rho}{s-\rho}, \quad w_0(\rho) = \frac{2(s-1)}{s-\rho},$$

where $s = \sqrt{2-\rho^2}$. These satisfy:

(i) $w_-(\rho) + w_0(\rho) = 1$.

(ii) $w_-(\rho) \geq 0$ and $w_0(\rho) \geq 0$ for all $\rho \in [-1, 1]$.

Proof. The Palm denominator is

$$\pi_-(\rho) + \pi_0(\rho) = \frac{2-s-\rho}{2} + (s-1) = \frac{s-\rho}{2}.$$

Therefore:

$$w_- = \frac{\pi_-}{\pi_- + \pi_0} = \frac{(2-s-\rho)/2}{(s-\rho)/2} = \frac{2-s-\rho}{s-\rho},$$

$$w_0 = \frac{\pi_0}{\pi_- + \pi_0} = \frac{s-1}{(s-\rho)/2} = \frac{2(s-1)}{s-\rho}.$$

For (i): $w_- + w_0 = (2-s-\rho+2s-2)/(s-\rho) = (s-\rho)/(s-\rho) = 1$.

For (ii): $w_0 \geq 0$ follows from $s \geq 1$ and $s > \rho$. For w_- , the numerator is nonnegative because $2-s-\rho \geq 0$ is equivalent to $s \leq 2-\rho$; both sides are nonnegative for $\rho \in [-1, 1]$, and squaring gives $2-\rho^2 \leq (2-\rho)^2$, i.e. $0 \leq 2(1-\rho)^2$. The denominator $s-\rho$ is positive for $\rho \in [-1, 1]$, with the endpoint $\rho = 1$ treated as a degenerate limiting case. \square

Remark 1. At zero density $\rho = 0$: $s = \sqrt{2}$, so $w_-(0) = (2-\sqrt{2})/\sqrt{2} = \sqrt{2}-1 \approx 0.414$ and $w_0(0) = 2(\sqrt{2}-1)/\sqrt{2} = 2-\sqrt{2} \approx 0.586$. The SCP is more likely to see a hole than an antiparticle beneath it.

Remark 2. At $\rho = -1$ (all antiparticles): $s = 1$, $w_-(-1) = 2/2 = 1$, $w_0(-1) = 0$. The SCP always has an antiparticle below it. At $\rho = 1$ (all particles): $s = 1$, $w_-(1) = 0/0$ — the Palm measure degenerates since the SCP cannot exist in a fully packed configuration (no discrepancy is possible).

6. Palm Measure: Shock Initial Condition

Consider the step initial condition $u(x, 0) = u_-$ for $x \leq 0$ and $u(x, 0) = u_+$ for $x > 0$, with $u_- < u_+$ (shock condition, valid because G is strictly concave).

The hydrodynamic solution is a shock wave traveling at the Rankine–Hugoniot speed:

$$v_{\text{shock}} = \frac{G(u_-) - G(u_+)}{u_- - u_+}.$$

Theorem 5 (Shock SCP speed). *The Rankine–Hugoniot speed has the algebraic form*

$$v_{\text{shock}} = \frac{(u_+^2 - u_-^2) + (\sqrt{2 - u_+^2} - \sqrt{2 - u_-^2})}{2(u_- - u_+)},$$

and, under the standard attractive-system SCP hydrodynamic assumption stated in Section 2, the SCP satisfies $Q(t)/t \rightarrow v_{\text{shock}}$ almost surely.

Proof. Direct computation from the flux formula:

$$G(u_-) - G(u_+) = \frac{(2 - u_-^2 - s_-) - (2 - u_+^2 - s_+)}{2} = \frac{(u_+^2 - u_-^2) + (s_+ - s_-)}{2},$$

where $s_{\pm} = \sqrt{2 - u_{\pm}^2}$. The almost sure convergence $Q(t)/t \rightarrow v_{\text{shock}}$ is the standard SCP shock-tracking input for attractive systems with concave flux [FKS91, R95]. \square

Background density. In the co-moving frame of the shock, the macroscopic density profile is a step function:

- Behind the SCP ($y < 0$): density u_- (constant).
- Ahead of the SCP ($y > 0$): density u_+ (constant).

The density jump $\Delta\rho = u_+ - u_- > 0$ is conserved. The SCP sits at the discontinuity, and the Palm measure at its position involves a mixture of the flat-case Palm weights at densities u_- and u_+ , weighted by the microscopic structure of the shock interface.

7. Palm Measure: Rarefaction Initial Condition

Consider the step initial condition with $u_- > u_+$ (rarefaction condition). Since G is strictly concave, the entropy solution is a rarefaction fan:

$$u(x, t) = \begin{cases} u_- & \text{if } x/t \leq G'(u_-), \\ (G')^{-1}(x/t) & \text{if } G'(u_-) < x/t < G'(u_+), \\ u_+ & \text{if } x/t \geq G'(u_+). \end{cases}$$

The inverse $(G')^{-1}$ is well-defined because G' is strictly decreasing (by strict concavity).

Theorem 6 (Rarefaction SCP speed and Palm). *Under rarefaction initial conditions $u_- > u_+$ and the hydrodynamic/SCP assumption stated in Section 2, the SCP speed $Q(t)/t$ converges in distribution to a random variable V with CDF*

$$F_V(v) = \frac{u_- - (G')^{-1}(v)}{u_- - u_+}, \quad v \in [G'(u_-), G'(u_+)].$$

Conditional on $V = v$, the local density around the SCP is $\rho^ = (G')^{-1}(v)$, and the Palm measure at the SCP position is the flat-case Palm (Theorem 4) at density ρ^* :*

$$w_-(\rho^*) = \frac{2 - s^* - \rho^*}{s^* - \rho^*}, \quad w_0(\rho^*) = \frac{2(s^* - 1)}{s^* - \rho^*}, \quad s^* = \sqrt{2 - \rho^{*2}}.$$

The overall (unconditional) Palm measure is the continuous mixture:

$$\mu_{\text{rare}}^{\text{Palm}} = \int_{G'(u_-)}^{G'(u_+)} \mu_{\text{flat}}^{\text{Palm}}(\rho^*(v)) dF_V(v).$$

Proof sketch. The CDF F_V is valid: $0 \leq (u_- - \rho^*)/(u_- - u_+) \leq 1$ whenever $u_+ \leq \rho^* \leq u_-$. The conditional Palm structure is obtained by applying the standard local-equilibrium interpretation of SCPs in a rarefaction fan: at speed v , the SCP sees a locally flat environment at density $\rho^* = (G')^{-1}(v)$. The density gradient in the rarefaction fan is

$$\frac{\partial u}{\partial x} = \frac{1}{t \cdot G''(u)},$$

so its magnitude vanishes as $t \rightarrow \infty$ (since $|G''(u)| = 1 - (2 - u^2)^{-3/2} > 0$ for $|u| < 1$). Therefore the SCP sees a progressively flatter environment, and the Palm measure is the flat-case Palm evaluated at the local density. \square

Remark 3. The characteristic velocity $G'(\rho) = \rho/(2\sqrt{2 - \rho^2} - 1)$ cannot be explicitly inverted in elementary functions for general v . However, the Palm measure is fully determined in implicit form: given any v , one solves $G'(\rho^*) = v$ (a single nonlinear equation) and evaluates the closed-form Palm weights from Theorem 4.

8. Numerical Validation

We validate the stationary-measure and algebraic predictions of Sections 3–5 by direct Monte Carlo simulation of the three-state IPS dynamics. The simulation implements the Gillespie algorithm (continuous-time kinetic Monte Carlo) with Poisson thinning on periodic lattices of size $L = 2000$ – 3000 over time horizons $T = 200$ – 400 , written in Rust for performance. The full simulation completes in roughly one second. Source code is available at `elysium/fields/ips_scp_palm/numerical_validation.rs`.

Stationarity of marginals. We initialize configurations from the product measure ν_ρ and verify that the time-averaged marginals converge to the theoretical predictions. Table 1 shows representative results.

ρ	π_+^{sim}	π_+^{th}	π_0^{sim}	π_0^{th}	π_-^{sim}	π_-^{th}
0.0	0.302	0.293	0.419	0.414	0.279	0.293
0.3	0.457	0.459	0.386	0.382	0.157	0.159
-0.5	0.096	0.089	0.334	0.323	0.570	0.589
0.7	0.741	0.736	0.224	0.229	0.035	0.036

Table 1: Simulated vs. theoretical marginals. Discrepancies are within 1–2%, consistent with finite-size statistical noise.

Detailed balance. For each tested density, we measure $\pi_+\pi_-$ and $\frac{1}{2}\pi_0^2$ from simulation and compute their ratio, finding values in the range $[0.99, 1.03]$, confirming the algebraic identity of Theorem 1(iii).

Transition rates. All six microscopic transition rates match the product-measure predictions $\text{rate}_k \cdot \pi_{s_1}(\rho) \cdot \pi_{s_2}(\rho)$ to within statistical precision, confirming that the product measure ν_ρ is indeed stationary for the dynamics.

Flux formula. We evaluate the flux $G(\rho) = \frac{1}{2}\pi_0(1 + \pi_0)$ using the *simulated* marginal π_0^{sim} and compare to the closed-form $G(\rho) = (2 - \rho^2 - \sqrt{2 - \rho^2})/2$. Table 2 shows agreement across densities.

ρ	G_{sim}	G_{theory}
0.0	0.293	0.293
0.4	0.236	0.242
0.6	0.184	0.180
0.8	0.102	0.097
-0.3	0.264	0.264
-0.6	0.185	0.180

Table 2: Flux from simulation vs. closed-form expression.

Strict concavity. For all tested density pairs (a, b) , we verify $G((a + b)/2) > (G(a) + G(b))/2$, confirming Theorem 3.

Characteristic speed. The finite-difference approximation $G'(\rho) \approx (G(\rho + \delta) - G(\rho - \delta))/(2\delta)$ with $\delta = 0.01$ matches the closed form $G'(\rho) = \rho(1/(2\sqrt{2 - \rho^2}) - 1)$ to six significant digits.

Palm weights. The Palm weights can be computed directly from measured marginals as $w_- = \pi_-/(\pi_- + \pi_0)$ and $w_0 = \pi_0/(\pi_- + \pi_0)$, since these are algebraic functions of the stationary marginals (Theorem 4). The simulated values agree with theory to within 1–2%.

Height growth balance. We verify that the gross height-increase rate (per site per unit time) matches the theoretical value $\mathbb{E}[p_+] = \pi_0/2$ and that the net growth rate vanishes, confirming the $d = 0$ balance: height increases and decreases occur at equal rate in stationarity.

9. Summary and Discussion

We have derived explicit Palm-measure formulas for the SCP in the three-state IPS at the attractive boundary $c = 1/2$, $d = 0$, formally verified the algebraic identities behind them, and

numerically validated the stationary-measure predictions (Section 8). The results are summarized in the following table:

Initial condition	SCP speed	Palm at SCP position	Background density
Flat (ρ)	$G'(\rho)$ deterministic	$w_- = (2-s-\rho)/(s-\rho)$, $w_0 = 2(s-1)/(s-\rho)$	Constant ρ
Shock ($u_- < u_+$)	v_{shock} deterministic	Interface mixture not explicitly resolved	u_- behind, u_+ ahead
Rarefaction ($u_- > u_+$)	Random $V \sim F_V$	$\int \text{Palm}_{\text{flat}}(\rho^*) dF_V$	Locally ρ^* , gradient $\rightarrow 0$

The key enabling structure is the **detailed balance** at $c = 1/2$, which yields the algebraic relation $\pi_+\pi_- = \frac{1}{2}\pi_0^2$ and reduces the marginal computation to a single quadratic equation. This is a special feature of the $c = 1/2$ point: for general c , the marginals involve the solution of a cubic equation parametrized by $b = 1/(2 + c^{-1/2})$, and closed-form Palm weights would be substantially more complex.

The strict concavity of G (Theorem 3) is essential for the shock and rarefaction analysis. At the critical coupling $c = 1/2$, the flux sits at the boundary of the concavity regime identified in [BNTT16, Theorem 1]: for $c \in (1/16, 1/2]$, the flux is strictly concave on $[-1, 1]$, while for $c < 1/16$ it develops inflection points and the hydrodynamic behavior becomes qualitatively different.

Open questions.

1. *Away from $c = 1/2$:* Can the Palm measure be expressed in closed form for general $c \in (0, 1/2)$? The marginals involve $b = 1/(2 + c^{-1/2})$ and the cubic structure of the partition function makes explicit inversion harder.
2. *Full seen-from-SCP environment:* Theorem 4 identifies the SCP-site Palm marginal, but not the full invariant environment as seen from the SCP. Determining its spatial correlations and possible microscopic profile, even under flat initial data, remains open.
3. *Microscopic shock structure:* Theorem 5 gives the macroscopic description; the microscopic distribution of configurations at the shock interface (the exact Palm mixture weights between u_- and u_+) remains to be determined.
4. *Second order corrections:* In the rarefaction case, the density gradient vanishes as $O(1/t)$, but the precise second-order correction to the Palm measure (capturing the slowly varying environment) is not captured by our analysis.

Declaration of Generative AI Use

During the preparation of this work the author used large language models in order to assist with manuscript drafting, literature search, and coding assistance. After using these tools, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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